Notes on Intuitionistic Fuzzy Sets
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System of intuitionistic fuzzy differential equations with intuitionistic fuzzy initial values

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Abstract: In this paper, we have studied the system of differential equations with intuitionistic fuzzy initial values under the interpretation of (i,ii)-GH differentiability concepts and Zadeh's extension principle interpretation. And we have given some numerical examples.
Keywords: Intuitionistic fuzzy sets, Strongly generalized Hukuhara differentiability, Intuitionistic fuzzy initial value problems, Intuitionistic Zadeh's extension principle.
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1 Introduction

Fuzzy set theory was firstly introduced by L. A. Zadeh in 1965 [46]. He defined fuzzy set concept by introducing every element with a function $\mu : X \to [0, 1]$, called membership function. Later, some extensions of fuzzy set theory were proposed [8, 30, 37]. One of these extensions is Atanassov's intuitionistic fuzzy set (IFS) theory [8].

In 1983, Atanassov [7] introduced the concept of intuitionistic fuzzy sets and carried out rigorous researches to develop the theory [8–16]. In this set concept, Apart from the membership function, he introduced a new degree $\nu : X \to [0, 1]$, called non-membership function, such that the sum $\mu + \nu$ is less than or equal to 1. Hence the difference $1 - (\mu + \nu)$ is regarded as degree of hesitation. Since intuitionistic fuzzy set theory contains membership function, non-membership function and the degree of hesitation, it can be regarded as a tool which is more flexible and closer to human reasoning in handling uncertainty due to imprecise knowledge or data.

Intuitionistic fuzzy set and fuzzy set theory have very compelling applications in various fields of science and engineering [1, 3–6, 17, 23, 24, 27–29, 31, 33–36, 38–40, 42, 45].

In literature, there are different approaches for solving fuzzy differential equations. Each method has advantages and disadvantages in the applications. One of the commonly used method is based on Zadeh's extension principle. In this method, the fuzzy solution is obtained from the crisp solution by using the well-known Zadeh's extension principle [22]. However there is no definition of fuzzy derivative in this approach . Hence some other methods based on fuzzy derivative concept were also proposed and used. One of the earliest method is Hukuhara differentiability concept [41]. However, this approach has also a weak point which is that the solution becomes fuzzier as time passes by [25]. Hence the length of the support of the fuzzy solution increases. To overcome this disadvantage some methods such as differential inclusions [32] and strongly generalized differentiability concept [19] were coined. The method based on strongly generalized differentiability concept allows us to obtain the solutions with decreasing length of support [18–21]. Hence the drawback of Hukuhara differentiability can be overcome with strongly generalized Hukuhara differentiability concept. Besides, this approach shows to be more favorable in applications [21].

The main goal of this paper is to give solutions to system of differential equations under the special cases of strongly generalized differentiability (GH) concept, (i.e. (i)-, (ii)-GH differentiability) [18–21] and under intuitionistic Zadeh's extension principle [16].

This paper is prepared as follows. In Section 2, some fundamental definitions and theorems in fuzzy sets and intuitionistic fuzzy sets are given. In Section 3, some definitions and theorems related to (i)-GH and (ii)-GH are extended from fuzzy case to intuitionistic fuzzy case by using the definitions and theorems in Section 2. In Section 4, we study system of differential equation on intuitionistic fuzzy environment under (i)-GH and (ii)-GH differentiability and the intuitionistic Zadeh's extension principle interpretation. Besides we give some numerical examples in this section. Finally we conclude the paper by giving summary and results in Section 6.

2 Preliminaries

Definition 2.1. [8] Let $\mu_A, \nu_A : \mathbb{R}^n \to [0,1]$ be two functions such that for each $x \in \mathbb{R}^n$, $0 \le \mu_A(x) + \nu_A(x) \le 1$ holds. The set

$$\tilde{A}^i = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in \mathbb{R}^n; \mu_A, \nu_A : \mathbb{R}^n \to [0,1]\}$$

is called an intuitionistic fuzzy set in $\mathbb{R}^n$. Here μ_A and ν_A are called membership and non-membership functions, respectively.

We will denote set of all intuitionistic fuzzy sets in $\mathbb{R}^n$ by $IF(\mathbb{R}^n)$.

Definition 2.2. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The α-cut of $\tilde{A}^i$ is defined as follows:

For $\alpha \in (0,1]$

$$A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \ge \alpha\},$$

and for $\alpha = 0$

$$A(0) = cl\left(\bigcup_{\alpha\in(0,1]} A(\alpha)\right).$$

Definition 2.3. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The β-cut of $\tilde{A}^i$ is defined as follows:

For $\beta \in (0, 1)$

$$A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\},$$

and for $\beta = 1$

$$A^*(1) = cl\left(\bigcup_{\beta \in [0,1)} A^*(\beta)\right).$$

Definition 2.4. [8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. For α and $\beta \in [0, 1]$ with $0 \leq \alpha + \beta \leq 1$, the set

$$A(\alpha, \beta) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$$

is called (α, β)-cut of $\tilde{A}^i$

Theorem 2.1. *[8] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. Then*

$$A(\alpha, \beta) = A(\alpha) \cap A^*(\beta)$$

holds.

Definition 2.5. [43] Let X be a topological space. Let f be a function from X to $\mathbb{R} \cup \{-\infty, \infty\}$.

i) f is called an upper semi-continuous function if for all $k \in \mathbb{R}$, the set $\{x \in X \mid f(x) < k\}$ is an open set.

ii) f is called an lower semi-continuous function if for all $k \in \mathbb{R}$, the set $\{x \in X \mid f(x) > k\}$ is an open set.

Definition 2.6. [44] Let f be a function defined on a convex subset K of $\mathbb{R}^n$.

i) f is called a quasi-concave function on K if for all $x, y \in K$ and $t \in [0, 1]$, it holds that $f(tx + (1 - ty)) \geq \min\{f(x), f(y)\}$.

ii) f is called a quasi-convex function on K if for all $x, y \in K$ and $t \in [0, 1]$, it holds that $f(tx + (1 - ty)) \leq \max\{f(x), f(y)\}$.

Definition 2.7. An intuitionistic fuzzy set $\tilde{A}^i \in IF(\mathbb{R}^n)$ satisfying the following properties is called an intuitionistic fuzzy number in $\mathbb{R}^n$.

1. $\tilde{A}^i$ is a normal set, i.e., $A(1) \neq \varnothing$ and $A^*(0) \neq \varnothing$.
2. $A(0)$ and $A^*(1)$ are bounded sets in $\mathbb{R}^n$.
3. $\mu_A : \mathbb{R}^n \to [0, 1]$ is an upper semi-continuous function; i.e., $\forall k \in [0, 1]$, $\{x \in A \mid \mu_A(x) < k\}$ is an open set.
4. $\nu_A : \mathbb{R}^n \to [0, 1]$ is a lower semi-continuous function; i.e., $\forall k \in [0, 1]$, $\{x \in A \mid \nu_A(x) > k\}$ is an open set.

5. The membership function μ_A is quasi-concave; i.e., $\forall\lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

6. The non-membership function ν_A is quasi-convex; i.e.,$\forall\lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$

$$\nu_A(\lambda x + (1-\lambda)y) \leq \max(\nu_A(x), \nu_A(y)); \forall\lambda \in [0,1],$$

We will denote the set of all intuitionistic fuzzy numbers in $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

Definition 2.8. [39] A triangular intuitionistic fuzzy number (TIFN) $\bar{A}^i \in IF_N(\mathbb{R})$ is defined with the following membership and non-membership functions:

$$\mu_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1};\ a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2};\ a_2 \leq x \leq a_3 \\ \quad 0;\ \text{otherwise} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1^*};\ a_1^* \leq x \leq a_2 \\ \dfrac{x-a_2}{a_3^*-a_2};\ a_2 \leq x \leq a_3^* \\ \quad 1;\ \text{otherwise} \end{cases}$$

Here $a_1^* \leq a_1 \leq a_2 \leq a_3 \leq a_3^*$ and it is denoted by $\bar{A}^i = (a_1, a_2, a_3; a_1^*, a_2, a_3^*)$. Note that its α and β cuts can be obtained as

$$A(\alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_3 - a_2)]$$

and

$$A^*(\beta) = [a_2 + \beta(a_2 - a_1^*), a_2 + \beta(a_3^* - a_2)].$$

Definition 2.9. [26] Let A and B be two nonempty subsets of $\mathbb{R}^n$ and $c \in \mathbb{R}$. The Minkowski addition and scalar multiplication of sets are defined as follows:

$$\begin{aligned} A + B &= \{a + b : a \in A \text{ and } b \in B\} \\ cA &= \{ca : a \in A\} \end{aligned}$$

Theorem 2.2. *[26] The family of all compact and convex subsets of $\mathbb{R}^n$ is closed under Minkowski addition and scalar multiplication.*

Definition 2.10. [2] Let $\bar{A}^i$, $\bar{B}^i \in IF_N(\mathbb{R}^n)$ and $c \in \mathbb{R} - \{0\}$. Addition and scalar multiplication of fuzzy numbers in $IF_N(\mathbb{R}^n)$ is defined as follows:

$$\begin{aligned} \bar{A}^i + \bar{B}^i &= \bar{C}^i \Leftrightarrow C(\alpha) = A(\alpha) + B(\alpha) \text{ and } C^*(\beta) = A^*(\beta) + B^*(\beta) \\ c(\bar{A}^i) &= \bar{D}^i \Leftrightarrow D(\alpha) = cA(\alpha) \text{ and } D^*(\beta) = cA^*(\beta) \end{aligned}$$

Definition 2.11. [26] Let a_1, a_2, b_1 and $b_2 \in \mathbb{R}$ such that $A = [a_1, a_2]$ and $B = [b_1, b_2]$. Basic end-point arithmetic operations of these closed and bounded intervals are as follows:

1. **Addition:** $A + B = [a_1 + b_1, a_2 + b_2]$
2. **Substraction:** $A - B = [a_1 - b_2, a_2 - b_1]$
3. **Multiplication:** $A.B = [min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$
4. **Division:** Assume the interval B does not contain zero. Then

   $A/B = [min\{a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2\}, max\{a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2\}]$

Theorem 2.3. [2] *Let $\bar{A}^i$, $\bar{B}^i \in IF_N(\mathbb{R}^n)$. Let us define the following distance functions*

$$\begin{aligned} D_1(\bar{A}^i, \bar{B}^i) &= \sup\{d_H(A(\alpha), B(\alpha)) : \alpha \in [0,1]\}, \\ D_2(\bar{A}^i, \bar{B}^i) &= \sup\{d_H(A(\beta), B(\beta)) : \beta \in [0,1]\}. \end{aligned}$$

Here d_H is Hausdorff metric [49]. The function

$$D(\bar{A}^i, \bar{B}^i) = \max\{D_1(\bar{A}^i, \bar{B}^i), D_2(\bar{B}^i, \bar{A}^i)\}$$

defines a metric on $IF_N(R^n)$. Hence $(IF_N(R^n), D)$ is a metric space.

Definition 2.12. Let $\bar{A}^i$, $\bar{B}^i \in IF_N(\mathbb{R}^n)$. The Hukuhara difference of $\bar{A}^i$ and $\bar{B}^i$ is $\bar{C}^i$, if it exists, such that

$$\bar{A}^i \ominus_H \bar{B}^i = \bar{C}^i \Longleftrightarrow \bar{A}^i = \bar{B}^i + \bar{C}^i$$

Definition 2.13. Let $\bar{A}^i$, $\bar{B}^i \in IF_N(\mathbb{R}^n)$. The generalized Hukuhara difference of $\bar{A}^i$ and $\bar{B}^i$ is $\bar{C}^i$, if it exists, such that

$$\bar{A}^i \ominus_{gH} \bar{B}^i = \bar{C}^i \Longleftrightarrow \bar{A}^i = \bar{B}^i + \bar{C}^i \text{ or } \bar{B}^i = \bar{A}^i + (-1)\bar{C}^i$$

Definition 2.14. Let $f : (a, b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $x_0, x_0 + h \in (a, b)$. f is called Hukuhara differentiable at x_0 if there exists an element $f'_H(x_0) \in IF_N(\mathbb{R})$ such that for all $h > 0$ the following is satisfied

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_H(x_0).$$

Definition 2.15. Let $f : (a, b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $x_0, x_0 + h \in (a, b)$. f is called strongly generalized Hukuhara differentiable at x_0 if there exists an element $f'_{GH}(x_0) \in IF_N(\mathbb{R})$ such that for all $h > 0$ at least one of the followings is satisfied:

1. $f(x_0 + h) \ominus_H f(x_0)$ and $f(x_0) \ominus_H f(x_0 - h)$ exist and the following limits exist such that

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_{GH}(x_0)$$

2. $f(x_0) \ominus_H f(x_0 + h)$ and $f(x_0 - h) \ominus_H f(x_0)$ exist and the following limits exist such that

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_{GH}(x_0)$$

3. $f(x_0+h)\ominus_H f(x_0)$ and $f(x_0-h)\ominus_H f(x_0)$ exist and the following limits exist such that

$$\lim_{h\to 0^+}\frac{f(x_0+h)\ominus_H f(x_0)}{h}=\lim_{h\to 0^+}\frac{f(x_0-h)\ominus_H f(x_0)}{-h}=f'_{GH}(x_0)$$

4. $f(x_0)\ominus_H f(x_0+h)$ and $f(x_0)\ominus_H f(x_0-h)$ exist and the following limits exist such that

$$\lim_{h\to 0^+}\frac{f(x_0)\ominus_H f(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{f(x_0)\ominus_H f(x_0-h)}{h}=f'_{GH}(x_0)$$

Definition 2.16. [1] Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a real valued function. The following function

$$\theta(f(x))=\begin{cases}1, & f(x)\geq 0\\ 0, & f(x)<0\end{cases}$$

is called the Heaviside step function.

