Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy implications revisited. Part 1

Krassimir Atanassov[1], Eulalia Szmidt[2], Janusz Kacprzyk[2] and Nora Angelova[3]

[1] Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

[2] Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warsaw, Poland
and
Warsaw School of Information Technology
ul. Newelska 6, 01-447 Warsaw, Poland
e-mails: {szmidt,kacprzyk}@ibspan.waw.pl

[3] Faculty of Mathematics and Informatics, Sofia University
5 James Bourchier Blvd., Sofia 1164, Bulgaria
e-mail: nora.angelova@fmi.uni-sofia.bg

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Abstract: Definitions of two intuitionistic fuzzy implications, published during the last 15 years, are revisited. It is checked which existing by the moment intuitionistic fuzzy implications (190 in number) satisfy a well-known tautology in first order logic.
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1 Introduction

In [2], 185 different intuitionisic fuzzy implications are given. After its publishing, five new implications are introduced in [2–8]. In [2], the problem for correctness of these implications is discussed. Some criteria are shown as suitable for this problem.

The pair $A = \langle a, b\rangle$ is called an *intuitionisic fuzzy pair,* if $a, b, a + b \in [0, 1]$. It is:
a *tautology* if and only if (iff) $a = 1$ and $b = 0$, and
an *intuitionistic fuzzy tautology (IFT)* iff $a \geq b$.

For the implications, for example, it is obligatory that they satisfy the following condition

$$\langle 0,1\rangle \to \langle 0,1\rangle = \langle 1,0\rangle,$$

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$$\langle 1,0\rangle \to \langle 0,1\rangle = \langle 0,1\rangle,$$

$$\langle 1,0\rangle \to \langle 1,0\rangle = \langle 1,0\rangle,$$

i.e., they behave as standard tautologies.

In the present research, we will formulate new condition for correctness of the intuitionisic fuzzy implications.

2 Main results

Let $A = \langle a, b\rangle$, and $B = \langle c, d\rangle$ where $a, b, c, d, a + b, c + d \in [0, 1]$, be intuitionisic fuzzy pairs. Initially, following [11], we must mention that two of implications from [2] coincide in spite of the fact that they have different records there. They are

$$\langle a, b\rangle \to_{40} \langle c, d\rangle = \langle 1 - \mathrm{sg}(a + d - 1), 1 - \overline{\mathrm{sg}}(a + d - 1)\rangle$$

and

$$\langle a, b\rangle \to_{173} \langle c, d\rangle = \langle \overline{\mathrm{sg}}(a + d - 1), \mathrm{sg}(a + d - 1)\rangle,$$

where

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\overline{\mathrm{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Really, for each variable x:

$$1 - \mathrm{sg}(x) = \overline{\mathrm{sg}}(x).$$

In the first part of our research, we will check which implications from [2] satisfy the well-known formula

$$((\neg\neg A \to A) \to (\neg\neg A \vee \neg A)) \to (\neg\neg A \vee \neg A) \tag{1}$$

that in first order logic (see, e.g., [10]) is a tautology.

Theorem 1. Implication $\to_i$ satisfy (1) for $i = 2, 3, 8, 11, 14, 15, 16, 19, 20, 23, 31, 32, 34, 37, 40,$ $41, 42, 43, 44, 45, 47, 48,\ 52, 55, 56, 57,\ 62, 63, 65, 68, 69,\ 74, 77, 83, 84, 88,\ 90, 97, 99,\ 171, 172,$ $174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 185$ as a standard tautology.

Proof. We will check Theorem 1 for $i = 2$. The remaining checks are analogous. The implication $\to_2$ and the negation $\neg_2$, generated by it, have the forms (see [2]):

