Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 22, 2016, No. 1, 78–88

Modal operator $F_{\alpha,\beta}$ on intuitionistic fuzzy BG-algebras

S. R. Barbhuiya

Department of Mathematics, Srikishan Sarda College
Hailakandi–788151, Assam, India
e-mail: saidurbarbhuiya@gmail.com

Received: 22 October 2015 **Accepted:** 11 May 2016

Abstract: In this paper we study the effect of modal operator(s) in intuitionistic fuzzy BG-algebras and the effect of modal operator(s) on intuitionistic fuzzy BG-algebras under homomorphism and obtained some interesting properties.
Keywords: BG-algebra, Intuitionistic fuzzy set, Modal operator, (α, β)-Modal operator, Subalgebra, Normal subalgebra, Homomorphism.
AMS Classification: 06F35, 03E72, 03G25, 08A72.

1 Introduction

In 1966, Imai and Iseki [11] introduced two classes of abstract algebras, viz., BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. Neggers and Kim [15] introduced a new concept, called B-algebras, which are related to several classes of algebras such as BCI/BCK-algebras. Kim and Kim [13] introduced the notion of BG-algebra which is a generalization of B-algebra. The concept of intuitionistic fuzzy subset(IFS) was introduced by Atanassov [3], which is a generalization of the notion of fuzzy sets [17]. The intuitionistic fuzzy modal operators $\Box$ and $\Diamond$ were introduced by Atanassov [3] which are analogous to the modal logic operator of necessity and possibility and have no counterparts in ordinary fuzzy set theory. The extension on both the operators $\Box$ and $\Diamond$ is the new operator D_α which represents both of them. Further the extension of all the operators is the operator $F_{\alpha,\beta}$ called (α, β)-modal operator. The effect of all the modal operator on IFSs is again an IFSs. The modal operators play a very significant role in the study of IFSs. A lot of operators were defined and studied in [2, 4–10, 14, 16]. The concept of fuzzy subalgebras of BG-algebras was introduced by Ahn and Lee in [1]. Here in this paper, we study the effect of modal operators in particular (α, β)-modal operator on intuitionistic fuzzy BG-algebra.

2 Preliminaries

Definition 2.1. *A BG-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $(x * y) * (0 * y) = x,\ \forall\, x, y \in X$.

For brevity, we also call X a BG-algebra.

Example 2.2. *Let X = {0, 1, 2, 3, 4} with the following cayley table*

Table 1: Example of BG-algebra.

$*$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then (X, $*$, 0) is a BG-algebra.

Definition 2.3. *A non-empty subset S of a BG-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.*

Definition 2.4. *A fuzzy subset μ of a BG-algebra X is called a fuzzy subalgebra of X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.*

Definition 2.5. *An intuitionistic fuzzy set (IFS) A of a non empty set X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in set A. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\ x \in X\}$. The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. is called the degree of uncertainty of $x \in A$. The class of IFSs on a universe X is denoted by $IFS(X)$.*

Definition 2.6. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$ *are any two IFSs of a set* X*, then*

$A \subseteq B$ *if and only if for all* $x \in X, \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$*,*

$A = B$ *if and only if for all* $x \in X, \mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$*,*

$A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x)\rangle | x \in X\}$*,*

where $(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ *and* $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$*,*

$A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle | x \in X\}$*,*

where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ *and* $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$*.*

Definition 2.7. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$ *are any two IFSs of a set* X*, then their cartesian product is defined by*

$A \times B = \{\langle (x, y), (\mu_A \times \mu_B)(x, y), (\nu_A \times \nu_B)(x, y)\rangle | x, y \in X\}$*,*

where $(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ *and* $(\nu_A \times \nu_B)(x, y) = \max\{\nu_A(x), \nu_B(y)\}$*.*

Definition 2.8. *For any IFS* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ *of* X *and* $\alpha \in [01]$*, the operators* $\Box : IFS(X) \to IFS(X), \Diamond : IFS(X) \to IFS(X), D_\alpha : IFS(X) \to IFS(X)$ *are defined as*