Definition 2.17. [16] Let X and Y be two sets and $f: X\to Y$ be a function. Let $\bar{A}^i$ be an intuitionistic fuzzy set over X. Then $f(\bar{A}^i)$ is an intuitionistic fuzzy set over Y such that for every $y\in Y$

$$\mu_{f(\bar{A}^i)}(y)=\begin{cases}\sup\{\mu_A(x): f(x)=y\}; & y\in f(X)\\ 0; & y\notin f(X)\end{cases}$$

$$\nu_{f(\bar{A}^i)}(y)=\begin{cases}\inf\{\nu_A(x): f(x)=y\}; & y\in f(X)\\ 1; & y\notin f(X)\end{cases}$$

3 (i)-GH and (ii)-GH differentiability in an intuitionistic fuzzy environment

Definition 3.1. Let $f:[a,b]\to IF_N(\mathbb{R})$ and its α and β cuts be given by $f(t,\alpha)=[f_1(t,\alpha),f_2(t,\alpha)]$ and $f^*(t,\beta)=[f_1^*(t,\beta),f_2^*(t,\beta)]$ such that the end-points of its α and β cuts are differentiable at $t_0\in(a,b)$.

1. If

$$\begin{aligned}f'_{GH}(t_0,\alpha) &= [f'_1(t_0,\alpha),f'_2(t_0,\alpha)]\\ (f^*)'_{GH}(t_0,\beta) &= [(f_1^*)'(t_0,\beta),(f_2^*)'(t_0,\beta)],\end{aligned}$$

then f is called (i)-GH differentiable at x_0 for each $\alpha,\beta\in[0,1]$ and is denoted by $f'_{(i)-GH}$.

2. If

$$\begin{aligned}f'_{GH}(t_0,\alpha) &= [f'_2(t_0,\alpha),f'_1(t_0,\alpha)]\\ (f^*)'_{GH}(t_0,\beta) &= [(f_2^*)'(t_0,\beta),(f_1^*)'(t_0,\beta)],\end{aligned}$$

then f is called (ii)-GH differentiable at t_0 for each $\alpha,\beta\in[0,1]$ and is denoted by $f'_{(ii)-GH}$.

Theorem 3.1. [2] *Let $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ and $\alpha, \beta \in [0,1]$ such that the α and β cuts of $\tilde{A}^i$ given by $A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\}$ and $A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\}$. Then the followings hold.*

1. *For every $\alpha \in [0,1]$, $A(\alpha)$ is a non-empty compact and convex set in $\mathbb{R}^n$*
2. *If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A(\alpha_2) \subseteq A(\alpha_1)$.*
3. *If (α_n) is a non-decreasing sequence in $[0,1]$ converging to α then*
$$\bigcap_{n=1}^{\infty} A(\alpha_n) = A(\alpha).$$
4. *If (α_n) is a non-increasing sequence in $[0,1]$ converging to 0 then*
$$cl\left(\bigcup_{n=1}^{\infty} A(\alpha_n)\right) = A(0).$$
5. *For every $\beta \in [0,1]$, $A^*(\beta)$ is a non-empty compact and convex set in $\mathbb{R}^n$.*
6. *If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $A^*(\beta_1) \subseteq A^*(\beta_2)$.*
7. *If (β_n) is a non-increasing sequence in $[0,1]$ converging to β then*
$$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$
8. *If (β_n) is a non-decreasing sequence in $[0,1]$ converging to 1 then*
$$cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) = A^*(1).$$

Lemma 3.2. *Let $f : (a,b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $t_0 \in (a,b)$.*

1. *If for every $\alpha \in [0,1]$ the interval $[A_1(\alpha), A_2(\alpha)]$ satisfies the properties (1)-(4) in Theorem 3.1 and for every $\beta \in [0,1]$ the interval $[A_1^*(\beta), A_2^*(\beta)]$ satisfies the properties (5)-(8) in Theorem 3.1, and*
2. *If for every $\alpha, \beta \in [0,1]$, $\lim_{x \to x_0} f(t,\alpha) = [A_1(\alpha), A_2(\alpha)]$ and $\lim_{x \to x_0} f^*(t,\beta) = [A_1^*(\beta), A_2^*(\beta)]$*

then there exists an intuitionistic fuzzy number $\bar{A}^i$ such that $\lim_{t \to t_0} f(t) = \bar{A}^i$. And the α and β cuts of $\bar{A}^i$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$, respectively.

Proof. Assume the conditions (1) and (2) are satisfied. Since for every $\alpha \in [0,1]$ the interval $[A_1(\alpha), A_2(\alpha)]$ satisfies the properties (1)-(4) in Theorem 3.1 and for every $\beta \in [0,1]$ the interval $[A_1^*(\beta), A_2^*(\beta)]$ satisfies the properties (5)-(8) in Theorem 3.1 then there exists an intuitionistic fuzzy number $\bar{A}^i$ such that the α and β cuts of $\bar{A}^i$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$ [2]. Furthermore, since

$$\lim_{t\to t_0} f(t;\alpha) = A(\alpha) \Rightarrow \lim_{t\to t_0} D_1(f(t;\alpha), A(\alpha)) = 0$$

and

$$\lim_{t\to t_0} f^*(t;\beta) = A^*(\beta) \Rightarrow \lim_{t\to t_0} D_2(f^*(t;\beta), A^*(\beta)) = 0,$$

then by the definition of the metric D we can write that $\lim_{t\to t_0} D(f(t), \bar{A}^i) = 0$. Hence we obtain that $\lim_{t\to t_0} f(t) = \bar{A}^i$. □

Theorem 3.3. *Let $f : (a,b) \to IF_N(\mathbb{R})$ and its α and β-cuts be given by*

$$f(t,\alpha) = [f_1(t,\alpha), f_2(t,\alpha)] \text{ and } f^*(t,\beta) = [f_1^*(t,\beta), f_2^*(t,\beta)].$$

1. *If f is strongly generalized differentiable at $t_0 \in (a,b)$ as in case (1) of Definition 2.15 then for every $\alpha, \beta \in [0,1]$*

$$\begin{aligned} f'(t_0,\alpha) &= [f_1'(t_0,\alpha), f_2'(t_0,\alpha)] \\ (f^*)'(t_0,\beta) &= [(f_1^*)'(t_0,\beta), (f_2^*)'(t_0,\beta)] \end{aligned}$$

2. *If f is strongly generalized differentiable at $t_0 \in (a,b)$ as in case (2) of Definition 2.15 then for every $\alpha, \beta \in [0,1]$*

$$\begin{aligned} f'(t_0,\alpha) &= [f_2'(t_0,\alpha), f_1'(t_0,\alpha)] \\ (f^*)'(t_0,\beta) &= [(f_2^*)'(t_0,\beta), (f_1^*)'(t_0,\beta)] \end{aligned}$$

Proof. 1. Let $f : (a,b) \to IF_N(\mathbb{R})$, $t_0 \in (a,b)$ be strongly generalized differentiable as in case (1). Then $f(t+h) \ominus_H f(t)$ exists. Let $C(t,h) \in IF_N(\mathbb{R})$ such that $f(t+h) \ominus_H f(t) = C(t,h)$. Then $f(t+h) = f(t) + C(t,h)$ implies $f(t+h,\alpha) = f(t,\alpha) + C(t,h;\alpha)$ and $f^*(t+h,\beta) = f^*(t,\beta) + C^*(t,h;\beta)$. So by the interval operations we can write that

$$\begin{aligned} [f_1(t+h,\alpha), f_2(t+h,\alpha)] &= [f_1(t,\alpha), f_2(t,\alpha)] + [C_1(t,h;\alpha), C_2(t,h;\alpha)] \\ &= [f_1(t,\alpha) + C_1(t,h;\alpha), f_2(t,\alpha) + C_2(t,h;\alpha)] \end{aligned}$$

and

$$\begin{aligned} [f_1^*(t+h,\beta), f_2^*(t+h,\beta)] &= [f_1^*(t,\beta), f_2^*(t,\beta)] + [C_1^*(t,h;\beta), C_2^*(t,h;\beta)] \\ &= [f_1^*(t,\beta) + C_1^*(t,h;\beta), f_2^*(t,\beta) + C_2^*(t,h;\beta)] \end{aligned}$$

So by the equality of intervals we obtain that

$$C_1(t,h;\alpha) = f_1(t+h,\alpha) - f_1(t,\alpha), C_2(t,h;\alpha) = f_2(t+h,\alpha) - f_2(t,\alpha)$$

and

$$C_1^*(t,h;\beta) = f_1^*(t+h,\beta) - f_1^*(t,\beta), C_2^*(t,h;\beta) = f_2^*(t+h,\beta) - f_2^*(t,\beta).$$

Similar results can be obtained for $f(t) \ominus_H f(t-h)$. Hence we obtain

$$\begin{aligned}
\lim_{h\to 0^+} \frac{C_1(t,h;\alpha)}{h} &= \lim_{h\to 0^+} \frac{f_1(t+h,\alpha)-f_1(t,\alpha)}{h} = f_1'(t,\alpha) \\
\lim_{h\to 0^+} \frac{C_2(t,h;\alpha)}{h} &= \lim_{h\to 0^+} \frac{f_2(t+h,\alpha)-f_2(t,\alpha)}{h} = f_2'(t,\alpha) \\
\lim_{h\to 0^+} \frac{C_1^*(t,h;\beta)}{h} &= \lim_{h\to 0^+} \frac{f_1^*(t+h,\beta)-f_1^*(t,\beta)}{h} = (f_1^*)'(t,\beta) \\
\lim_{h\to 0^+} \frac{C_2^*(t,h;\beta)}{h} &= \lim_{h\to 0^+} \frac{f_2^*(t+h,\beta)-f_2^*(t,\beta)}{h} = (f_2^*)'(t,\beta)
\end{aligned}$$

So by Lemma 3.2 we can write that

$$\begin{aligned}
f'(t_0,\alpha) &= [f_1'(t_0,\alpha), f_2'(t_0,\alpha)] \\
(f^*)'(t_0,\beta) &= [(f_1^*)'(t_0,\beta), (f_2^*)'(t_0,\beta)]
\end{aligned}$$

2. The proof can be done in a similar way. □

Theorem 3.4. *Let $f:(a,b)\to IF_N(\mathbb{R})$, $t_0\in(a,b)$. If f is strongly generalized differentiable at t_0 as in case (3) or (4) of Definition 2.17. Then $f'(t)\in\mathbb{R}$ for all $t\in(a,b)$.*

Theorem 3.5. *Let f and g be intuitionistic fuzzy number valued functions. If f and g are both (i)-GH differentiable or both (ii)-GH differentiable, then*

1. $(f+g)'_{(i)-GH} = f'_{(i)-GH} + g'_{(i)-GH}$
2. $(f+g)'_{(ii)-GH} = f'_{(ii)-GH} + g'_{(ii)-GH}$

Proof. 1. Let $f,g\in IF_N(\mathbb{R})$. Let $f(t,\alpha)=[f_1(t,\alpha),f_2(t,\alpha)]$ and $f^*(t,\beta)=[f_1^*(t,\beta),f_2^*(t,\beta)]$ be α and β cuts of f ; and $g(t,\alpha)=[g_1(t,\alpha),g_2(t,\alpha)]$ and $g^*(t,\beta)=[g_1^*(t,\beta),g_2^*(t,\beta)]$ be α and β cuts of g. Assume f and g be (i)-GH differentiable then we can write that

$$\begin{aligned}
(f+g)'(t,\alpha) &= [(f+g)_1'(t,\alpha),(f+g)_2'(t,\alpha)] \\
&= [f_1'(t,\alpha)+g_1'(t,\alpha), f_2'(t,\alpha)+g_2'(t,\alpha)] \\
&= [f_1'(t,\alpha),f_2'(t,\alpha)]+[g_1'(t,\alpha),g_2'(t,\alpha)] \\
&= f'(t,\alpha)+g'(t,\alpha)
\end{aligned}$$

and

$$\begin{aligned}
((f+g)^*)'(t,\beta) &= [((f+g)_1^*)'(t,\beta),((f+g)_2^*)'(t,\beta)] \\
&= [(f_1^*)'(t,\beta)+(g_1^*)'(t,\beta),(f_2^*)'(t,\beta)+(g_2^*)'(t,\beta)] \\
&= [(f_1^*)'(t,\beta),(f_2^*)'(t,\beta)]+[(g_1^*)'(t,\beta),(g_2^*)'(t,\beta)] \\
&= (f^*)'(t,\beta)+(g^*)'(t,\beta)
\end{aligned}$$

2. The proof can be done in a similar way. □

4 Application to a system of intuitionistic fuzzy differential equations

In this section we will study the following system of first order differential equations in intuitionistic fuzzy environment under (i,ii)-GH differentiability and the intuitionistic Zadeh's extension principle interpretation.

