

Intuitionistic fuzzy implications revisited. Part 1

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Abstract: Definitions of two intuitionistic fuzzy implications, published during the last 15 years, are revisited. It is checked which existing by the moment intuitionistic fuzzy implications (190 in number) satisfy a well-known tautology in first order logic.

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1 Introduction

In [3], 185 different intuitionistic fuzzy implications are given. After its publishing, five new implications are introduced in [2–8]. In [3], the problem for correctness of these implications is

discussed. Some criteria are shown as suitable for this problem.

The pair $A = \langle a, b \rangle$ is called an *intuitionistic fuzzy pair*, if $a, b, a + b \in [0, 1]$. It is: a *tautology* if and only if (iff) $a = 1$ and $b = 0$, and an *intuitionistic fuzzy tautology (IFT)* iff $a \geq b$.

For the implications, for example, it is obligatory that they satisfy the following condition

$$\begin{aligned}\langle 0, 1 \rangle \rightarrow \langle 0, 1 \rangle &= \langle 1, 0 \rangle, \\ \langle 0, 1 \rangle \rightarrow \langle 1, 0 \rangle &= \langle 1, 0 \rangle, \\ \langle 1, 0 \rangle \rightarrow \langle 0, 1 \rangle &= \langle 0, 1 \rangle, \\ \langle 1, 0 \rangle \rightarrow \langle 1, 0 \rangle &= \langle 1, 0 \rangle,\end{aligned}$$

i.e., they behave as standard tautologies.

In the present research, we will formulate new condition for correctness of the intuitionistic fuzzy implications.

2 Main results

Let $A = \langle a, b \rangle$, and $B = \langle c, d \rangle$ where $a, b, c, d, a + b, c + d \in [0, 1]$, be intuitionistic fuzzy pairs. Initially, following [12], we must mention that two of implications from [3] coincide in spite of the fact that they have different records there. They are

$$\langle a, b \rangle \rightarrow_{40} \langle c, d \rangle = \langle 1 - \text{sg}(a + d - 1), 1 - \overline{\text{sg}}(a + d - 1) \rangle$$

and

$$\langle a, b \rangle \rightarrow_{173} \langle c, d \rangle = \langle \overline{\text{sg}}(a + d - 1), \text{sg}(a + d - 1) \rangle,$$

where

$$\begin{aligned}\text{sg}(x) &= \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \\ \overline{\text{sg}}(x) &= \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}\end{aligned}$$

Really, for each variable x :

$$1 - \text{sg}(x) = \overline{\text{sg}}(x).$$

In the first part of our research, we will check which implications from [3] satisfy the well-known formula

$$((\neg\neg A \rightarrow A) \rightarrow (\neg\neg A \vee \neg A)) \rightarrow (\neg\neg A \vee \neg A) \quad (1)$$

that in first order logic (see, e.g., [11]) is a tautology.

Theorem 1. Implication \rightarrow_i satisfy (1) for $i = 2, 3, 8, 11, 14, 15, 16, 19, 20, 23, 31, 32, 34, 37, 40, 41, 42, 43, 44, 45, 47, 48, 52, 55, 56, 57, 62, 63, 65, 68, 69, 74, 77, 83, 84, 88, 90, 97, 99, 171, 172, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 185$ as a standard tautology.

Proof. We will check Theorem 1 for $i = 2$. The remaining checks are analogous. The implication \rightarrow_2 and the negation \neg_2 , generated by it, have the forms (see [3]):

$$A \rightarrow_2 B = \langle \overline{\text{sg}}(a - c), d \text{sg}(a - c) \rangle,$$