(i) $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$ *is called necessity operator*

(ii) $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in X\}$ *is called possibility operator*

(iii) $D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x)\rangle | x \in X\}$ *is called* α*-modal operator.*

Clearly $\Box(A) \subseteq A \subseteq \Diamond(A)$ *and the equality hold, when A is a fuzzy set also* $D_0(A) = \Box(A)$ *and* $D_1(A) = \Diamond(A)$*. Therefore the* α*-model operator* $D_\alpha(A)$ *is an extension of necessity operator* $\Box(A)$ *and possibility operator* $\Diamond(A)$*.*

Definition 2.9. *For any IFS* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ *of* X *and for any* $\alpha, \beta \in [01]$ *such that* $\alpha + \beta \leq 1$*, the* (α, β)*-modal operator* $F_{\alpha,\beta} : IFS(X) \to IFS(X)$ *is defined as* $F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle | x \in X\}$*, where* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *for all* $x \in X$*. Therefore we can write*

$F_{\alpha,\beta}(A)$ *as* $F_{\alpha,\beta}(A)(x) = (\mu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x))$

where $\mu_{F_{\alpha,\beta}}(x) = \mu_A(x) + \alpha\pi_A(x)$ *and* $\nu_{F_{\alpha,\beta}(A)}(x)) = \nu_A(x) + \beta\pi_A(x)$*.*

Clearly, $F_{0,1}(A) = \Box(A), F_{1,0}(A) = \Diamond(A)$ *and* $F_{\alpha,1-\alpha}(A) = D_\alpha(A)$*.*

Definition 2.10. *Let X and Y be two non empty sets and* $f : X \longrightarrow Y$ *be a mapping. Let* A *and* B *be IFS's of X and Y respectively . Then the image of A under the map f is denoted by f(A) and is defined by* $f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y))$*, where*

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\} \\ 0 \qquad \textit{otherwise} \end{cases} \qquad \nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\} \\ 1 \qquad \textit{otherwise} \end{cases}$$

also pre image of B under f is denoted by $f^{-1}(B)$ *and is defined as* $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)) = (\mu_B(f(x)), \nu_B(f(x))); \forall x \in X$.

Remark 2.11. $\mu_A(x) \leq \mu_{f(A)}(f(x))$ *and* $\nu_A(x) \geq \nu_{f(A)}(f(x)) \quad \forall x \in X$ *however equality hold when the map f is bijective.*

Definition 2.12. *An IFS A of a BG-algebra X is said to be an IF BG-subalgebra of X if*

(i) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \; \forall \; x, y \in X$.

Example 2.13. *Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:*

Table 2: Example of IF BG-subalgebra.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ given by $\mu_A(0) = \mu_A(1) = 0.6, \mu_A(2) = 0.2$ and $\nu_A(0) = \nu_A(1) = 0.3, \nu_A(2) = 0.5$ is an IF BG-subalgebra of X.

Definition 2.14. *An IFS A of a BG-algebra X is said to be an IF normal subalgebra of X if*

(i) $\mu_A((x * a) * (y * b)) \geq \min\{\mu_A(x * y), \mu_A(a * b)\}$,

(ii) $\nu_A((x * a) * (y * b)) \leq \max\{\nu_A(x * y), \nu_A(a * b)\}, \; \forall \; x, y \in X$.

3 Modal operator $F_{\alpha,\beta}$ on intuitionistic fuzzy subalgebras

In this section, we study the effect of modal operator on IF subalgebra of BG-algebra X.