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

4.1 Solving a system of intuitionistic fuzzy differential equations under (i,ii)-GH differentiability interpretation

Let A and B be non-zero real numbers. Let us consider the following system of intuitionistic fuzzy differential equation:

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

with the following triangular intuitionistic fuzzy initial values $x(t_0) = (a_1, a_2, a_3; a_1^*, a_2, a_3^*)$ and $y(t_0) = (b_1, b_2, b_3; b_1^*, b_2, b_3^*)$. So in terms of α and β cuts we can write this system as follows:

$$\begin{aligned}
\frac{dx(t,\alpha)}{dt} &= Ay(t,\alpha) \\
\frac{dy(t,\alpha)}{dt} &= Bx(t,\alpha) \\
x(t_0,\alpha) &= [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)] \\
y(t_0,\alpha) &= [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]
\end{aligned}$$

and

$$\begin{aligned}
\frac{dx^*(t,\beta)}{dt} &= Ay^*(t,\beta) \\
\frac{dy^*(t,\beta)}{dt} &= Bx^*(t,\beta) \\
x^*(t_0,\beta) &= [a_2 + \beta(a_1^* - a_2), a_2 + \beta(a_3^* - a_2)] \\
y^*(t_0,\beta) &= [b_2 + \beta(b_1^* - b_2), b_2 + \beta(b_3^* - b_2)]
\end{aligned}$$

Case 1: If x and y are (i)-differentiable, we can write that

$$\begin{aligned}
[x_1'(t,\alpha), x_2'(t,\alpha)] &= A[y_1(t,\alpha), y_2(t,\alpha)] \\
[y_1'(t,\alpha), y_2'(t,\alpha)] &= B[x_1(t,\alpha), x_2(t,\alpha)] \\
x(t_0,\alpha) &= [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)] \\
y(t_0,\alpha) &= [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)].
\end{aligned}$$

So by using Heaviside step function we can obtain that

$$
\begin{aligned}
x_1'(t) &= \theta(A)(y_1(t,\alpha) - y_2(t,\alpha)) + y_2(t,\alpha)\\
x_2'(t) &= \theta(A)(y_2(t,\alpha) - y_1(t,\alpha)) + y_1(t,\alpha)\\
y_1'(t) &= \theta(B)(x_1(t,\alpha) - x_2(t,\alpha)) + x_2(t,\alpha)\\
y_2'(t) &= \theta(B)(x_2(t,\alpha) - x_1(t,\alpha)) + x_1(t,\alpha)\\
x_1(t_0,\alpha) &= a_1 + \alpha(a_2 - a_1)\\
x_2(t_0,\alpha) &= a_3 + \alpha(a_2 - a_3)\\
y_1(t_0,\alpha) &= b_1 + \alpha(b_2 - b_1)\\
y_2(t_0,\alpha) &= b_3 + \alpha(b_2 - b_3)
\end{aligned}
$$

and

$$
\begin{aligned}
(x_1^*)'(t) &= \theta(A)(y_1^*(t,\beta) - y_2^*(t,\beta)) + y_2^*(t,\beta)\\
(x_2^*)'(t) &= \theta(A)(y_2^*(t,\beta) - y_1^*(t,\beta)) + y_1^*(t,\beta)\\
(y_1^*)'(t) &= \theta(B)(x_1^*(t,\beta) - x_2^*(t,\beta)) + x_2^*(t,\beta)\\
(y_2^*)'(t) &= \theta(B)(x_2^*(t,\beta) - x_1^*(t,\beta)) + x_1^*(t,\beta)\\
x_1^*(t_0,\beta) &= a_2 + \beta(a_1^* - a_2)\\
x_2^*(t_0,\beta) &= a_2 + \beta(a_3^* - a_2)\\
y_1^*(t_0,\beta) &= b_2 + \beta(b_1^* - b_2)\\
y_2^*(t_0,\beta) &= b_2 + \beta(b_3^* - b_2)
\end{aligned}
$$

Case 2: If x and y are (ii)-differentiable we can write that

$$
\begin{aligned}
[x_2', x_1'] &= A[y_1(t,\alpha), y_2(t,\alpha)]\\
[y_2', y_1'] &= B[x_1(t,\alpha), x_2(t,\alpha)]\\
x(t_0,\alpha) &= [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]\\
x(t_0,\alpha) &= [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]
\end{aligned}
$$

So by using Heaviside step function we can obtain that

$$
\begin{aligned}
x_2'(t) &= \theta(A)(y_1(t,\alpha) - y_2(t,\alpha)) + y_2(t,\alpha)\\
x_1'(t) &= \theta(A)(y_2(t,\alpha) - y_1(t,\alpha)) + y_1(t,\alpha)\\
y_2'(t) &= \theta(B)(x_1(t,\alpha) - x_2(t,\alpha)) + x_2(t,\alpha)\\
y_1'(t) &= \theta(B)(x_2(t,\alpha) - x_1(t,\alpha)) + x_1(t,\alpha)\\
x_1(t_0,\alpha) &= a_1 + \alpha(a_2 - a_1)\\
x_2(t_0,\alpha) &= a_3 + \alpha(a_2 - a_3)\\
y_1(t_0,\alpha) &= b_1 + \alpha(b_2 - b_1)\\
y_2(t_0,\alpha) &= b_3 + \alpha(b_2 - b_3)
\end{aligned}
$$

and

$$
\begin{aligned}
(x_2^*)'(t) &= \theta(A)(y_1^*(t,\beta) - y_2^*(t,\beta)) + y_2^*(t,\beta)\\
(x_1^*)'(t) &= \theta(A)(y_2^*(t,\beta) - y_1^*(t,\beta)) + y_1^*(t,\beta)\\
(y_2^*)'(t) &= \theta(B)(x_1^*(t,\beta) - x_2^*(t,\beta)) + x_2^*(t,\beta)\\
(y_1^*)'(t) &= \theta(B)(x_2^*(t,\beta) - x_1^*(t,\beta)) + x_1^*(t,\beta)
\end{aligned}
$$

$$
\begin{aligned}
x_1^*(t_0,\beta) &= a_2+\beta(a_1^*-a_2)\\
x_2^*(t_0,\beta) &= a_2+\beta(a_3^*-a_2)\\
y_1^*(t_0,\beta) &= b_2+\beta(b_1^*-b_2)\\
y_2^*(t_0,\beta) &= b_2+\beta(b_3^*-b_2)
\end{aligned}
$$

Case 3: If x is (i)- and y is (ii)-differentiable we can write that

$$
\begin{aligned}
[x_1',x_2'] &= A[y_1(t,\alpha),y_2(t,\alpha)]\\
[y_2',y_1'] &= B[x_1(t,\alpha),x_2(t,\alpha)]\\
x(t_0,\alpha) &= [a_1+\alpha(a_2-a_1),a_3+\alpha(a_2-a_3)]\\
y(t_0,\alpha) &= [b_1+\alpha(b_2-b_1),b_3+\alpha(b_2-b_3)]
\end{aligned}
$$

So by using Heaviside step function we can obtain that

$$
\begin{aligned}
x_1'(t) &= \theta(A)(y_1(t,\alpha)-y_2(t,\alpha))+y_2(t,\alpha)\\
x_2'(t) &= \theta(A)(y_2(t,\alpha)-y_1(t,\alpha))+y_1(t,\alpha)\\
y_2'(t) &= \theta(B)(x_1(t,\alpha)-x_2(t,\alpha))+x_2(t,\alpha)\\
y_1'(t) &= \theta(B)(x_2(t,\alpha)-x_1(t,\alpha))+x_1(t,\alpha)\\
x_1(t_0,\alpha) &= a_1+\alpha(a_2-a_1)\\
x_2(t_0,\alpha) &= a_3+\alpha(a_2-a_3)\\
y_1(t_0,\alpha) &= b_1+\alpha(b_2-b_1)\\
y_2(t_0,\alpha) &= b_3+\alpha(b_2-b_3)
\end{aligned}
$$

and

$$
\begin{aligned}
(x_1^*)'(t) &= \theta(A)(y_1^*(t,\beta)-y_2^*(t,\beta))+y_2^*(t,\beta)\\
(x_2^*)'(t) &= \theta(A)(y_2^*(t,\beta)-y_1^*(t,\beta))+y_1^*(t,\beta)\\
(y_2^*)'(t) &= \theta(B)(x_1^*(t,\beta)-x_2^*(t,\beta))+x_2^*(t,\beta)\\
(y_1^*)'(t) &= \theta(B)(x_2^*(t,\beta)-x_1^*(t,\beta))+x_1^*(t,\beta)\\
x_1^*(t_0,\beta) &= a_2+\beta(a_1^*-a_2)\\
x_2^*(t_0,\beta) &= a_2+\beta(a_3^*-a_2)\\
y_1^*(t_0,\beta) &= b_2+\beta(b_1^*-b_2)\\
y_2^*(t_0,\beta) &= b_2+\beta(b_3^*-b_2)
\end{aligned}
$$

Case 4: If x is (ii)- and y is (i)-differentiable we can write that

$$
\begin{aligned}
[x_2',x_1'] &= A[y_1(t,\alpha),y_2(t,\alpha)]\\
[y_1',y_2'] &= B[x_1(t,\alpha),x_2(t,\alpha)]\\
x(t_0,\alpha) &= [a_1+\alpha(a_2-a_1),a_3+\alpha(a_2-a_3)]\\
x(t_0,\alpha) &= [b_1+\alpha(b_2-b_1),b_3+\alpha(b_2-b_3)]
\end{aligned}
$$

In terms of α cuts we obtain

$$\begin{aligned}
x_2'(t) &= \theta(A)(y_1(t,\alpha) - y_2(t,\alpha)) + y_2(t,\alpha)\\
x_1'(t) &= \theta(A)(y_2(t,\alpha) - y_1(t,\alpha)) + y_1(t,\alpha)\\
y_1'(t) &= \theta(B)(x_1(t,\alpha) - x_2(t,\alpha)) + x_2(t,\alpha)\\
y_2'(t) &= \theta(B)(x_2(t,\alpha) - x_1(t,\alpha)) + x_1(t,\alpha)\\
x_1(t_0,\alpha) &= a_1 + \alpha(a_2 - a_1)\\
x_2(t_0,\alpha) &= a_3 + \alpha(a_2 - a_3)\\
y_1(t_0,\alpha) &= b_1 + \alpha(b_2 - b_1)\\
y_2(t_0,\alpha) &= b_3 + \alpha(b_2 - b_3)
\end{aligned}$$

and

$$\begin{aligned}
(x_2^*)'(t) &= \theta(A)(y_1^*(t,\beta) - y_2^*(t,\beta)) + y_2^*(t,\beta)\\
(x_1^*)'(t) &= \theta(A)(y_2^*(t,\beta) - y_1^*(t,\beta)) + y_1^*(t,\beta)\\
(y_1^*)'(t) &= \theta(B)(x_1^*(t,\beta) - x_2^*(t,\beta)) + x_2^*(t,\beta)\\
(y_2^*)'(t) &= \theta(B)(x_2^*(t,\beta) - x_1^*(t,\beta)) + x_1^*(t,\beta)\\
x_1^*(t_0,\beta) &= a_2 + \beta(a_1^* - a_2)\\
x_2^*(t_0,\beta) &= a_2 + \beta(a_3^* - a_2)\\
y_1^*(t_0,\beta) &= b_2 + \beta(b_1^* - b_2)\\
y_2^*(t_0,\beta) &= b_2 + \beta(b_3^* - b_2)
\end{aligned}$$

Example 4.1.1: Let us find the intuitionistic fuzzy solution of the following system of differential equations

$$\frac{dx(t)}{dt} = 5y(t), \quad \frac{dy(t)}{dt} = 3x(t)$$

with the following triangular intuitionistic fuzzy initial values $x(0) = (1, 2, 3; -2, 2, 6)$ and $y(0) = (-2, -1, 0; -4, -1, 2)$ under (i,ii)-GH differentiability. For not being too repetitive we will give the solutions only for Case 1 and Case 3.