$$A \to_2 B = \langle \overline{\text{sg}}(a - c), d\,\text{sg}(a - c)\rangle,$$

where $B = \langle c, d\rangle$ and

$$\neg_2 A = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle.$$

Let

$$\begin{aligned}
Z &\equiv ((\neg_2\neg_2 A \to_2 A) \to_2 (\neg_2\neg_2 A \vee \neg_2 A)) \to_2 (\neg_2\neg_2 A \vee \neg_2 A)\\
&= ((\neg_2\neg_2\langle a, b\rangle \to_2 \langle a, b\rangle) \to_2 (\neg_2\neg_2\langle a, b\rangle \vee \neg_2\langle a, b\rangle)) \to_2 (\neg_2\neg_2\langle a, b\rangle \vee \neg_2\langle a, b\rangle)\\
&= ((\neg_2\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \to_2 \langle a, b\rangle) \to_2 (\neg_2\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle))\\
&\quad \to_2 (\neg_2\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle)\\
&= (((\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a))\rangle \to_2 \langle a, b\rangle) \to_2 (\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a))\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle))\\
&\quad \to_2 (\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a))\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle)\\
&= (((\langle \text{sg}(a), 1 - \text{sg}(a)\rangle \to_2 \langle a, b\rangle) \to_2 (\langle \text{sg}(a), 1 - \text{sg}(a)\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle))\\
&\quad \to_2 (\langle \text{sg}(a), 1 - \text{sg}(a)\rangle \vee \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle)\\
&= (\langle \overline{\text{sg}}(\text{sg}(a) - a), b\,\text{sg}(\text{sg}(a) - a)\rangle \to_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a))\rangle)\\
&\quad \to_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a))\rangle\\
&= (\langle \overline{\text{sg}}(\text{sg}(a) - a), b\,\text{sg}(\text{sg}(a) - a)\rangle \to_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a))\rangle)\\
&\quad \to_2 \langle \max(\text{sg}(a), \overline{\text{sg}}(a)), \min(1 - \text{sg}(a), \text{sg}(a))\rangle.
\end{aligned}$$

If $a = 0$, then $\text{sg}(a) = 0$, $\overline{\text{sg}}(a) = 1$ and

$$Z = (\langle \overline{\text{sg}}(0), b\,\text{sg}(0)\rangle \to \langle \max(0, 1), \min(1, 0)\rangle) \to \langle \max(0, 1), \min(1, 0)\rangle$$

$$\begin{aligned}
&= (\langle 1, 0\rangle \to \langle 1, 0\rangle) \to \langle 1, 0\rangle\\
&= \langle \overline{\text{sg}}(1 - 1), 0.\,\text{sg}(1 - 1)\rangle \to \langle 1, 0\rangle\\
&= \langle 1, 0\rangle \to \langle 1, 0\rangle\\
&= \langle \overline{\text{sg}}(1 - 1), 0.\,\text{sg}(1 - 1)\rangle = \langle 1, 0\rangle.
\end{aligned}$$

Therefore, Z is a tautology. □

Now, we will give an example in which, e.g., implication

$$A \to_{12} B = \langle \max(b, c), 1 - \max(b, c)\rangle$$

and the negation

$$\neg_{10} A = \langle b, 1 - b\rangle$$

generated by it, do not satisfy (1).

Let

$$Z \equiv ((\neg_4\neg_4 A \to_{12} A) \to_{12} (\neg_4\neg_4 A \vee \neg_4 A)) \to_{12} (\neg_4\neg_4 A \vee \neg_{12} A)$$

$$= ((\neg_4\neg_4\langle a, b\rangle \to_{12} \langle a, b\rangle) \to_{12} (\neg_4\neg_4\langle a, b\rangle \vee \neg_4\langle a, b\rangle))$$

$$\to_{12} (\neg_4\neg_4\langle a, b\rangle \vee \neg_4\langle a, b\rangle)$$

$$= ((\neg_4\langle b, 1-b\rangle \to_{12} \langle a, b\rangle) \to_{12} (\neg_4\langle b, 1-b\rangle \vee \langle b, 1-b\rangle)) \to_{12} (\neg_4\langle b, 1-b\rangle \vee \langle b, 1-b\rangle)$$

$$= (((\langle 1-b, b\rangle \to_{12} \langle a, b\rangle) \to_{12} (\langle 1-b, b\rangle \vee \langle b, 1-b\rangle)) \to_{12} (\langle 1-b, b\rangle \vee \langle b, 1-b\rangle)$$

$$= (\langle \max(a, b), 1-\max(a, b)\rangle \to_{12} \langle \max(b, 1-b), \min(b, 1-b)\rangle)$$