where $B = \langle c, d \rangle$ and

$$\neg_2 A = \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle.$$

Let

$$\begin{aligned} Z &\equiv ((\neg_2 \neg_2 A \rightarrow_2 A) \rightarrow_2 (\neg_2 \neg_2 A \vee \neg_2 A)) \rightarrow_2 (\neg_2 \neg_2 A \vee \neg_2 A) \\ &= ((\neg_2 \neg_2 \langle a, b \rangle \rightarrow_2 \langle a, b \rangle) \rightarrow_2 (\neg_2 \neg_2 \langle a, b \rangle \vee \neg_2 \langle a, b \rangle)) \rightarrow_2 (\neg_2 \neg_2 \langle a, b \rangle \vee \neg_2 \langle a, b \rangle) \\ &= ((\neg_2 \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle \rightarrow_2 \langle a, b \rangle) \rightarrow_2 (\neg_2 \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle)) \\ &\quad \rightarrow_2 (\neg_2 \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle) \\ &= ((\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a)) \rangle \rightarrow_2 \langle a, b \rangle) \rightarrow_2 (\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a)) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle)) \\ &\quad \rightarrow_2 (\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a)) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle) \\ &= ((\langle \text{sg}(a), 1 - \text{sg}(a) \rangle \rightarrow_2 \langle a, b \rangle) \rightarrow_2 (\langle \text{sg}(a), 1 - \text{sg}(a) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle)) \\ &\quad \rightarrow_2 (\langle \text{sg}(a), 1 - \text{sg}(a) \rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a) \rangle) \\ &= (\langle \overline{\text{sg}}(\text{sg}(a) - a), b \text{sg}(\text{sg}(a) - a) \rangle \rightarrow_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a)) \rangle) \\ &\quad \rightarrow_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a)) \rangle \\ &= (\langle \overline{\text{sg}}(\text{sg}(a) - a), b \text{sg}(\text{sg}(a) - a) \rangle \rightarrow_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a)) \rangle) \\ &\quad \rightarrow_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a)) \rangle. \end{aligned}$$

If $a = 0$, then $\text{sg}(a) = 0$, $\overline{\text{sg}}(a) = 1$ and

$$\begin{aligned} Z &= (\langle \overline{\text{sg}}(0), b \text{sg}(0) \rangle \rightarrow \langle \max(0, 1), \min(1, 0) \rangle) \rightarrow \langle \max(0, 1), \min(1, 0) \rangle \\ &= (\langle 1, 0 \rangle \rightarrow \langle 1, 0 \rangle) \rightarrow \langle 1, 0 \rangle \\ &= \langle \overline{\text{sg}}(1 - 1), 0. \text{sg}(1 - 1) \rangle \rightarrow \langle 1, 0 \rangle \\ &= \langle 1, 0 \rangle \rightarrow \langle 1, 0 \rangle \\ &= \langle \overline{\text{sg}}(1 - 1), 0. \text{sg}(1 - 1) \rangle = \langle 1, 0 \rangle. \end{aligned}$$

Therefore, Z is a tautology. □

Now, we will give an example in which, e.g., implication

$$A \rightarrow_{12} B = \langle \max(b, c), 1 - \max(b, c) \rangle$$

and the negation

$$\neg_{10} A = \langle b, 1 - b \rangle$$

generated by it, do not satisfy (1).

Let

$$\begin{aligned}
Z &\equiv ((\neg_4\neg_4A \rightarrow_{12} A) \rightarrow_{12} (\neg_4\neg_4A \vee \neg_4A)) \rightarrow_{12} (\neg_4\neg_4A \vee \neg_{12}A) \\
&= ((\neg_4\neg_4\langle a, b \rangle \rightarrow_{12} \langle a, b \rangle) \rightarrow_{12} (\neg_4\neg_4\langle a, b \rangle \vee \neg_4\langle a, b \rangle)) \\
&\quad \rightarrow_{12} (\neg_4\neg_4\langle a, b \rangle \vee \neg_4\langle a, b \rangle) \\
&= ((\neg_4\langle b, 1-b \rangle \rightarrow_{12} \langle a, b \rangle) \rightarrow_{12} (\neg_4\langle b, 1-b \rangle \vee \langle b, 1-b \rangle)) \rightarrow_{12} (\neg_4\langle b, 1-b \rangle \vee \langle b, 1-b \rangle) \\
&= ((\langle 1-b, b \rangle \rightarrow_{12} \langle a, b \rangle) \rightarrow_{12} (\langle 1-b, b \rangle \vee \langle b, 1-b \rangle)) \rightarrow_{12} (\langle 1-b, b \rangle \vee \langle b, 1-b \rangle) \\
&= (\langle \max(a, b), 1 - \max(a, b) \rangle \rightarrow_{12} \langle \max(b, 1-b), \min(b, 1-b) \rangle) \\
&\quad \rightarrow_{12} \langle \max(b, 1-b), \min(b, 1-b) \rangle \\
&= (\langle \max(1 - \max(a, b), \max(b, 1-b)), 1 - \max(1 - \max(a, b), \max(b, 1-b)) \rangle) \\
&\quad \rightarrow_{12} \langle \max(b, 1-b), \min(b, 1-b) \rangle \\
&= (\langle \max(\min(1-a, 1-b), \max(b, 1-b)), 1 - \max(\min(1-a, 1-b), \max(b, 1-b)) \rangle) \\
&\quad \rightarrow_{12} \langle \max(b, 1-b), \min(b, 1-b) \rangle
\end{aligned}$$