Case 1: If x and y are both (i)-GH differentiable we can write that α and β cuts of the system as follows:

$$\begin{aligned}
x_1'(t) &= 5y_1(t,\alpha)\\
x_2'(t) &= 5y_2(t,\alpha)\\
y_1'(t) &= 3x_1(t,\alpha)\\
y_2'(t) &= 3x_2(t,\alpha)\\
x_1(t_0,\alpha) &= 1 + \alpha\\
x_2(t_0,\alpha) &= 3 - \alpha\\
y_1(t_0,\alpha) &= -2 + \alpha\\
y_2(t_0,\alpha) &= -\alpha
\end{aligned}$$

and

$$
\begin{array}{rcl}
(x_1^*)'(t) &=& 5y_1^*(t,\beta) \\
(x_2^*)'(t) &=& 5y_2^*(t,\beta) \\
(y_1^*)'(t) &=& 3x_1^*(t,\beta) \\
(y_2^*)'(t) &=& 3x_2^*(t,\beta) \\
x_1^*(t_0,\beta) &=& 2-4\beta \\
x_2^*(t_0,\beta) &=& 2+4\beta \\
y_1^*(t_0,\beta) &=& -1-3\beta \\
y_2^*(t_0,\beta) &=& -1+3\beta
\end{array}
$$

By solving these two systems as in classical ordinary differential systems we obtain the solutions as follows:

$$
\begin{array}{rcl}
x_1(t,\alpha) &=& \frac{1}{10}e^{-3\sqrt{5}t-t}(-3\sqrt{5}\alpha e^t+5\alpha e^t+3\sqrt{5}\alpha e^{6\sqrt{5}t+t}+5\alpha e^{6\sqrt{5}t+t}+\sqrt{5}e^t-5e^t \\
&& +15e^{3\sqrt{5}t}+5e^{3\sqrt{5}t+2t}-3\sqrt{5}e^{6\sqrt{5}t+t}-5e^{6\sqrt{5}t+t}) \\
x_2(t,\alpha) &=& -\frac{1}{10}e^{-3\sqrt{5}t-t}(-3\sqrt{5}\alpha e^t+5\alpha e^t+3\sqrt{5}\alpha e^{6\sqrt{5}t+t}+5\alpha e^{6\sqrt{5}t+t}+3\sqrt{5}e^t-5e^t \\
&& -15e^{3\sqrt{5}t}-5e^{3\sqrt{5}t+2t}-3\sqrt{5}e^{6\sqrt{5}t+t}-5e^{6\sqrt{5}t+t}) \\
y_1(t,\alpha) &=& \frac{1}{6}e^{-3\sqrt{5}t-t}(-\sqrt{5}\alpha e^t+3\alpha e^t+\sqrt{5}\alpha e^{6\sqrt{5}t+t}+3\alpha e^{6\sqrt{5}t+t}+\sqrt{5}e^t-3e^t \\
&& -9e^{3\sqrt{5}t}+3e^{3\sqrt{5}t+2t}-\sqrt{5}e^{6\sqrt{5}t+t}-3e^{6\sqrt{5}t+t}) \\
y_2(t,\alpha) &=& -\frac{1}{6}e^{-3\sqrt{5}t-t}(-\sqrt{5}\alpha e^t+3\alpha e^t+\sqrt{5}\alpha e^{6\sqrt{5}t+t}+3\alpha e^{6\sqrt{5}t+t}+\sqrt{5}e^t-3e^t \\
&& +9e^{3\sqrt{5}t}-3e^{3\sqrt{5}t+2t}-\sqrt{5}e^{6\sqrt{5}t+t}-3e^{6\sqrt{5}t+t})
\end{array}
$$

and

$$
\begin{array}{rcl}
x_1^*(t,\beta) &=& -\frac{1}{10}e^{-3\sqrt{5}t-t}(-9\sqrt{5}\beta e^t+20\beta e^t+9\sqrt{5}\beta e^{6\sqrt{5}t+t}+20\beta e^{6\sqrt{5}t+t} \\
&& -15e^{3\sqrt{5}t}-5e^{3\sqrt{5}t+2t}) \\
x_2^*(t,\beta) &=& \frac{1}{10}e^{-3\sqrt{5}t-t}(-9\sqrt{5}\beta e^t+20\beta e^t+9\sqrt{5}\beta e^{6\sqrt{5}t+t}+20\beta e^{6\sqrt{5}t+t} \\
&& +15e^{3\sqrt{5}t}+5e^{3\sqrt{5}t+2t}) \\
y_1^*(t,\beta) &=& -\frac{1}{6}e^{-3\sqrt{5}t-t}(-4\sqrt{5}\beta e^t+9\beta e^t+4\sqrt{5}\beta e^{6\sqrt{5}t+t}+9\beta e^{6\sqrt{5}t+t} \\
&& +9e^{3\sqrt{5}t}-3e^{3\sqrt{5}t+2t}) \\
y_2^*(t,\beta) &=& \frac{1}{6}e^{-3\sqrt{5}t-t}(-4\sqrt{5}\beta e^t+9\beta e^t+4\sqrt{5}\beta e^{6\sqrt{5}t+t}+9\beta e^{6\sqrt{5}t+t} \\
&& -9e^{3\sqrt{5}t}+3e^{3\sqrt{5}t+2t})
\end{array}
$$

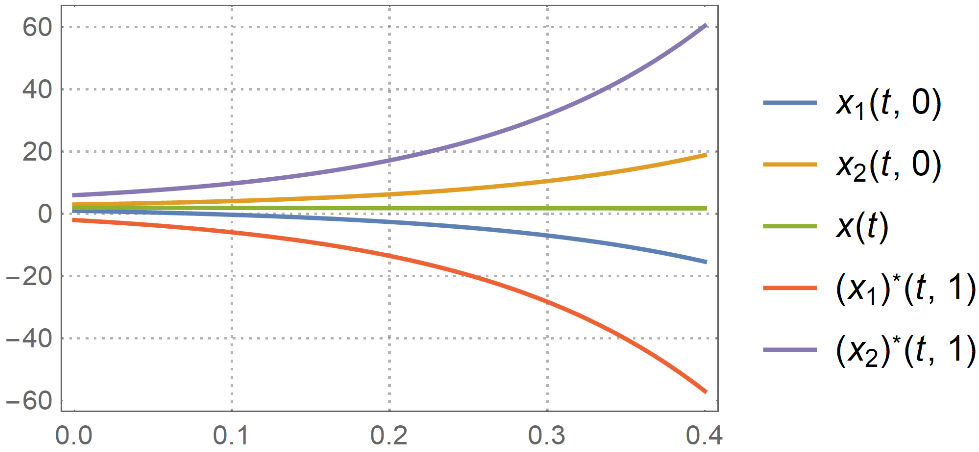

Figure 1. End points of $x(t, 0)$ and $x^*(t, 1)$ for Case 1.

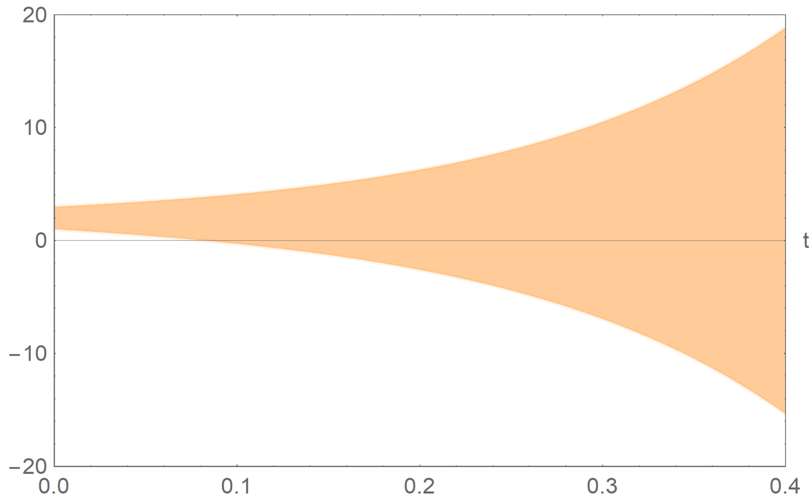

Figure 2. Graph of α cut of the solution x for Case 1.

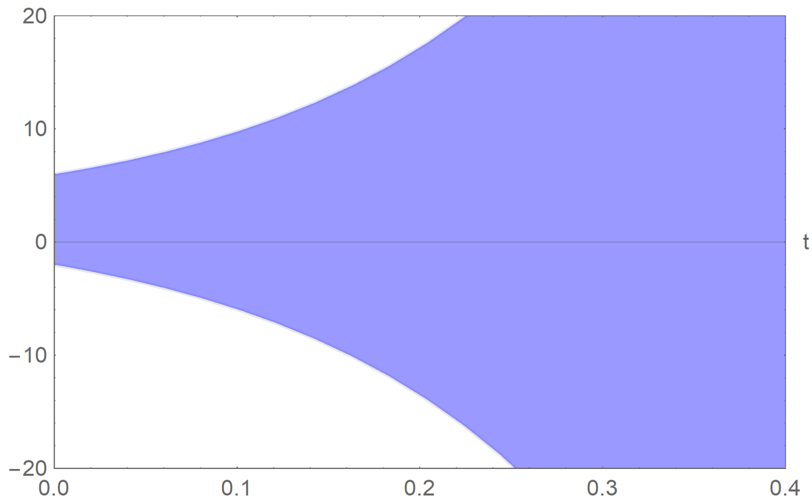

Figure 3. Graph of β cut of the solution x for Case 1.

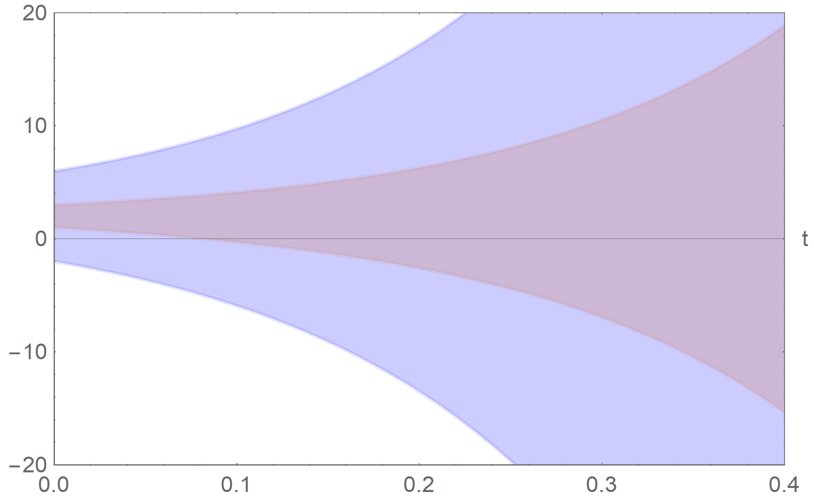

Figure 4. The intersected region is (α, β) cut of the solution x for Case 1.

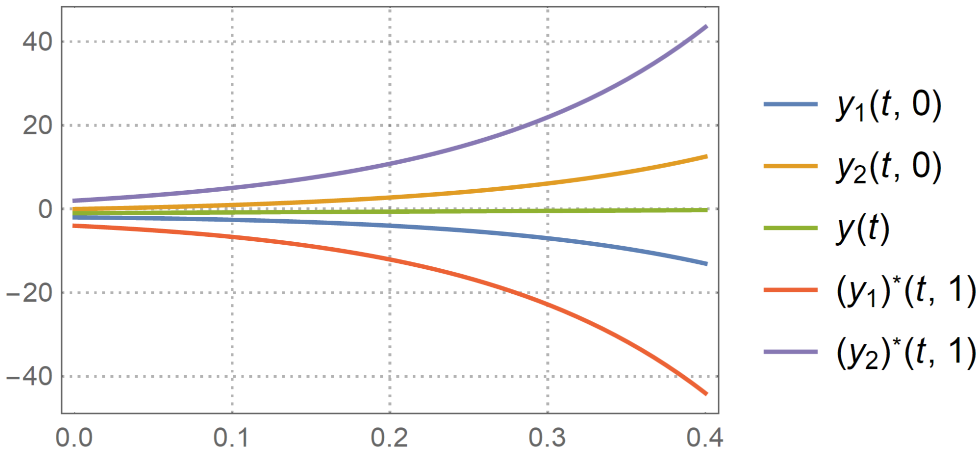

Figure 5. End points of $y(t, 0)$ and $y^*(t, 1)$ for Case 1.

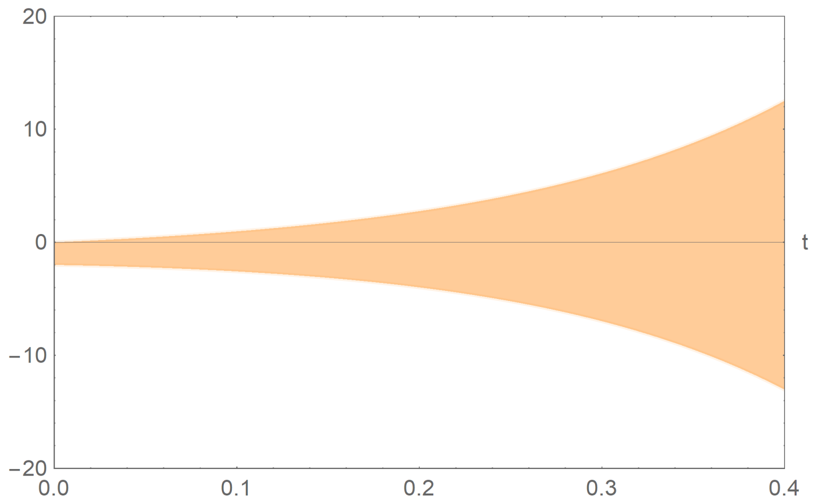

Figure 6. Graph of β cut of the solution y for Case 1.