Theorem 3.1. *If A is an IF subalgebra of BG-algebra X, then $F_{\alpha,\beta}(A)$ is also an IF subalgebra of BG-algebra X.*
Proof: Let $x, y \in X$, then $F_{\alpha,\beta}(x*y) = (\mu_{F_{\alpha,\beta}(A)}(x*y), \nu_{F_{\alpha,\beta}(A)}(x*y))$. Where $\mu_{F_{\alpha,\beta}(A)}(x*y) = \mu_A(x * y) + \alpha\pi_A(x * y)$ and $\nu_{F_{\alpha,\beta}(A)}(x * y) = \nu_A(x * y) + \beta\pi_A(x * y)$
Now

$$\begin{aligned}
\mu_{F_{\alpha,\beta}(A)}(x * y) &= \mu_A(x * y) + \alpha\pi_A(x * y) \\
&= \mu_A(x * y) + \alpha(1 - \mu_A(x * y) - \nu_A(x * y)) \\
&= \alpha + (1 - \alpha)\mu_A(x * y) - \alpha\nu_A(x * y) \\
&\geq \alpha + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) - \alpha\max(\nu_A(x), \nu_A(y)) \\
&= \alpha\{1 - \max(\nu_A(x), \nu_A(y))\} + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) \\
&= \alpha\min(1 - \nu_A(x), 1 - \nu_A(y))\} + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) \\
&= \min\{\alpha(1 - \nu_A(x)) + (1 - \alpha)\mu_A(x), \alpha(1 - \nu_A(y)) + (1 - \alpha)\mu_A(y)\} \\
&= \min\{\mu_A(x) + \alpha(1 - \mu_A(x) - \nu_A(x)), \mu_A(y) + \alpha(1 - \mu_A(y) - \nu_A(y))\} \\
&= \min\{\mu_{F_{\alpha,\beta}(A)}(x), \mu_{F_{\alpha,\beta}(A)}(y)\}
\end{aligned}$$

$$\therefore \mu_{F_{\alpha,\beta}(A)}(x*y) \geq \min\{\mu_{F_{\alpha,\beta}(A)}(x), \mu_{F_{\alpha,\beta}(A)}(y)\}$$

Similarly we can prove

$$\nu_{F_{\alpha,\beta}(A)}(x*y) \leq \max\{\nu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(y)\}$$

Hence $F_{\alpha,\beta}(A)$ is an IF subalgebra of BG-algebra X.

Remark 3.2. *The converse of above Theorem need not be true as shown in Example below.*

Example 3.3. *Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:*

Table 3: Illustration of converse of Theorem 3.1.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The IF subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ given by $\mu_A(0) = 0.48, \mu_A(1) = 0.5, \mu_A(2) = 0.3$ and $\nu_A(0) = 0.3, \nu_A(1) = 0.4, \nu_A(2) = 0.5$ is not an IF BG-subalgebra of X. Since $\mu_A(0) = 0.48 \ngeq \min\{\mu_A(1), \mu_A(1)\} = \mu_A(1) = 0.5$.

Now take $\alpha = 0.7, \beta = 0.3, \alpha + \beta \leq 1$, then $F_{\alpha,\beta}(A) = \{\langle x, \mu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x)\rangle | x \in X\}$ is $\mu_{F_{0.7,0.3}(A)}(0) = 0.63, \mu_{F_{0.7,0.3}(A)}(1) = 0.57, \mu_{F_{0.7,0.3}(A)}(2) = 0.44$ and $\nu_{F_{0.7,0.3}(A)}(0) = 0.36, \nu_{F_{0.7,0.3}(A)}(1) = 0.43, \nu_{F_{0.7,0.3}(A)}(2) = 0.56$. It can easily verified that $F_{0.7,0.3}(A)$ is an IF BG-subalgebra of X.

Corollary 3.4. *If A is an IF subalgebra of BG-algebra X, then*
(i) $\Box(A)$ is also an IF subalgebra of BG-algebra X;
(ii) $\Diamond(A)$ is also an IF subalgebra of BG-algebra X;
(iii) $D_\alpha(A)$ is also an IF subalgebra of BG-algebra X.