(because, for $a, b, a+b \in [0, 1]$: $\min(1-a, 1-b) \leq 1-b \leq \max(b, 1-b)$)

$$\begin{aligned}
&= \langle \max(b, 1-b), 1 - \max(b, 1-b) \rangle \rightarrow_{12} \langle \max(b, 1-b), \min(b, 1-b) \rangle \\
&= \langle \max(1 - \max(b, 1-b), \max(b, 1-b)), 1 - \max(1 - \max(b, 1-b), \max(b, 1-b)) \rangle \\
&= \langle \max(\min(b, 1-b), \max(b, 1-b)), 1 - \max(\min(b, 1-b), \max(b, 1-b)) \rangle \\
&= \langle \max(b, 1-b), 1 - \max(b, 1-b) \rangle \\
&= \langle \max(b, 1-b), \min(b, 1-b) \rangle \neq \langle 1, 0 \rangle
\end{aligned}$$

for $0 \leq b < 1$, i.e., (1) is not a tautology.

Theorem 2. Implication \rightarrow_i satisfies (1) for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57, 61, 62, 63, 64, 65, 66, 67, 68, 69, 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 88, 89, 90, 91, 94, 95, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 141, 146, 147, 148, 151, 158, 159, 160, 161, 166, 167, 168, 169, 170, 171, 172, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185$ as IFT.

Proof. We will check Theorem 2 for $i = 1$. The remaining checks are analogous. The implication \rightarrow_1 and the negation \neg_1 generated by it, have the forms (see [3]):

$$A \rightarrow_1 B = \langle \max(b, \min(a, c)), \min(a, d) \rangle,$$

where B is defined above and

$$\neg_1 A = \langle b, a \rangle.$$

Let

$$\begin{aligned}
Z &\equiv ((\neg_1 \neg_1 A \rightarrow_1 A) \rightarrow_1 (\neg_1 \neg_1 A \vee \neg_1 A)) \rightarrow_1 (\neg_1 \neg_1 A \vee \neg_1 A) \\
&= ((\neg_1 \neg_1 \langle a, b \rangle \rightarrow_1 \langle a, b \rangle) \rightarrow_1 (\neg_1 \neg_1 \langle a, b \rangle \vee \neg_1 \langle a, b \rangle)) \\
&\quad \rightarrow_1 (\neg_1 \neg_1 \langle a, b \rangle \vee \neg_1 \langle a, b \rangle) \\
&= ((\neg_1 \langle b, a \rangle \rightarrow_1 \langle a, b \rangle) \rightarrow_1 (\neg_1 \langle b, a \rangle \vee \langle b, a \rangle)) \\
&\quad \rightarrow_1 (\neg_1 \langle b, a \rangle \vee \langle b, a \rangle) \\
&= ((\langle a, b \rangle \rightarrow_1 \langle a, b \rangle) \rightarrow_1 (\langle a, b \rangle \vee \langle b, a \rangle)) \\
&\quad \rightarrow_1 (\langle a, b \rangle \vee \langle b, a \rangle) \\
&= (\langle \max(b, \min(a, a)), \min(a, b) \rangle \rightarrow_1 \langle \max(a, b), \min(a, b) \rangle) \rightarrow_1 \langle \max(a, b), \min(a, b) \rangle \\
&= (\langle \max(a, b), \min(a, b) \rangle \rightarrow_1 \langle \max(a, b), \min(a, b) \rangle) \rightarrow_1 \langle \max(a, b), \min(a, b) \rangle \\
&= (\langle \max(\min(a, b), \min(\max(a, b), \max(a, b))), \min(\max(a, b), \min(a, b)) \rangle \\
&\quad \rightarrow_1 \langle \max(a, b), \min(a, b) \rangle) \\
&= \langle \max(\min(\max(a, b), \min(a, b)), \\
&\quad \min(\max(\min(a, b), \min(\max(a, b), \max(a, b))), \max(a, b))), \\
&\quad \min(\max(\min(a, b), \min(\max(a, b), \max(a, b))), \min(a, b)) \rangle \\
&= \langle \max(\min(a, b), \min(\max(\min(a, b), \max(a, b)), \max(a, b))), \\
&\quad \min(\max(\min(a, b), \max(a, b)), \min(a, b)) \rangle \\
&= \langle \max(\min(a, b), \min(\max(a, b), \max(a, b))), \\
&\quad \min(\max(\min(a, b), \max(a, b)), \min(a, b)) \rangle \\
&= \langle \max(\min(a, b), \max(a, b)), \min(\max(a, b), \min(a, b)) \rangle \\
&= \langle \max(a, b), \min(a, b) \rangle.
\end{aligned}$$