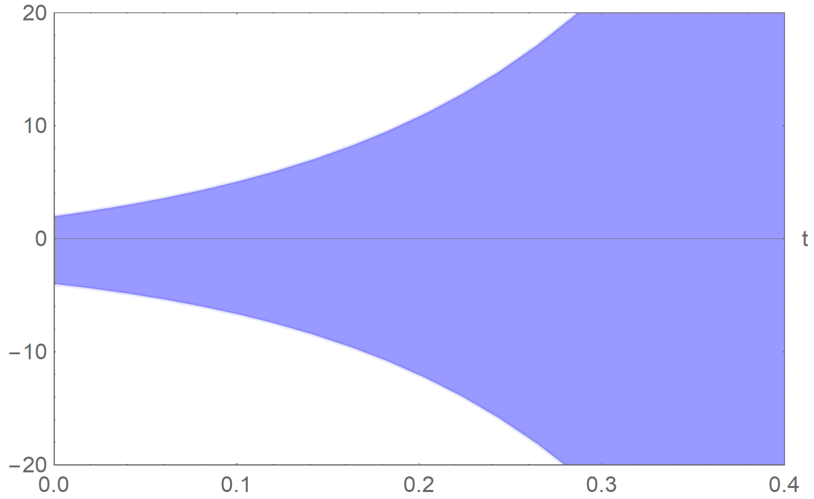

Figure 7. Graph of β cut of the solution y for Case 1.

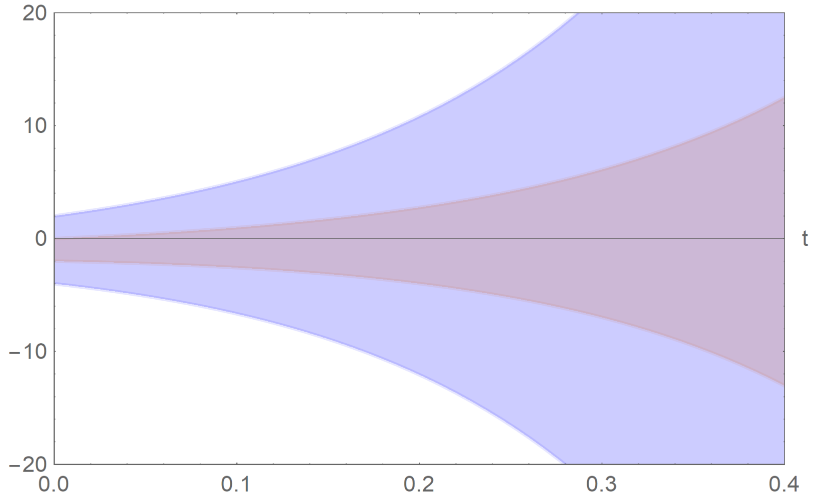

Figure 8. The intersected region is (α, β) cuts of the solution y for Case 1.

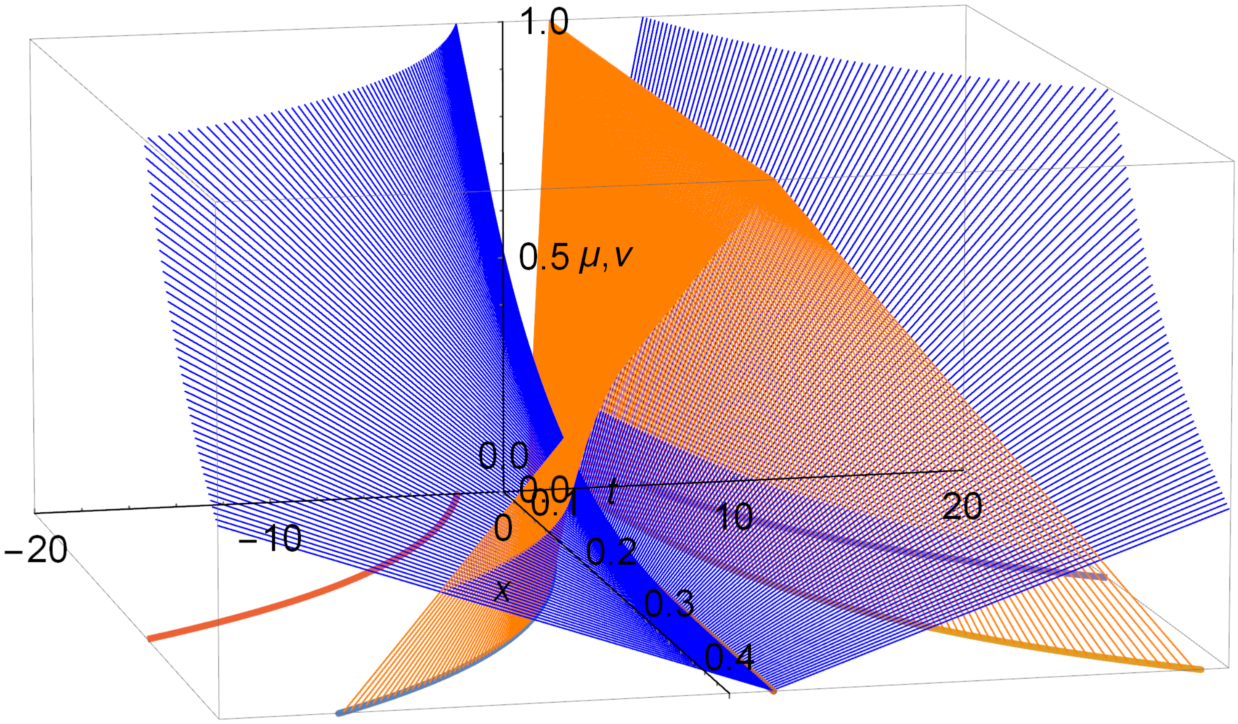

Figure 9. Membership μ and non-membership ν functions of the solution x for Case 1.

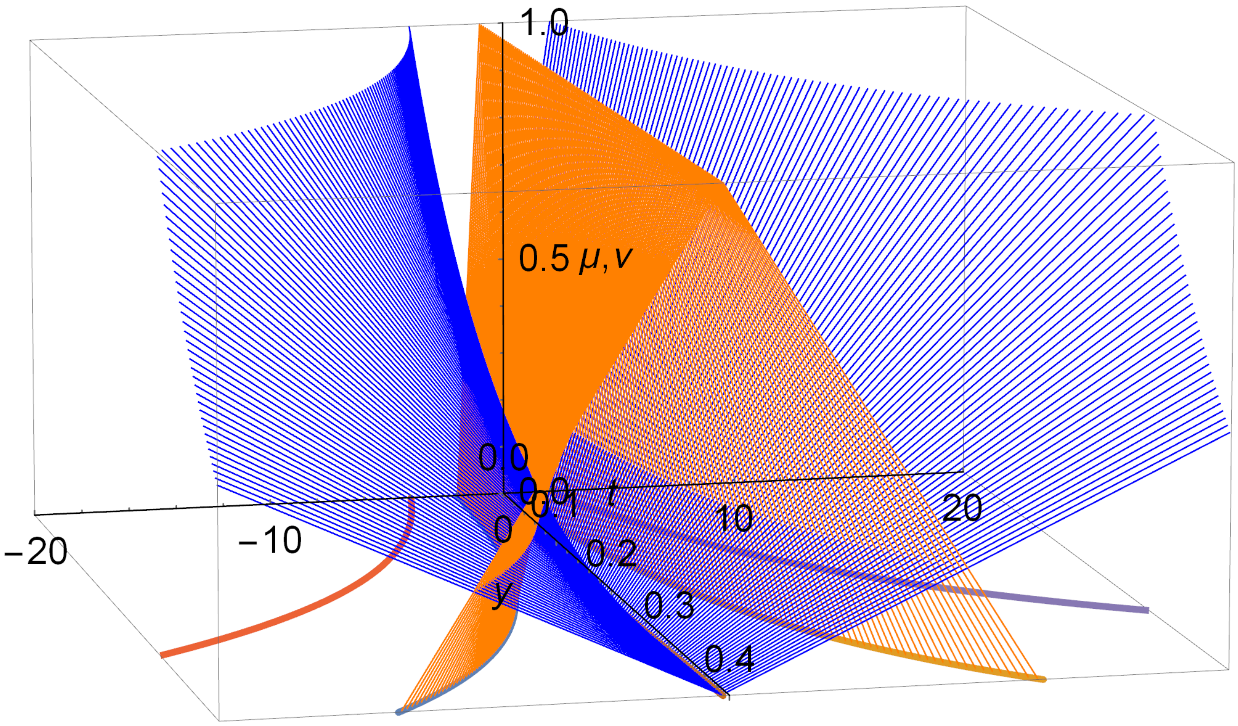

Figure 10. Membership μ and non-membership ν functions of the solution y for Case 1.

Case 3: Let x be (i)-GH differentiable and y be (ii)-GH differentiable. Then we can write that α and β cuts of the system as follows:

$$\begin{aligned}
x_1'(t) &= 5y_1(t,\alpha)\\
x_2'(t) &= 5y_2(t,\alpha)\\
y_1'(t) &= 3x_2(t,\alpha)\\
y_2'(t) &= 3x_1(t,\alpha)\\
x_1(t_0,\alpha) &= 1+\alpha\\
x_2(t_0,\alpha) &= 3-\alpha\\
y_1(t_0,\alpha) &= -2+\alpha\\
y_2(t_0,\alpha) &= -\alpha
\end{aligned}$$

and

$$\begin{aligned}
(x_1^*)'(t) &= 5y_1^*(t,\beta)\\
(x_2^*)'(t) &= 5y_2^*(t,\beta)\\
(y_1^*)'(t) &= 3x_2^*(t,\beta)\\
(y_2^*)'(t) &= 3x_1^*(t,\beta)\\
x_1^*(t_0,\beta) &= 2-4\beta\\
x_2^*(t_0,\beta) &= 2+4\beta\\
y_1^*(t_0,\beta) &= -1-3\beta\\
y_2^*(t_0,\beta) &= -1+3\beta
\end{aligned}$$

By solving these two systems as in classical ordinary differential systems we obtain the solutions as follows:

$$\begin{aligned}
x_1(t,\alpha) &= \frac{1}{10}e^{-t}(6\sqrt{5}\alpha e^t\sin\left(3\sqrt{5}t\right)+10\alpha e^t\cos\left(3\sqrt{5}t\right)+5e^{2t}-6\sqrt{5}e^t\sin\left(3\sqrt{5}t\right)\\
&\quad -10e^t\cos\left(3\sqrt{5}t\right)+15)\\
x_2(t,\alpha) &= \frac{1}{10}e^{-t}(-6\sqrt{5}\alpha e^t\sin\left(3\sqrt{5}t\right)-10\alpha e^t\cos\left(3\sqrt{5}t\right)+5e^{2t}+6\sqrt{5}e^t\sin\left(3\sqrt{5}t\right)\\
&\quad +10e^t\cos\left(3\sqrt{5}t\right)+15)\\
y_1(t,\alpha) &= \frac{1}{6}e^{-t}(-2\sqrt{5}\alpha e^t\sin\left(3\sqrt{5}t\right)+6\alpha e^t\cos\left(3\sqrt{5}t\right)+3e^{2t}+2\sqrt{5}e^t\sin\left(3\sqrt{5}t\right)\\
&\quad -6e^t\cos\left(3\sqrt{5}t\right)-9)\\
y_2(t,\alpha) &= \frac{1}{6}e^{-t}(2\sqrt{5}\alpha e^t\sin\left(3\sqrt{5}t\right)-6\alpha e^t\cos\left(3\sqrt{5}t\right)+3e^{2t}-2\sqrt{5}e^t\sin\left(3\sqrt{5}t\right)\\
&\quad +6e^t\cos\left(3\sqrt{5}t\right)-9)
\end{aligned}$$

and

$$\begin{aligned}
x_1^*(t,\beta) &= \frac{1}{10}e^{-t}(-18\sqrt{5}\beta e^t\sin\left(3\sqrt{5}t\right)-40\beta e^t\cos\left(3\sqrt{5}t\right)+5e^{2t}+15)\\
x_2^*(t,\beta) &= \frac{1}{10}e^{-t}(18\sqrt{5}\beta e^t\sin\left(3\sqrt{5}t\right)+40\beta e^t\cos\left(3\sqrt{5}t\right)+5e^{2t}+15)\\
y_1^*(t,\beta) &= \frac{1}{6}e^{-t}(8\sqrt{5}\beta e^t\sin\left(3\sqrt{5}t\right)-18\beta e^t\cos\left(3\sqrt{5}t)+3e^{2t}-9\right)\\
y_2^*(t,\beta) &= \frac{1}{6}e^{-t}(-8\sqrt{5}\beta e^t\sin\left(3\sqrt{5}t\right)+18\beta e^t\cos\left(3\sqrt{5}t)+3e^{2t}-9\right)
\end{aligned}$$