Theorem 3.5. *If A is an IF subalgebra of BG-algebra X, then*
(i) $\mu_{F_{\alpha,\beta}(A)}(0) \geq \mu_{F_{\alpha,\beta}(A)}(x)$
(ii) $\nu_{F_{\alpha,\beta}(A)}(0) \leq \nu_{F_{\alpha,\beta}(A)}(x) \quad \forall x \in X$.
Proof: We have

$$\begin{aligned} \mu_{F_{\alpha,\beta}(A)}(0) &= \mu_{F_{\alpha,\beta}(A)}(x*x) \\ &\geq \min\{\mu_{F_{\alpha,\beta}(A)}(x), \mu_{F_{\alpha,\beta}(A)}(x)\} \\ &= \mu_{F_{\alpha,\beta}(A)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{F_{\alpha,\beta}(A)}(0) &= \nu_{F_{\alpha,\beta}(A)}(x*x) \\ &\leq \max\{\nu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x)\} \\ &= \nu_{F_{\alpha,\beta}(A)}(x) \quad \forall x \in X. \end{aligned}$$

Theorem 3.6. *If A and B are two IF subalgebras of BG-algebra X, then*
(i) $A \cap B$ is also an IF subalgebra of BG-algebra X.
(ii) $A \times B$ is also an IF subalgebra of BG-algebra $X \times X$.
Proof: (i) We have $A \cap B = \{\langle x, \mu_{(A\cap B)}(x), \nu_{(A\cup B)}(x)\rangle | x \in X\}$,
where $\mu_{(A\cap B)}(x) = (\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and
$\nu_{(A\cup B)}(x) = (\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,
Let $x, y \in X$. Since both A, B are subalgebras of X, therefore
$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$
Also $\mu_B(x * y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x * y) \leq \max\{\nu_B(x), \nu_B(y)\}$
Now

$$\begin{aligned}
\mu_{(A\cap B)}(x * y) &= \min\{\mu_A(x * y), \mu_A(x * y)\} \\
&\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}\} \\
&= \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\
&= \min\{\mu_{(A\cap B)}(x), \mu_{(A\cap B)}(y)\} \\
\Rightarrow \mu_{(A\cap B)}(x * y) &\geq \min\{\mu_{(A\cap B)}(x), \mu_{(A\cap B)}(y)\}
\end{aligned}$$

Similarly we can prove

$$\nu_{(A\cup B)}(x * y) \leq \max\{\nu_{(A\cup B)}(x), \nu_{(A\cup B)}(y)\}$$

Hence $A \cap B$ is also an IF subalgebra of BG-algebra X.
(ii) Similar to proof of (i)

Theorem 3.7. *If A and B are two IF subalgebras of BG-algebra X, then*
(i) $F_{\alpha,\beta}(A \cap B)$ is also an IF subalgebra of BG-algebra X.
(ii) $F_{\alpha,\beta}(A \times B)$ is also an IF subalgebra of BG-algebra $X \times X$.

Proof: (i) We have $F_{\alpha,\beta}(A \cap B)(x) = \{\langle x, \mu_{F_{\alpha,\beta}(A\cap B)}(x), \nu_{F_{\alpha,\beta}(A\cup B)}(x)\rangle | x \in X\}$, where $\mu_{(A\cap B)}(x) = (\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and
$\nu_{(A\cup B)}(x) = (\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,
Let $x, y \in X$. Since both A, B are subalgebras of X, therefore
$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$
Also $\mu_B(x * y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x * y) \leq \max\{\nu_B(x), \nu_B(y)\}$
Now

$$\begin{aligned}
&\mu_{F_{\alpha,\beta}(A\cap B)}(x * y) \\
&= \mu_{(A\cap B)}(x * y) + \alpha\pi_{(A\cap B)}(x * y) \\
&= \mu_{(A\cap B)}(x * y) + \alpha\{1 - \mu_{(A\cap B)}(x * y) - \nu_{(A\cup B)}(x * y)\} \\
&= \alpha + (1 - \alpha)\mu_{(A\cap B)}(x * y) - \alpha\nu_{(A\cup B)}(x * y) \\
&= \alpha(1 - \max(\nu_A(x * y), \nu_A(y * y))) + (1 - \alpha)\min(\mu_A(x * y), \mu_A(x * y)) \\
&\geq (1 - \alpha)\min\{\min(\mu_A(x), \mu_A(y)), \min(\mu_B(x), \mu_B(y))\} + \alpha - \\
&\qquad \alpha\max\{\max(\nu_A(x), \nu_A(y)), \max(\nu_B(x), \nu_B(y)))\}
\end{aligned}$$