$$\to_{12} \langle \max(b, 1-b), \min(b, 1-b)\rangle$$

$$= (\langle \max(1-\max(a, b), \max(b, 1-b)), 1-\max(1-\max(a, b), \max(b, 1-b))\rangle$$

$$\to_{12} \langle \max(b, 1-b), \min(b, 1-b)\rangle$$

$$= (\langle \max(\min(1-a, 1-b), \max(b, 1-b)), 1-\max(\min(1-a, 1-b), \max(b, 1-b))\rangle$$

$$\to_{12} \langle \max(b, 1-b), \min(b, 1-b)\rangle$$

(because, for $a, b, a+b \in [0, 1]$: $\min(1-a, 1-b) \le 1-b \le \max(b, 1-b)$)

$$= \langle \max(b, 1-b), 1-\max(b, 1-b)\rangle \to_{12} \langle \max(b, 1-b), \min(b, 1-b)\rangle$$

$$= \langle \max(1-\max(b, 1-b), \max(b, 1-b)), 1-\max(1-\max(b, 1-b), \max(b, 1-b))\rangle$$

$$= \langle \max(\min(b, 1-b), \max(b, 1-b)), 1-\max(\min(b, 1-b), \max(b, 1-b))\rangle$$

$$= \langle \max(b, 1-b), 1-\max(b, 1-b)\rangle$$

$$= \langle \max(b, 1-b), \min(b, 1-b)\rangle \neq \langle 1, 0\rangle$$

for $0 \le b < 1$, i.e., (1) is not a tautology.

Theorem 2. Implication $\to_i$ satisfies (1) for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29,\ 30, 31, 32, 33, 34, 35, 36, 37, 38,\ 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57,\ 61, 62, 63, 64, 65, 66, 67, 68, 69,\ 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 88, 89, 90, 91, 94, 95, 97, 98, 99,\ 100, 101, 102, 103, 104, 105, 106, 107, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120,\ 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137,\ 141, 146, 147, 148, 151, 158, 159, 160, 161, 166, 167, 168, 169, 170, 171, 172, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185$ as IFT.

Proof. We will check Theorem 2 for $i = 1$. The remining checks are analogous. The implication $\to_1$ and the negation $\neg_1$ generated by it, have the forms (see [2]):

$$A \to_1 B = \langle \max(b, \min(a, c)), \min(a, d)\rangle,$$

where B is defined above and

$$\neg_1 A = \langle b, a\rangle.$$

Let

$$Z \equiv ((\neg_1\neg_1 A \to_1 A) \to_1 (\neg_1\neg_1 A \vee \neg_1 A)) \to_1 (\neg_1\neg_1 A \vee \neg_1 A)$$
$$= ((\neg_1\neg_1\langle a,b\rangle \to_1 \langle a,b\rangle) \to_1 (\neg_1\neg_1\langle a,b\rangle \vee \neg_1\langle a,b\rangle))$$
$$\to_1 (\neg_1\neg_1\langle a,b\rangle \vee \neg_1\langle a,b\rangle)$$
$$= ((\neg_1\langle b,a\rangle \to_1 \langle a,b\rangle) \to_1 (\neg_1\langle b,a\rangle \vee \langle b,a\rangle))$$
$$\to_1 (\neg_1\langle b,a\rangle \vee \langle b,a\rangle)$$
$$= ((\langle a,b\rangle \to_1 \langle a,b\rangle) \to_1 (\langle a,b\rangle \vee \langle b,a\rangle))$$
$$\to_1 (\langle a,b\rangle \vee \langle b,a\rangle)$$
$$= (\langle \max(b,\min(a,a)),\min(a,b)\rangle \to_1 \langle \max(a,b),\min(a,b)\rangle) \to_1 \langle \max(a,b),\min(a,b)\rangle$$
$$= (\langle \max(a,b),\min(a,b)\rangle \to_1 \langle \max(a,b),\min(a,b)\rangle) \to_1 \langle \max(a,b),\min(a,b)\rangle$$
$$= (\langle \max(\min(a,b),\min(\max(a,b),\max(a,b))),\min(\max(a,b),\min(a,b))\rangle)$$
$$\to_1 \langle \max(a,b),\min(a,b)\rangle$$
$$= \langle \max(\min(\max(a,b),\min(a,b)),$$
$$\min(\max(\min(a,b),\min(\max(a,b),\max(a,b))),\max(a,b))),$$
$$\min(\max(\min(a,b),\min(\max(a,b),\max(a,b))),\min(a,b))\rangle$$
$$= \langle \max(\min(a,b),\min(\max(\min(a,b),\max(a,b)),\max(a,b))),$$
$$\min(\max(\min(a,b),\max(a,b)),\min(a,b))\rangle$$
$$= \langle \max(\min(a,b),\min(\max(a,b),\max(a,b))),$$
$$\min(\max(\min(a,b),\max(a,b)),\min(a,b))\rangle$$
$$= \langle \max(\min(a,b),\max(a,b)),\min(\max(a,b),\min(a,b))\rangle$$
$$= \langle \max(a,b),\min(a,b)\rangle.$$