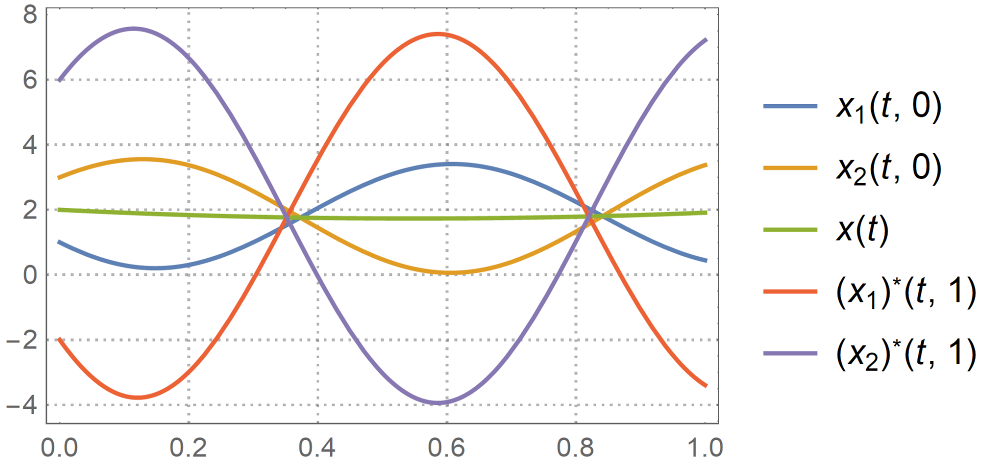

Figure 11. End points of $x(t,0)$ and $x^*(t,1)$ for Case 3.

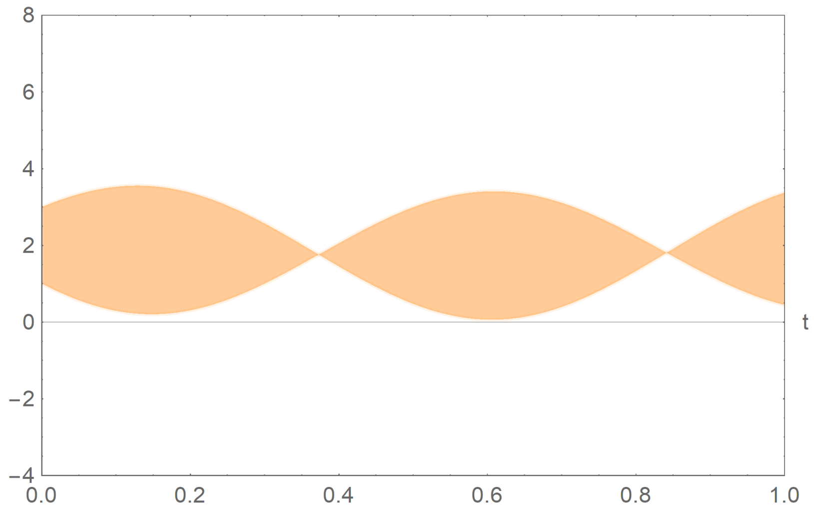

Figure 12. Graph of α cut of the solution x for Case 3.

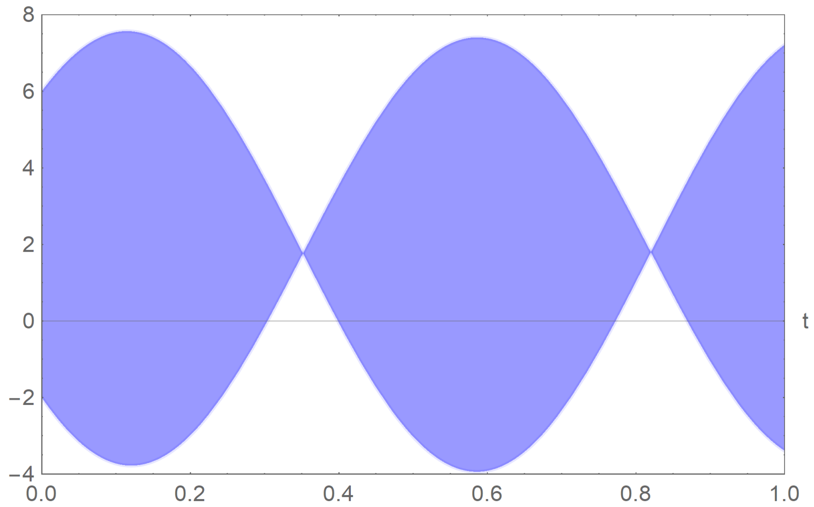

Figure 13. Graph of β cut of the solution x for Case 3.

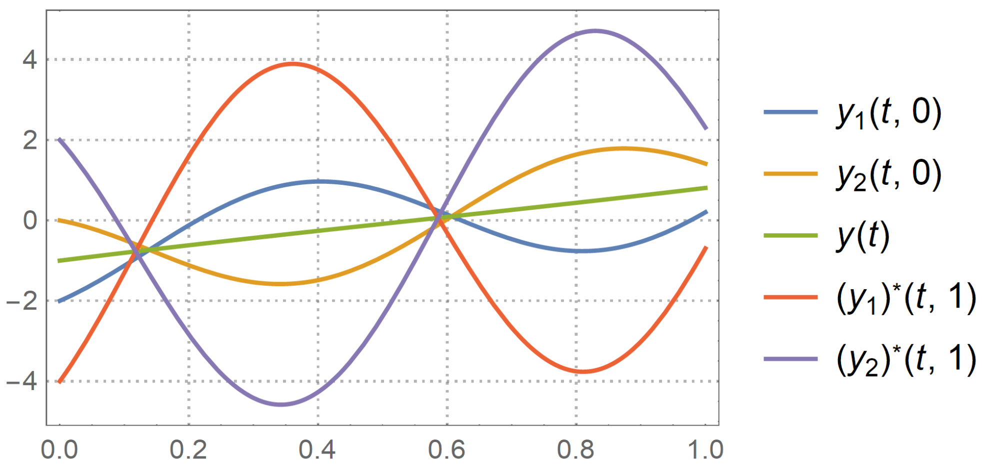

Figure 14. End points of $y(t, 0)$ and $y^*(t, 1)$ for Case 3.

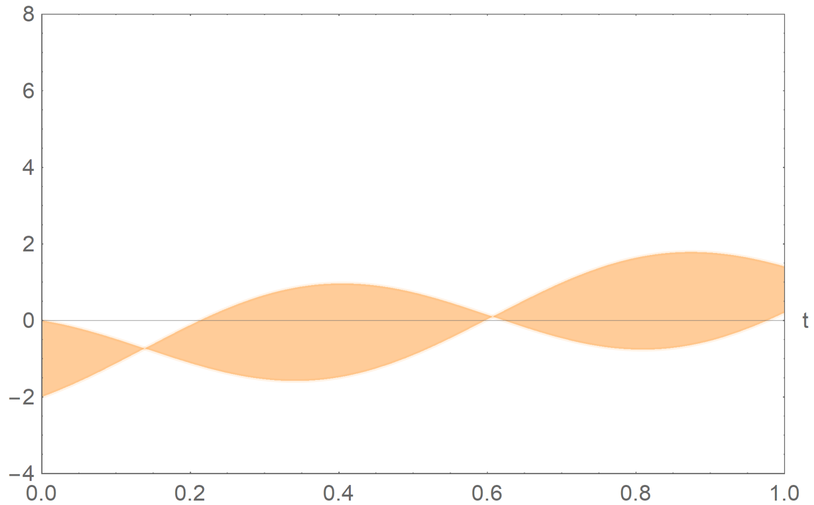

Figure 15. Graph of β cut of the solution y for Case 3.

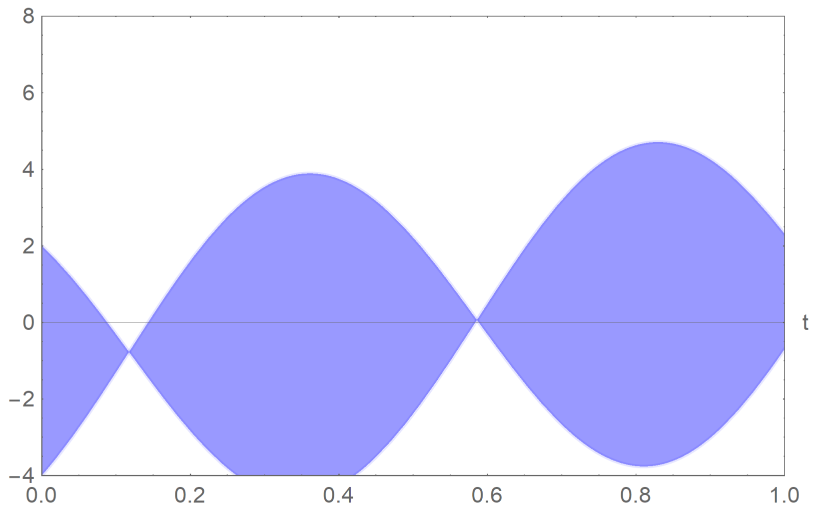

Figure 16. Graph of β cut of the solution y for Case 3.

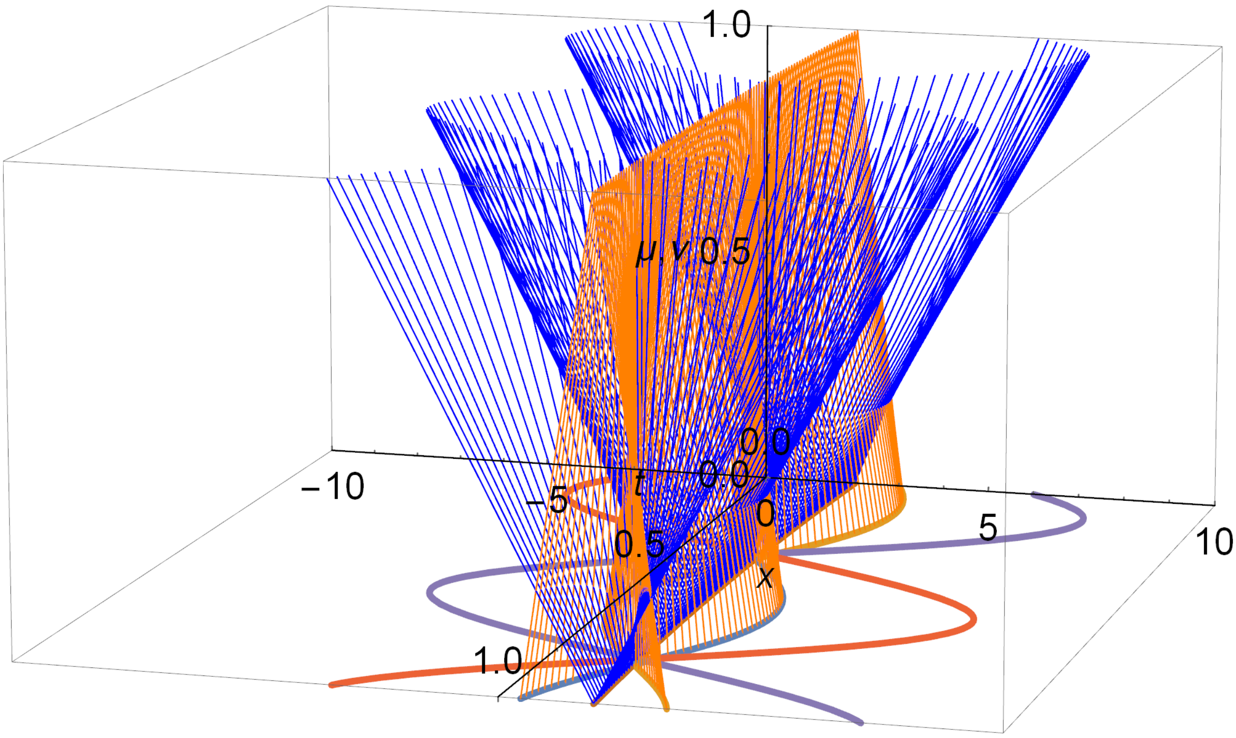

Figure 17. Membership μ and non-membership ν functions of the solution x for Case 3.