$= (1-\alpha)\min\{\min(\mu_A(x),\mu_B(x)),\min(\mu_A(y),\mu_B(y))\}+\alpha-$
$\alpha\max\{\max(\nu_A(x),\nu_B(x)),\max(\nu_A(y),\nu_B(y)))\}$
$= (1-\alpha)\min\{\mu_{(A\cap B)}(x),\mu_{(A\cap B)}(y)\}+\alpha-\alpha\max\{\nu_{(A\cup B)}(x),\nu_{(A\cup B)}(y)\}$
$= (1-\alpha)\min\{\mu_{(A\cap B)}(x),\mu_{(A\cap B)}(y)\}+\alpha\min\{1-\nu_{(A\cup B)}(x),1-\nu_{(A\cup B)}(y)\}$
$= \min\{(1-\alpha)\mu_{(A\cap B)}(x)+\alpha(1-\nu_{(A\cup B)}(x)),(1-\alpha)\mu_{(A\cap B)}(y)+\alpha(1-\nu_{(A\cup B)}(y)\})$
$= \min\{\mu_{F_{\alpha,\beta}(A\cap B)}(x),\mu_{F_{\alpha,\beta}(A\cap B)}(y)\}$
Therefore $\mu_{F_{\alpha,\beta}(A\cap B)}(x*y)\geq\min\{\mu_{F_{\alpha,\beta}(A\cap B)}(x),\mu_{F_{\alpha,\beta}(A\cap B)}(y)\}$

Similarly we can prove

$$\nu_{F_{\alpha,\beta}(A\cup B)}(x*y)\leq\max\{\nu_{F_{\alpha,\beta}(A\cup B)}(x),\nu_{F_{\alpha,\beta}(A\cup B)}(y)\}$$

(ii) Similar to proof of (i)

Theorem 3.8. *If $\{A_i : i=1,2,...,n\}$ be n IF subalgebras of X, then*
(i) $F_{\alpha,\beta}(\cap_{i=1}^n A_i : i=1,2,...,n\}$ is also an IF subalgebra of X.
(i) $F_{\alpha,\beta}(\times_{i=1}^n A_i : i=1,2,...,n\}$ is also an IF subalgebra of $\times_{i=1}^n X_i$.

Theorem 3.9. *If $F_{\alpha,\beta}(A)=\langle\mu_{F_{\alpha,\beta}A},\nu_{F_{\alpha,\beta}A}\rangle$ is an IF subalgebra of X. Then the sets*
$X_{\mu_{F_{\alpha,\beta}}}=\{x\in X|\mu_{F_{\alpha,\beta}(A)}(x)=\mu_{F_{\alpha,\beta}(A)}(0)\}$
$X_{\nu_{F_{\alpha,\beta}}}=\{x\in X|\nu_{F_{\alpha,\beta}(A)}(x)=\nu_{F_{\alpha,\beta}(A)}(0)\}$
are subalgebras of X.