Therefore, Z is an IFT. □

Now, we will give an example in which, e.g., the implication

$$A \to_{60} B = \langle \max(\overline{\text{sg}}(1-b),\overline{\text{sg}}(1-c)),\min(1-b,\overline{\text{sg}}(c))\rangle$$

and the negation, generated by it

$$\neg_{10} A = \langle \overline{\text{sg}}(1-b), 1-b\rangle$$

(see [2]) do not satisfy (1).

$$Z \equiv ((\neg_{10}\neg_{10} A \to_{60} A) \to_{60} (\neg_{10}\neg_{10} A \vee \neg_{10} A)) \to_{60} (\neg_{10}\neg_{10} A \vee \neg_{10} A)$$
$$= ((\neg_{10}\neg_{10}\langle a,b\rangle \to_{60} \langle a,b\rangle) \to_{60} (\neg_{10}\neg_{10}\langle a,b\rangle \vee \neg_{10}\langle a,b\rangle))$$
$$\to_{60} (\neg_{10}\neg_{10}\langle a,b\rangle \vee \neg_{10}\langle a,b\rangle)$$

$$= ((\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \rightarrow_{60} \langle a, b\rangle) \rightarrow_{60} (\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle))$$

$$\rightarrow_{60} (\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle)$$

$$= ((\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \rightarrow_{60} \langle a, b\rangle) \rightarrow_{60} (\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle))$$

$$\rightarrow_{60} (\neg_{10}\langle \overline{\rm sg}(1-b), 1-b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle)$$

$$= (((\langle \overline{\rm sg}(b), b\rangle \rightarrow_{60} \langle a, b\rangle) \rightarrow_{60} (\langle \overline{\rm sg}(b), b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle))$$

$$\rightarrow_{60} (\langle \overline{\rm sg}(b), b\rangle \vee \langle \overline{\rm sg}(1-b), 1-b\rangle)$$

$$= (\langle \max(\overline{\rm sg}(1-b), \overline{\rm sg}(1-a)), \min(1-b, \overline{\rm sg}(a))\rangle \rightarrow_{60} \langle \max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)), \min(b, 1-b)\rangle)$$

$$\rightarrow_{60} (\langle \max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)), \min(b, 1-b)\rangle$$

$$= \langle \max(\overline{\rm sg}(1-\min(1-b, \overline{\rm sg}(a))), \overline{\rm sg}(1-\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)))),$$

$$\min(1-\min(1-b, \overline{\rm sg}(a)), \overline{\rm sg}(\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b))))\rangle)$$

$$\rightarrow_{60} \langle \max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)), \min(b, 1-b)\rangle$$

$$= \langle \max(\overline{\rm sg}(1-\min(1-\min(1-b, \overline{\rm sg}(a)), \overline{\rm sg}(\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b))))),$$

$$\overline{\rm sg}(1-\langle \max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)))),$$

$$\min(1-\min(1-\min(1-b, \overline{\rm sg}(a)), \overline{\rm sg}(\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)))), \overline{\rm sg}(\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b))))\rangle.$$