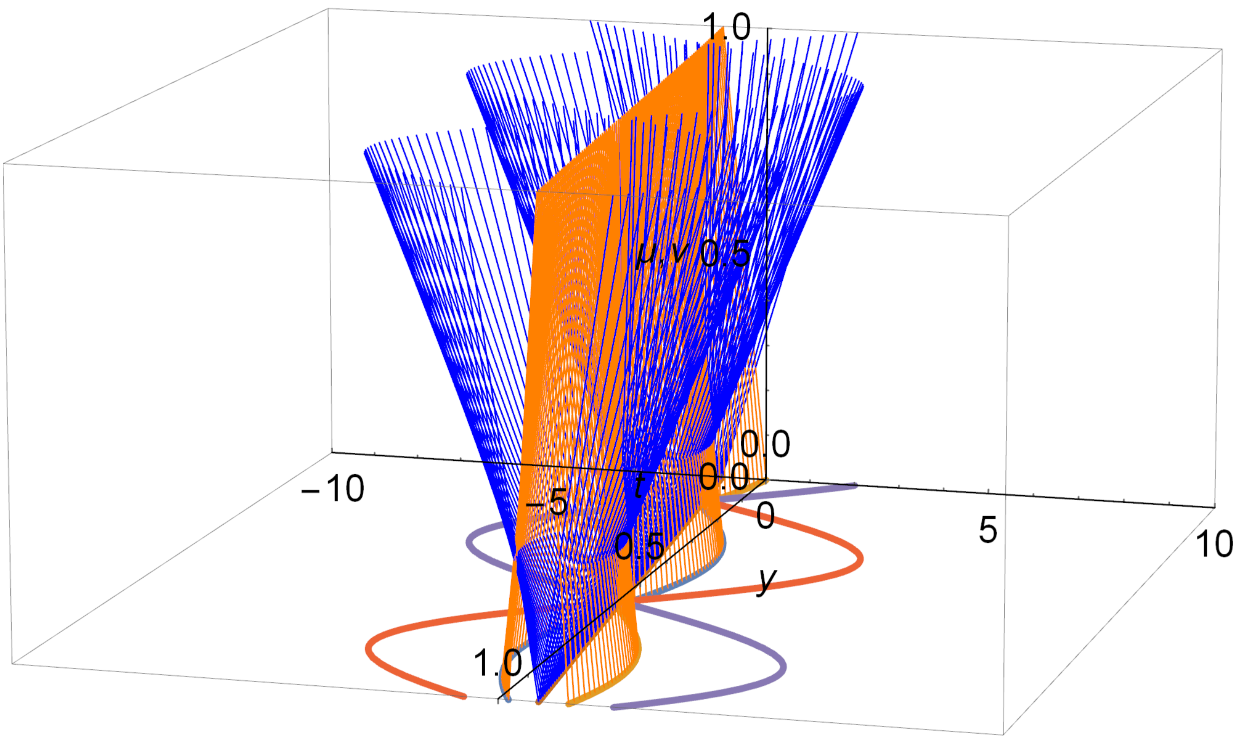

Figure 18. Membership μ and non-membership ν functions of the solution y for Case 3.

4.2 Solving a system of intuitionistic fuzzy differential equations under Zadeh's extension principle interpretation

Let A and B be positive real numbers. Now let us consider the following system of intuitionistic fuzzy differential equation

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

with the following triangular intuitionistic fuzzy numbers $x(t_0) = (a_1, a_2, a_3; a_1^*, a_2, a_3^*)$ and $y(t_0) = (b_1, b_2, b_3; b_1^*, b_2, b_3^*)$ under intuitionistic Zadeh's Extension Principle. Firstly we need to find the crisp solution of

$$\frac{dx(t)}{dt} = Ay(t), \quad \frac{dy(t)}{dt} = Bx(t)$$

with $x(t_0) = a_2$ and $y(t_0) = b_2$. The crisp solution of this system is

$$x(t) = \frac{e^{-\sqrt{AB}t}\left(a_2\sqrt{B}\left(e^{2\sqrt{AB}t}+1\right)+\sqrt{A}b_2\left(e^{2\sqrt{AB}t}-1\right)\right)}{2\sqrt{B}}$$

$$y(t) = \frac{e^{-\sqrt{AB}t}\left(a_2\sqrt{B}\left(e^{2\sqrt{AB}t}-1\right)+\sqrt{A}b_2\left(e^{2\sqrt{AB}t}+1\right)\right)}{2\sqrt{A}}$$

We know that when we replace crisp initial values with intuitionistic fuzzy ones we obtain the intuitionistic fuzzy solution by the intuitionistic Zadeh's extension principle. Hence we can write the following α and β cuts:

$$\begin{aligned} x(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})[a_1+(a_2-a_1)\alpha, a_3+(a_2-a_3)\alpha]+\sqrt{A}(-1+e^{2\sqrt{AB}t}) \\ &\quad [b_1+(b_2-b_1)\alpha, b_3+(b_2-b_3)\alpha]) \end{aligned}$$

$$\begin{aligned} y(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})[a_1+(a_2-a_1)\alpha, a_3+(a_2-a_3)\alpha]+\sqrt{A}(1+e^{2\sqrt{AB}t}) \\ &\quad [b_1+(b_2-b_1)\alpha, b_3+(b_2-b_3)\alpha]) \end{aligned}$$

and

$$\begin{aligned} x^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})[a_2+(a_1^*-a_2)\beta, a_2+(a_3^*-a_2)\beta]+\sqrt{A}(-1+e^{2\sqrt{AB}t}) \\ &\quad [b_2+(b_1^*-b_2)\beta, b_2+(b_3^*-b_2)\beta]) \end{aligned}$$

$$\begin{aligned} y^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})[a_2+(a_1^*-a_2)\beta, a_2+(a_3^*-a_2)\beta]+\sqrt{A}(1+e^{2\sqrt{AB}t}) \\ &\quad [b_2+(b_1^*-b_2)\beta, b_2+(b_3^*-b_2)\beta]) \end{aligned}$$

So by end-point interval arithmetics and Heaviside function we the following obtain the points of $x(t,\alpha)$ and $x^*(t,\beta)$ as follows:

$$\begin{aligned}
x_1(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(1+e^{2\sqrt{AB}t})(a_1+(a_2-a_1)\alpha-(a_3+(a_2-a_3)\alpha) \\
&\quad +a_3+(a_2-a_3)\alpha))) + (\sqrt{A}(-1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(-1+e^{2\sqrt{AB}t})(b_1+(b_2-b_1)\alpha \\
&\quad -(b_3+(b_2-b_3)\alpha)+b_3+(b_2-b_3)\alpha)))
\end{aligned}$$

$$\begin{aligned}
x_2(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(1+e^{2\sqrt{AB}t})(a_3+(a_2-a_3)\alpha-(a_1+(a_2-a_1)\alpha) \\
&\quad +a_1+(a_2-a_1)\alpha))) + (\sqrt{A}(-1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(-1+e^{2\sqrt{AB}t})(b_3+(b_2-b_3)\alpha \\
&\quad -(b_1+(b_2-b_1)\alpha)+b_1+(b_2-b_1)\alpha)))
\end{aligned}$$

$$\begin{aligned}
y_1(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(-1+e^{2\sqrt{AB}t})(a_1+(a_2-a_1)\alpha-(a_3+(a_2-a_3)\alpha) \\
&\quad +a_3+(a_2-a_3)\alpha))) + (\sqrt{A}(1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(1+e^{2\sqrt{AB}t})(b_1+(b_2-b_1)\alpha \\
&\quad -(b_3+(b_2-b_3)\alpha)+b_3+(b_2-b_3)\alpha)))
\end{aligned}$$

$$\begin{aligned}
y_2(t,\alpha) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(-1+e^{2\sqrt{AB}t})(a_3+(a_2-a_3)\alpha-(a_1+(a_2-a_1)\alpha) \\
&\quad +a_1+(a_2-a_1)\alpha))) + (\sqrt{A}(1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(1+e^{2\sqrt{AB}t})(b_3+(b_2-b_3)\alpha \\
&\quad -(b_1+(b_2-b_1)\alpha)+b_1+(b_2-b_1)\alpha)))
\end{aligned}$$

and

$$\begin{aligned}
x_1^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(1+e^{2\sqrt{AB}t})(a_2+(a_1^*-a_2)\beta-(a_2+(a_3^*-a_2)\beta) \\
&\quad +a_2+(a_3^*-a_2)\beta))) + (\sqrt{A}(-1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(-1+e^{2\sqrt{AB}t})(b_2+(b_1^*-b_2)\beta \\
&\quad -(b_2+(b_3^*-b_2)\beta)+b_2+(b_3^*-b_2)\beta)))
\end{aligned}$$

$$\begin{aligned}
x_2^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{B}}(\sqrt{B}(1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(1+e^{2\sqrt{AB}t})(a_2+(a_3^*-a_2)\beta-(a_2+(a_1^*-a_2)\beta) \\
&\quad +a_2+(a_1^*-a_2)\beta))) + (\sqrt{A}(-1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(-1+e^{2\sqrt{AB}t})(b_2+(b_3^*-b_2)\beta \\
&\quad -(b_2+(b_1^*-b_2)\beta)+b_2+(b_1^*-b_2)\beta)))
\end{aligned}$$

$$\begin{aligned}
y_1^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(-1+e^{2\sqrt{AB}t})(a_2+(a_1^*-a_2)\beta-(a_2+(a_3^*-a_2)\beta) \\
&\quad +a_2+(a_3^*-a_2)\beta))) + (\sqrt{A}(1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(1+e^{2\sqrt{AB}t})(b_2+(b_1^*-b_2)\beta \\
&\quad -(b_2+(b_3^*-b_2)\beta)+b_2+(b_3^*-b_2)\beta)))
\end{aligned}$$

$$
\begin{aligned}
y_2^*(t,\beta) &= \frac{e^{-\sqrt{AB}t}}{2\sqrt{A}}(\sqrt{B}(-1+e^{2\sqrt{AB}t})(\theta(\sqrt{B}(-1+e^{2\sqrt{AB}t})(a_2+(a_3^*-a_2)\beta-(a_2+(a_1^*-a_2)\beta) \\
&\quad +a_2+(a_1^*-a_2)\beta)))+(\sqrt{A}(1+e^{2\sqrt{AB}t})\theta(\sqrt{A}(1+e^{2\sqrt{AB}t})(b_2+(b_3^*-b_2)\beta \\
&\quad -(b_2+(b_1^*-b_2)\beta)+b_2+(b_1^*-b_2)\beta)))
\end{aligned}
$$

Example 4.2.1. Let us find the intuitionistic fuzzy solution of the following system of differential equations

$$\frac{dx(t)}{dt}=y(t), \quad \frac{dy(t)}{dt}=20x(t)$$

with the following triangular intuitionistic fuzzy initial values $x(0)=(1,1,3;-2,1,6)$ and $y(0)=(-2,3,0;-4,3,2)$. We can obtain the end points of α and β cuts of the solution as follows:

$$
\begin{aligned}
x_1(t,\alpha) &= -\frac{1}{6}e^{-3\sqrt{5}t-t}(-\sqrt{5}\alpha e^t+3\alpha e^t+\sqrt{5}\alpha e^{6\sqrt{5}t+t}+3\alpha e^{6\sqrt{5}t+t}+\sqrt{5}e^t \\
&\quad -3e^t+9e^{3\sqrt{5}t}-3e^{3\sqrt{5}t+2t}-\sqrt{5}e^{6\sqrt{5}t+t}-3e^{6\sqrt{5}t+t})x_1(t,\alpha) \\
&= \frac{e^{-2\sqrt{5}t}}{4\sqrt{5}}\left(\left(e^{4\sqrt{5}t}-1\right)\left(2\theta\left(-1+e^{4\sqrt{5}t}\right)-2\right)\right. \\
&\quad \left.+2\sqrt{5}\left(e^{4\sqrt{5}t}+1\right)\left(2\theta\left(2\sqrt{5}\left(1+e^{4\sqrt{5}t}\right)\right)+1\right)\right)y_1(t,\alpha) \\
&= \frac{1}{2}e^{-2\sqrt{5}t}\left(\left(e^{4\sqrt{5}t}+1\right)\left(3-5\theta\left(1+e^{4\sqrt{5}t}\right)\right)+2\sqrt{5}\left(e^{4\sqrt{5}t}-1\right)\right)y_2(t,\alpha) \\
&= \frac{1}{2}e^{-2\sqrt{5}t}\left(\left(e^{4\sqrt{5}t}+1\right)\left(5\theta\left(1+e^{4\sqrt{5}t}\right)-2\right)+2\sqrt{5}\left(e^{4\sqrt{5}t}-1\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
x_1^*(t,\beta) &= \frac{e^{-2\sqrt{5}t}}{4\sqrt{5}} \\
&\quad \left(\left(e^{4\sqrt{5}t}-1\right)\left(2-6\theta\left(-1+e^{4\sqrt{5}t}\right)\right)+2\sqrt{5}\left(e^{4\sqrt{5}t}+1\right)\left(6-8\theta\left(2\sqrt{5}\left(1+e^{4\sqrt{5}t}\right)\right)\right)\right) \\
x_2^*(t,\beta) &= \frac{e^{-2\sqrt{5}t}}{4\sqrt{5}}\left(2\sqrt{5}\left(e^{4\sqrt{5}t}+1\right)\left(8\theta\left(2\sqrt{5}\left(1+e^{4\sqrt{5}t}\right)\right)-2\right)+2\left(e^{4\sqrt{5}t}-1\right)\right) \\
y_1^*(t,\beta) &= \frac{1}{2}e^{-2\sqrt{5}t} \\
&\quad \left(2\sqrt{5}\left(e^{4\sqrt{5}t}-1\right)\left(6-8\theta\left(2\sqrt{5}\left(-1+e^{4\sqrt{5}t}\right)\right)\right)+\left(e^{4\sqrt{5}t}+1\right)\left(2-6\theta\left(1+e^{4\sqrt{5}t}\right)\right)\right) \\
y_2^*(t,\beta) &= \frac{1}{2}e^{-2\sqrt{5}t} \\
&\quad \left(2\sqrt{5}\left(e^{4\sqrt{5}t}-1\right)\left(8\theta\left(2\sqrt{5}\left(-1+e^{4\sqrt{5}t}\right)\right)-2\right)+\left(e^{4\sqrt{5}t}+1\right)\left(6\theta\left(1+e^{4\sqrt{5}t}\right)-4\right)\right)
\end{aligned}
$$