Proof: Let $x,y\in\mu_{F_{\alpha,\beta}(A)}$, then $\mu_{F_{\alpha,\beta}(A)}(x)=\mu_{F_{\alpha,\beta}(A)}(y)=\mu_{F_{\alpha,\beta}(A)}(0)$
Now

$$\begin{aligned}
\mu_{F_{\alpha,\beta}(A)}(x*y) &\geq \min\{\mu_{F_{\alpha,\beta}(A)}(x),\mu_{F_{\alpha,\beta}(A)}(y)\}\\
&= \min\{\mu_{F_{\alpha,\beta}(A)}(0),\mu_{F_{\alpha,\beta}(A)}(0)\}\\
&= \mu_{F_{\alpha,\beta}(A)}(0)\\
\Rightarrow \mu_{F_{\alpha,\beta}(A)}(x*y) &\geq \mu_{F_{\alpha,\beta}(A)}(0)\\
\text{Also}\quad \mu_{F_{\alpha,\beta}(A)}(0) &\geq \mu_{F_{\alpha,\beta}(A)}(x*y) \text{ by Theorem 3.5}\\
\text{Therefore}\quad \mu_{F_{\alpha,\beta}(A)}(x*y) &= \mu_{F_{\alpha,\beta}(A)}(0)\\
\Rightarrow x*y &\in X_{\mu_{F_{\alpha,\beta}}}\\
\text{Again let}\quad x,y\in\nu_{F_{\alpha,\beta}(A)} &\quad \text{then}\quad \nu_{F_{\alpha,\beta}(A)}(x)=\nu_{F_{\alpha,\beta}(A)}(y)=\nu_{F_{\alpha,\beta}(A)}(0)\\
\nu_{F_{\alpha,\beta}(A)}(x*y) &\leq \max\{\nu_{F_{\alpha,\beta}(A)}(x),\nu_{F_{\alpha,\beta}(A)}(y)\}\\
&= \max\{\nu_{F_{\alpha,\beta}(A)}(0),\nu_{F_{\alpha,\beta}(A)}(0)\}\\
&= \nu_{F_{\alpha,\beta}(A)}(0)\\
\Rightarrow \nu_{F_{\alpha,\beta}(A)}(x*y) &\leq \nu_{F_{\alpha,\beta}(A)}(0)\\
\text{also}\quad \nu_{F_{\alpha,\beta}(A)}(0) &\leq \nu_{F_{\alpha,\beta}(A)}(x*y) \text{ by Theorem 3.5}\\
\text{Therefore}\quad \nu_{F_{\alpha,\beta}(A)}(x*y) &= \nu_{F_{\alpha,\beta}(A)}(0)\\
\Rightarrow x*y &\in X_{\nu_{F_{\alpha,\beta}}}
\end{aligned}$$

Hence $X_{\mu_{F_{\alpha,\beta}}}$ and $X_{\nu_{F_{\alpha,\beta}}}$ are subalgebras of X.

Proposition 3.10. *If A and B be two IFS sets of X and Y respectively and $f : X \longrightarrow Y$ be a mapping, then*
(i) $f^{-1}(F_{\alpha,\beta}(B)) = F_{\alpha,\beta}(f^{-1}(B))$
(ii)$f(F_{\alpha,\beta}(A)) \subseteq F_{\alpha,\beta}(f(A))$

Theorem 3.11. *If A is an IF fuzzy normal subalgebra of BG-algebra X, then $F_{\alpha,\beta}(A)$ is also an IF normal subalgebra of X.*
Proof: Let $x, y, a, b \in X$, then $F_{\alpha,\beta}((x*a)*(y*b)) = (\mu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)), \nu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)))$ where $\mu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)) = \mu_A((x*a)*(y*b)) + \alpha\pi_A((x*a)*(y*b))$ and $\nu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)) = \nu_A((x*a)*(y*b)) + \beta\pi_A((x*a)*(y*b))$
Now