Therefore, Z is an IFT. □

Now, we will give an example in which, e.g., the implication

$$A \rightarrow_{60} B = \langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min(1 - b, \overline{\text{sg}}(c)) \rangle$$

and the negation, generated by it

$$\neg_{10} A = \langle \overline{\text{sg}}(1 - b), 1 - b \rangle$$

(see [3]) do not satisfy (1).

$$\begin{aligned}
Z &\equiv ((\neg_{10} \neg_{10} A \rightarrow_{60} A) \rightarrow_{60} (\neg_{10} \neg_{10} A \vee \neg_{10} A)) \rightarrow_{60} (\neg_{10} \neg_{10} A \vee \neg_{10} A) \\
&= ((\neg_{10} \neg_{10} \langle a, b \rangle \rightarrow_{60} \langle a, b \rangle) \rightarrow_{60} (\neg_{10} \neg_{10} \langle a, b \rangle \vee \neg_{10} \langle a, b \rangle)) \\
&\quad \rightarrow_{60} (\neg_{10} \neg_{10} \langle a, b \rangle \vee \neg_{10} \langle a, b \rangle)
\end{aligned}$$

$$\begin{aligned}
&= ((\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \rightarrow_{60} \langle a, b \rangle) \rightarrow_{60} (\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle)) \\
&\quad \rightarrow_{60} (\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle) \\
&= ((\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \rightarrow_{60} \langle a, b \rangle) \rightarrow_{60} (\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle)) \\
&\quad \rightarrow_{60} (\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle) \\
&\quad = ((\langle \overline{\text{sg}}(b), b \rangle \rightarrow_{60} \langle a, b \rangle) \rightarrow_{60} (\langle \overline{\text{sg}}(b), b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle)) \\
&\quad \rightarrow_{60} (\langle \overline{\text{sg}}(b), b \rangle \vee \langle \overline{\text{sg}}(1-b), 1-b \rangle) \\
&= (\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-a)), \min(1-b, \overline{\text{sg}}(a)) \rangle \rightarrow_{60} \langle \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)), \min(b, 1-b) \rangle) \\
&\quad \rightarrow_{60} (\langle \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)), \min(b, 1-b) \rangle) \\
&= \langle \max(\overline{\text{sg}}(1 - \min(1-b, \overline{\text{sg}}(a))), \overline{\text{sg}}(1 - \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle, \\
&\quad \min(1 - \min(1-b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle) \\
&\quad \rightarrow_{60} \langle \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)), \min(b, 1-b) \rangle \\
&= \langle \max(\overline{\text{sg}}(1 - \min(1 - \min(1-b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle, \\
&\quad \overline{\text{sg}}(1 - \langle \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)) \rangle), \\
&\min(1 - \min(1 - \min(1-b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle, \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle).
\end{aligned}$$

