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An intuitionistic fuzzy evaluation of the "subset" relation between two crisp sets

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Abstract: In this paper we propose an intuitionistic evaluation method for the is subset of relationship between two crisp sets. The subset comparison operator is an important component for different information management tasks, including querying, decision support and data quality handling. Considering it in an intuitionistic fuzzy logic framework provides us with a tool for developing semantic richer information management techniques.
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1 Introduction

The first idea for an intuitionistic fuzzy relation is discussed in [1]. The concept of an intuitionistic fuzzy relation was introduced formally and studied in details by Humberto Bustince-Sola and

Pedro Burillo in [6, 7]. Here, a new idea for an intuitionistic fuzzy relation "subset" is discussed for the case where this relation is applied over two crisp sets.

For the needs of the proposed research, we need the following definitions.

An Intuitionistic Fuzzy Pair (IFP) is an object of the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$ (see [5]). When $a + b = 1$, i.e., the IFP has the form $\langle a, 1 - a\rangle$, it can be called a fuzzy pair.

We will call the IFP $\langle a, b\rangle$ a strong IFP if and only it $a > 0$ and $b = 0$.

The first element (a) of the IFP $\langle a, b\rangle$ corresponds to the degree of membership, of validity, etc, the second element (b) – to the degree of non-membership, of non-validity, etc, while $1-a-b$ – to the degree of uncertainty.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. We define the relations

$$\begin{array}{lll} x < y & \text{if and only it} & a < c \text{ and } b > d, \\ x \leq y & \text{if and only it} & a \leq c \text{ and } b \geq d, \\ x > y & \text{if and only it} & a > c \text{ and } b < d, \ . \\ x \geq y & \text{if and only it} & a \geq c \text{ and } b \leq, \\ x = y & \text{if and only it} & a = c \text{ and } b = d \end{array}$$

For two crisp sets A and B it is valid that (see, e.g. [8, 9]):

$$A = B \text{ if and only if } \forall x (x \in A \equiv x \in B),$$

$$A \neq B \text{ if and only if it is not valid that } A = B,$$

$$A \subseteq B \text{ if and only if } \forall x (x \in A \to x \in B),$$

$$A \subset B \text{ if and only if } \forall x (x \in A \to x \in B) \wedge (A \neq B),$$

$$A \not\subset B \text{ if and only if it is not valid that } A \subset B.$$

Each one of these relations can be interpreted as a predicate that has truth-value "true" (or 1), if it is valid and "false" (or 0), otherwise. So, below, we will change the Boolean evaluation of a predicate with IFPs. We will denote this predicate by "IR", not by the more correct abbreviation "IFR" only because in the literature on intuitionistic fuzziness, the latest abreviation is used usually to denote an "intuitionistic fuzzy relation" in the sence of H. Bustince and P. Burillo [6, 7] (see, also [3, 4]).

2 Main results

Let A and B be finite crisp sets and let $|X|$ be the cardinality of (finite) set X. Then we define that the new relation is of the form:

$$IR(A, B) = \left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle. \tag{1}$$

Therefore, the degree of uncertainty will be equal to $\dfrac{|A - B|}{|A \cup B|}$.

First, we must prove

Theorem 1. The result of equation (1) is an IFP.

Proof. Let A and B be two crisp sets. Then, from set theory it is well known that

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$$

and

$$(A - B) \cap (A \cap B) = (A - B) \cap (B - A) = (A \cap B) \cap (B - A) = \emptyset.$$

Therefore,

$$|A \cup B| = |A - B| + |A \cap B| + |B - A|$$

and hence

$$\frac{|A \cap B|}{|A \cup B|} + \frac{|B - A|}{|A \cup B|} = 1 - \frac{|A - B|}{|A \cup B|} \leq 1.$$

Therefore, $\left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle$ is an IFP. □

Second, we prove the following two theorems.

Theorem 2. For two crisp sets A and B it holds that $A \subset B$ if and only if $IR(A, B)$ is a fuzzy pair.

Proof. For two crisp sets A and B, let $A \subset B$. Then

$$A \cap B = A, \ A \cup B = B, \ A - B = \emptyset$$

and hence

$$IR(A, B) = \left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle = \left\langle \frac{|A|}{|B|}, \frac{|B - A|}{|B|} \right\rangle$$

and

$$\frac{|A|}{|B|} + \frac{|B - A|}{|B|} = 1 - \frac{|A - B|}{|B|} = 1,$$

i.e., the pair $\left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle$ is a fuzzy pair.