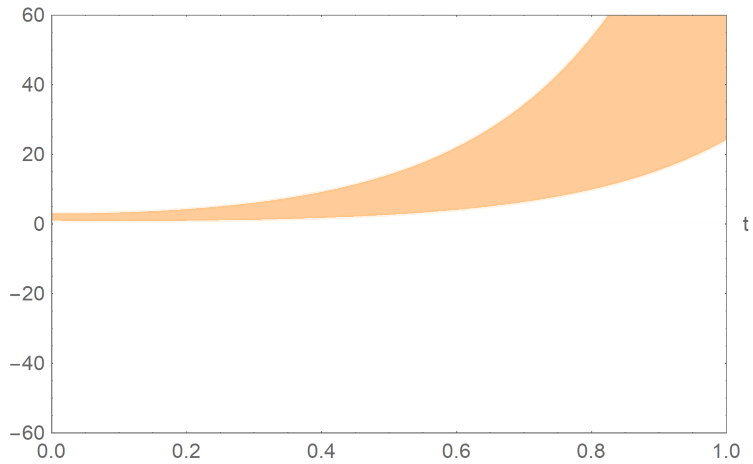

Figure 19. Graph of α cut of the solution x for Example 4.2.1.

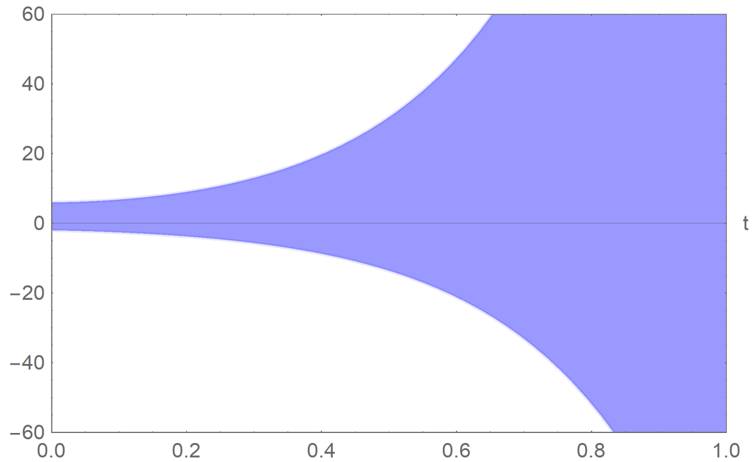

Figure 20. Graph of β cut of the solution x for Example 4.2.1.

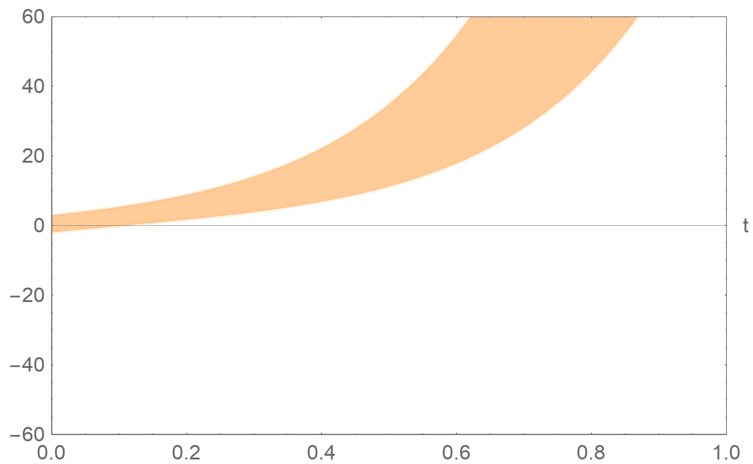

Figure 21. Graph of α cut of the solution y for Example 4.2.1.

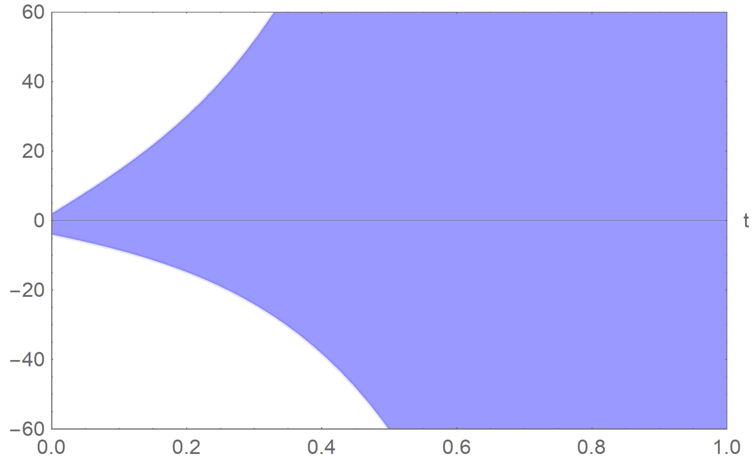

Figure 22. Graph of β cut of the solution y for Example 4.2.1.

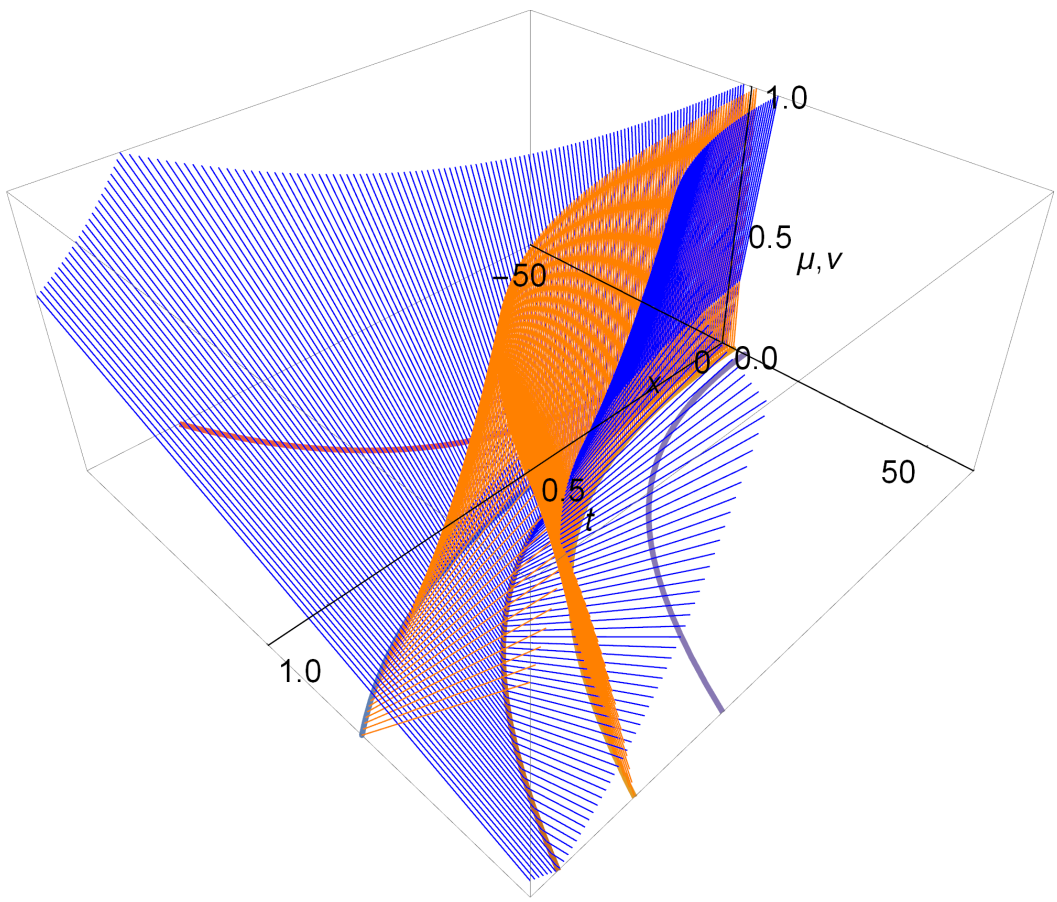

Figure 23. Membership μ and non-membership ν functions of the solution x for Example 4.2.1

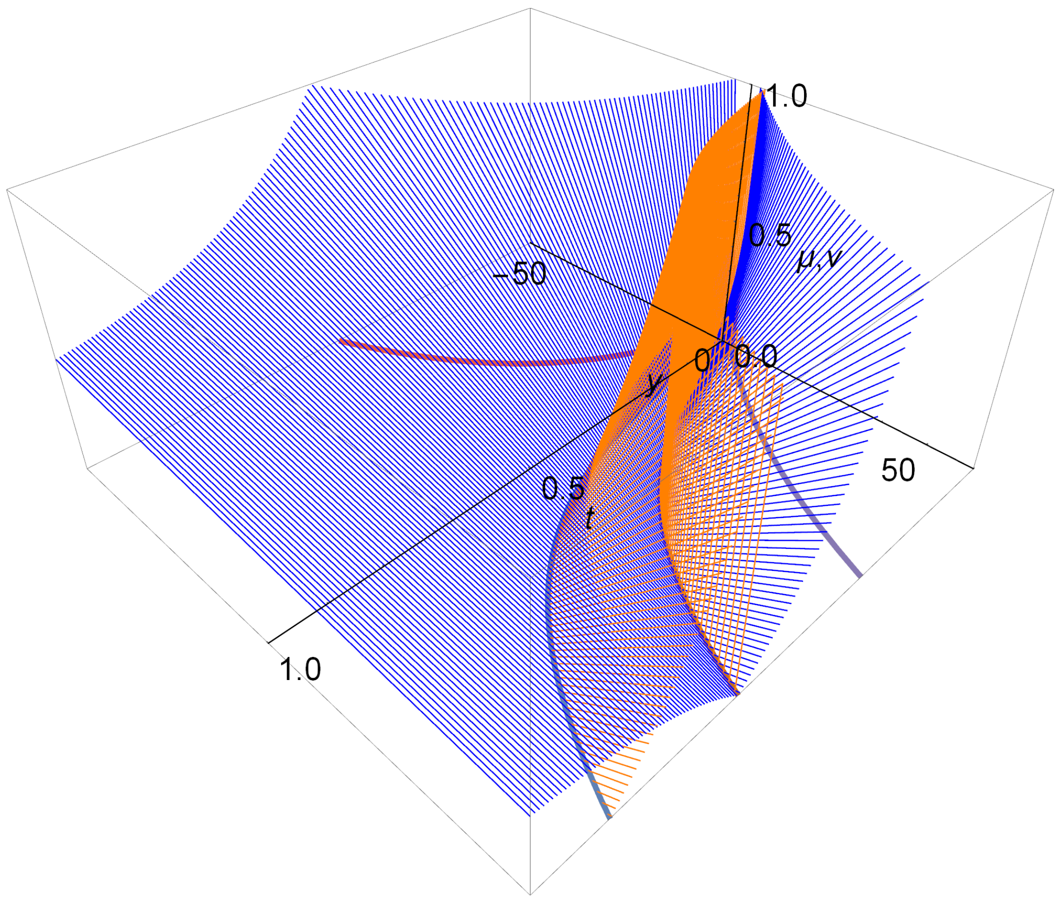

Figure 24. Membership μ and non-membership ν functions of the solution y for Example 4.2.1

5 Summary and conclusions

The main goal of this paper is to give solutions to system of differential equations with intuitionistic fuzzy initial values under the interpretation of (i,ii)-GH differentiability and intuitionistic Zadeh's extension principle concept. To do this we have firstly extended some theorems and definitions about (i,ii)-GH differentiability in fuzzy set theory in Section 3. Later, we have given a procedure to find the solutions to system of ordinary differential equations with triangular intuitionistic fuzzy initial values in Section 4. And we have given some numerical results in Section 4.

Under (i,ii)-GH differentiability concept or Zadeh's extension principle interpretation, we have observed that the endpoints of α or β of solutions may switch on subintervals where crisp solution exists. That is why, this fact makes the solution to be exists locally on subintervals. To cope with this Heaviside function can be used to write the endpoints of α or β of the solutions.

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