$$\begin{aligned}
&\mu_{F_{\alpha,\beta}(A)}((x*a)*(y*b))\\
&= \mu_A((x*a)*(y*b)) + \alpha\pi_A((x*a)*(y*b))\\
&= \mu_A((x*a)*(y*b)) + \alpha(1 - \mu_A((x*a)*(y*b)) - \nu_A((x*a)*(y*b)))\\
&= \alpha + (1-\alpha)\mu_A((x*a)*(y*b)) - \alpha\nu_A((x*a)*(y*b))\\
&\geq \alpha + (1-\alpha)\min(\mu_A(x*y), \mu_A(a*b)) - \alpha\max(\nu_A(x*y), \nu_A(a*b))\\
&= \alpha\{1 - \max(\nu_A(x*y), \nu_A(a*b))\} + (1-\alpha)\min(\mu_A(x*y), \mu_A(a*b))\\
&= \alpha\min(1 - \nu_A(x*y), 1 - \nu_A(a*b))\} + (1-\alpha)\min(\mu_A(x*y), \mu_A(a*b))\\
&= \min\{\alpha(1 - \nu_A(x*y)) + (1-\alpha)\mu_A(x*y), \alpha(1 - \nu_A(a*b)) + (1-\alpha)\mu_A(a*b)\}\\
&= \min\{\mu_A(x*y) + \alpha(1 - \mu_A(x*y) - \nu_A(x*y)), \mu_A(a*b)\\
&\quad + \alpha(1 - \mu_A(a*b) - \nu_A(a*b))\}\\
&= \min\{\mu_{F_{\alpha,\beta}(A)}(x*y), \mu_{F_{\alpha,\beta}(A)}(a*b)\}
\end{aligned}$$

$\therefore \mu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)) \geq \min\{\mu_{F_{\alpha,\beta}(A)}(x*y), \mu_{F_{\alpha,\beta}(A)}(a*b)\}$
Similarly we can prove
$\nu_{F_{\alpha,\beta}(A)}((x*a)*(y*b)) \leq \max\{\nu_{F_{\alpha,\beta}(A)}(x*y), \nu_{F_{\alpha,\beta}(A)}(a*b)\}$

Hence $F_{\alpha,\beta}(A)$ is an IF normal subalgebra of BG-algebra X.

Theorem 3.12. *If A and B are two IF normal subalgebras of BG-algebra X, then*
(i) $F_{\alpha,\beta}(A \cap B)$ is also an IF normal subalgebra of BG-algebra X.
(ii) $F_{\alpha,\beta}(A \times B)$ is also an IF normal subalgebra of BG-algebra $X \times X$.

Theorem 3.13. *If $\{A_i : i = 1, 2, ..., n\}$ be n IF normal subalgebras of X, then*
(i) $F_{\alpha,\beta}(\cap_{i=1}^{n} A_i : i = 1, 2, ..., n\}$ is also an IF normal subalgebra of X.
(i) $F_{\alpha,\beta}(\times_{i=1}^{n} A_i : i = 1, 2, ..., n\}$ is also an IF normal subalgebra of $\times_{i=1}^{n} X_i$.

4 Effect of modal operators on intuitionistic fuzzy BG-algebra under homomorphism

Definition 4.1. *Let X and Y be two BG-algebras, then a mapping $f : X \longrightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y),\ \forall x, y \in X$.*

Theorem 4.2. *Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF subalgebra of Y, then $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF subalgebra of X.*

Proof: Since $f^{-1}(F_{\alpha,\beta}(A)) = F_{\alpha,\beta}(f^{-1}(A))$
It is enough to show that $F_{\alpha,\beta}(f^{-1}(A))$ is an IF BG subalgebras of X.
Let $x, y \in X$, then

$$\begin{aligned}
\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) &= \mu_{F_{\alpha,\beta}(A)} f(x * y) \\
&= \mu_{F_{\alpha,\beta}(A)}(f(x) * f(y)) \\
&\geq \min\{\mu_{F_{\alpha,\beta}(A)}(f(x)), \mu_{F_{\alpha,\beta}(A)}(f(y))\} \\
&= \min\{\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \mu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\} \\
\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) &\geq \min\{\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \mu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\} \\
\text{Similarly we can show} & \\
\nu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) &\leq \max\{\nu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \nu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\}
\end{aligned}$$

Hence $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF BG subalgebras of X.