In the opposite case, let the pair $\left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle$ be a fuzzy pair for two finite crisp sets A and B and let us assume that $A \not\subset B$. Then there exists a (finite crisp) set $C \subset A$, so that $C \cap B = \emptyset$ and $C \cap (B - A) = \emptyset$. Therefore, for the degree of uncertainty we obtain

$$\frac{|A - B|}{|A \cup B|} \geq \frac{|C|}{|A \cup B|} > 0,$$

i.e., $\left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle$ is not an IFP, that is a contradiction. Therefore, $C = \emptyset$ and hence $A \subset B$. □

Theorem 3. For two crisp sets A and $B \neq \emptyset$ it holds that $B \subseteq A$ if and only if $IR(A, B)$ is a strong IFP.

Proof. Let $B \subseteq A$ for the two crisp sets A and B. Then

$$IR(A,B) = \left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle = \left\langle \frac{|B|}{|A|}, \frac{|\emptyset|}{|A|} \right\rangle,$$

i.e., $IR(A, B)$ is a strong IFP, because from $B \neq \emptyset$ it follows that $\frac{|B|}{|A|} > 0$, while, obviously, $\frac{|\emptyset|}{|A|} = 0$.

Let the IFP $IF(A, B) = \langle a, 0 \rangle$ for

$$a = \frac{|A \cap B|}{|A \cup B|} > 0.$$

Therefore, $|B - A| = 0$ that is equivalent ot the validity of inclusion $B \subseteq A$. Hence $|A \cap B| = |B|$ and $|A \cup B| = |A|$, i.e., $a = \frac{|B|}{|A|}$, what proves Theorem 3. $\square$

By analogy with [2, 4], we define:

$$\langle a, b \rangle \text{ is an intuitionistic fuzzy tautology if and only if } a \geq b,$$

$$\langle a, b \rangle \text{ is a tautology if and only if } a = 1 \text{ and } b = 1.$$

Therefore, from Theorem 3 it follows that if for the two crisp sets A and $B \neq \emptyset$, $B \subseteq A$, then $IR(A, B)$ is an IFT. Furthermore, the following assertion is valid.

Theorem 4. For the two crisp sets $A, B \neq \emptyset$ it holds that $A = B$ if and only if $IR(A, B)$ is a tautology.

Now, we can extend the definition of relation IR to the form

$$IR(A,B) = \begin{cases} \left\langle \frac{|A \cap B|}{|A \cup B|}, \frac{|B - A|}{|A \cup B|} \right\rangle, & \text{if } A, B \neq \emptyset \\ \langle 1, 0 \rangle, & \text{if } A, B = \emptyset. \end{cases}$$

3 Conclusions

In this paper we proposed a novel definition for set comparison in an intuitionistic fuzzy framework. More specifically an evaluation operator for the "subset" relation between two crisp sets, resulting in an intuitionistic pair has been studied.

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References

[1] Atanassov, K. (1984) Intuitionistic fuzzy relations. *Third Int. Symp. "Automation and Sci. Instrumentation"*, Varna, Oct. 1984, Proc. part II, 56-57.

[2] Atanassov, K. (1988) Two variants of intuitonistic fuzzy propositional calculus. *Preprint IM-MFAIS-5-88*, Sofia; Reprinted in:

[3] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets*, Springer, Heidelberg.

[4] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.

[5] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013) On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[6] Burillo, P., & Bustince, H. (1995) Intuitionistic fuzzy relations. I, *Mathware and Soft Computing*, 2(1), 5–38.

[7] Burillo, P., & Bustince, H. (1995) Intuitionistic fuzzy relations. II: Effect of Atanassov's operators on the properties of the intuitionistic fuzzy sets, *Mathware and Soft Computing*, 2(2), 117–148.

[8] Fraenkel, A., & Bar-Hillel Y. (1958) *Foundations of Set Theory*, Amsterdam, North-Holland Publ. Co.

[9] Kuratowski K., & Mostowski, A. (1967) *Set Theory*, Amsterdam, North-Holland Publ. Co.