Now, we check the value of

$$\begin{aligned}
Y_1 &\equiv \max(\overline{\text{sg}}(1 - \min(1 - \min(1-b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))) \rangle, \\
&\quad \overline{\text{sg}}(1 - \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)))).
\end{aligned}$$

If $b = 0$, then

$$\begin{aligned}
Y_1 &= \max(\overline{\text{sg}}(1 - \min(1 - \min(1, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(1, 0))) \rangle, \overline{\text{sg}}(1 - \max(1, 0))) \\
&= \max(\overline{\text{sg}}(1 - \min(1 - \overline{\text{sg}}(a), \overline{\text{sg}}(1))) \rangle, \overline{\text{sg}}(1 - 1)) \\
&= \max(\overline{\text{sg}}(1 - \min(1 - \overline{\text{sg}}(a), 0)) \rangle, 1) \\
&= \max(\overline{\text{sg}}(1 - 0) \rangle, 1) = 1.
\end{aligned}$$

If $b = 1$, then

$$\begin{aligned}
Y_1 &= \max(\overline{\text{sg}}(1 - \min(1 - \min(0, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(\overline{\text{sg}}(1), \overline{\text{sg}}(0)))) \rangle, \overline{\text{sg}}(1 - \max(\overline{\text{sg}}(1), \overline{\text{sg}}(0)))) \\
&= \max(\overline{\text{sg}}(1 - \min(1 - \min(0, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(0, 1))) \rangle, \overline{\text{sg}}(1 - \max(0, 1))) \\
&= \max(\overline{\text{sg}}(1 - \min(1 - \min(0, \overline{\text{sg}}(a))), \overline{\text{sg}}(1)) \rangle, \overline{\text{sg}}(1 - 1)) \\
&= \max(\overline{\text{sg}}(1 - \min(1 - 0, 0)) \rangle, 1) \\
&= \max(\overline{\text{sg}}(1) \rangle, 1) = 1.
\end{aligned}$$

If $0 < b < 1$, then

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(0, 0))), \overline{\text{sg}}(1 - \max(0, 0)))$$

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(0)), \overline{\text{sg}}(1))$$

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), 1), 0)$$

$$Y_1 = \max(\overline{\text{sg}}(1 - (1 - \min(1 - b, \overline{\text{sg}}(a)))), 0)$$

$$Y_1 = \overline{\text{sg}}(\min(1 - b, \overline{\text{sg}}(a))).$$

Let

$$Y_2 \equiv \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1 - b))), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1 - b)))).$$

For $b = 0, 1$ we can see immediately that $Y_2 = 0$.

If $0 < b < 1$, then

$$Y_2 = \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(0, 0)), \overline{\text{sg}}(\max(0, 0)))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(\max(0, 0)), \overline{\text{sg}}(\max(0, 0)))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), \overline{\text{sg}}(0), \overline{\text{sg}}(0))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a))), 1, 1)$$

$$= \min(1 - (1 - \min(1 - b, \overline{\text{sg}}(a))), 1)$$

$$= \min(1 - b, \overline{\text{sg}}(a)).$$

Therefore,

$$Z = \langle \overline{\text{sg}}(\min(1 - b, \overline{\text{sg}}(a))), \min(1 - b, \overline{\text{sg}}(a)) \rangle.$$

If $a = 0$, then

$$Z = \langle \overline{\text{sg}}(\min(1 - b, 1)), \min(1 - b, 1) \rangle$$

$$= \langle \overline{\text{sg}}(1 - b), 1 - b \rangle$$

$$= \langle 0, 1 - b \rangle$$

that for $0 < b < 1$ is not an IFT. So, we have shown that Theorem 2 is not valid for \rightarrow_{60} .

3 Conclusion

In the second part of the present research, we will determine which intuitionistic fuzzy implications satisfy two other tautologies in first order logic.

Open Problem: In [2] six Cartesian products are defined (the last of them was published in [1]). For which indices i and j the i -th implication and j -th Cartesian product satisfies equality

$$(A \rightarrow_i B) \times_j C = (A \times_j C) \rightarrow_i (B \times_j C)$$

for every three IFSs A, B, C ?

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