Corollary 4.3. *Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras.*
(i) If $\Box(A)$ is an IF subalgebra of Y, then $f^{-1}(\Box(A))$ is also an IF subalgebra of X.
(ii) If $\Diamond(A)$ is an IF subalgebra of Y, then $f^{-1}(\Diamond(A))$ is also an IF subalgebra of X.

Theorem 4.4. *Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF normal subalgebra of Y, then $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF normal subalgebra of X.*

Theorem 4.5. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF fuzzy subalgebra of X, then $f(F_{\alpha,\beta}(A))$ is also an IF fuzzy subalgebra of Y.*

Proof: Let $y_1, y_2 \in Y$. Since f is onto, therefore there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$
$f(F_{\alpha,\beta}(A))(y_1 * y_2) = (\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2), \nu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2))$
Now $\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) = \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2)$ where $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$

$$\begin{aligned}
&\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \\
&= \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2) \\
&= \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2) \\
&= \mu_A(x_1 * x_2) + \alpha\pi_A(x_1 * x_2) \\
&= \mu_A(x_1 * x_2) + \alpha[1 - \mu_A(x_1 * x_2) - \nu_A(x_1 * x_2)]
\end{aligned}$$

$$= \alpha + (1-\alpha)\mu_A(x_1 * x_2) - \alpha\nu_A(x_1 * x_2)$$
$$\geq \alpha + (1-\alpha)\min\{\mu_A(x_1), \mu_A(x_2)\} - \alpha\max\{\nu_A(x_1), \nu_A(x_2)\}$$
$$= \alpha\{1 - \max\{\nu_A(x_1), \nu_A(x_2)\}\} + (1-\alpha)\min\{\mu_A(x_1), \mu_A(x_2)\}$$
$$= \alpha\min\{1-\nu_A(x_1), 1-\nu_A(x_2)\} + (1-\alpha)\min\{\mu_A(x_1), \mu_A(x_2)\}$$
$$= \min\{(1-\alpha)\mu_A(x_1) + \alpha(1-\nu_A(x_1)), (1-\alpha)\mu_A(x_2) + \alpha(1-\nu_A(x_2))\}$$
$$= \min\{\mu_A(x_1) + \alpha(1-\nu_A(x_1) - \mu_A(x_1)), \mu_A(x_2) + \alpha(1-\nu_A(x_2) - \mu_A(x_2))\}$$
$$= \min\{\mu_A(x_1) + \alpha\pi_A(x_1), \mu_A(x_2) + \alpha\pi_A(x_2)\}$$
$$= \min\{\mu_{F_{\alpha,\beta}(A)}(x_1), \mu_{F_{\alpha,\beta}(A)}(x_2)\}$$
$$= \min\{\mu_{f(F_{\alpha,\beta}(A))}(f(x_1)), \mu_{f(F_{\alpha,\beta}(A))}(f(x_2))\}$$
$$= \min\{\mu_{f(F_{\alpha,\beta}(A))}(y_1), \mu_{f(F_{\alpha,\beta}(A))}(y_2)\}$$
$$\therefore \mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \geq \min\{\mu_{f(F_{\alpha,\beta}(A))}(y_1), \mu_{f(F_{\alpha,\beta}(A))}(y_2)\}$$

Similarly we can show

$$\nu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \leq \max\{\nu_{f(F_{\alpha,\beta}(A))}(y_1), \nu_{f(F_{\alpha,\beta}(A))}(y_2)\}$$

Hence $f(F_{\alpha,\beta}(A))$ is an IF fuzzy subalgebra of Y.

Corollary 4.6. *Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras.*
(i) If $\Box(A)$ is an IF subalgebra of X, then $f(\Box(A))$ is also an IF subalgebra of Y.
(ii) If $\Diamond(A)$ is an IF subalgebra of X, then $f(\Diamond(A))$ is also an IF subalgebra of Y.

Theorem 4.7. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF fuzzy normal subalgebra of X, then $f(F_{\alpha,\beta}(A))$ is also an IF fuzzy normal subalgebra of Y.*

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