Now, we check the value of

$$Y_1 \equiv \max(\overline{\rm sg}(1-\min(1-\min(1-b, \overline{\rm sg}(a)), \overline{\rm sg}(\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b))))),$$

$$\overline{\rm sg}(1-\max(\overline{\rm sg}(b), \overline{\rm sg}(1-b)))).$$

If $b = 0$, then

$$Y_1 = \max(\overline{\rm sg}(1-\min(1-\min(1, \overline{\rm sg}(a)), \overline{\rm sg}(\max(1, 0)))), \overline{\rm sg}(1-\max(1, 0)))$$

$$= \max(\overline{\rm sg}(1-\min(1-\overline{\rm sg}(a), \overline{\rm sg}(1))), \overline{\rm sg}(1-1))$$

$$= \max(\overline{\rm sg}(1-\min(1-\overline{\rm sg}(a), 0)), 1)$$

$$= \max(\overline{\rm sg}(1-0), 1) = 1.$$

If $b = 1$, then

$$Y_1 = \max(\overline{\rm sg}(1-\min(1-\min(0, \overline{\rm sg}(a)), \overline{\rm sg}(\max(\overline{\rm sg}(1), \overline{\rm sg}(0))))), \overline{\rm sg}(1-\max(\overline{\rm sg}(1), \overline{\rm sg}(0))))$$

$$= \max(\overline{\rm sg}(1-\min(1-\min(0, \overline{\rm sg}(a)), \overline{\rm sg}(\max(0, 1)))), \overline{\rm sg}(1-\max(0, 1)))$$

$$= \max(\overline{\rm sg}(1-\min(1-\min(0, \overline{\rm sg}(a)), \overline{\rm sg}(1))), \overline{\rm sg}(1-1))$$

$$= \max(\overline{\rm sg}(1-\min(1-0, 0)), 1)$$

$$= \max(\overline{\rm sg}(1), 1) = 1.$$

If $0 < b < 1$, then

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(0, 0)))), \overline{\text{sg}}(1 - \max(0, 0)))$$

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(0))), \overline{\text{sg}}(1))$$

$$Y_1 = \max(\overline{\text{sg}}(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), 1)), 0)$$

$$Y_1 = \max(\overline{\text{sg}}(1 - (1 - \min(1 - b, \overline{\text{sg}}(a)))), 0)$$

$$Y_1 = \overline{\text{sg}}(\min(1 - b, \overline{\text{sg}}(a))).$$

Let

$$Y_2 \equiv \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1 - b)))), \overline{\text{sg}}(\max(\overline{\text{sg}}(b), \overline{\text{sg}}(1 - b)))).$$

For $b = 0, 1$ we can see immediately that $Y_2 = 0$.
If $0 < b < 1$, then

$$Y_2 = \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(0, 0))), \overline{\text{sg}}(\max(0, 0)))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(\max(0, 0))), \overline{\text{sg}}(\max(0, 0)))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), \overline{\text{sg}}(0)), \overline{\text{sg}}(0))$$

$$= \min(1 - \min(1 - \min(1 - b, \overline{\text{sg}}(a)), 1), 1)$$

$$= \min(1 - (1 - \min(1 - b, \overline{\text{sg}}(a))), 1)$$

$$= \min(1 - b, \overline{\text{sg}}(a)).$$

Therefore,

$$Z = \langle \overline{\text{sg}}(\min(1 - b, \overline{\text{sg}}(a))), \min(1 - b, \overline{\text{sg}}(a)) \rangle.$$

If $a = 0$, then

$$Z = \langle \overline{\text{sg}}(\min(1 - b, 1)), \min(1 - b, 1) \rangle$$

$$= \langle \overline{\text{sg}}(1 - b), 1 - b \rangle$$

$$= \langle 0, 1 - b \rangle$$

that for $0 < b < 1$ is not an IFT. So, we have shown that Theorem 2 is not valid for $\rightarrow_{60}$.

3 Conclusion

In the second part of the present research, we will determine which intuitionistic fuzzy implications satisfy two other tautologies in first order logic.
Open Problem: In [**?**] six Cartesian products are defined (the last of them was published in [1]). For which indices i and j the i-th implication and j-th Cartesian product satisfies equality

$$(A \rightarrow_i B) \times_j C = (A \times_j C) \rightarrow_i (B \times_j C)$$

for every three IFSs A, B, C?

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