Notes on Intuitionistic Fuzzy Sets
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On some methods of probability

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Abstract: The paper contains a review of some methods for building probability theory on intuitionistic fuzzy sets. They are based on some representation of states by Kolmogorov probability spaces as well as the embedding of IF-spaces into the MV-algebras.
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1 Introduction

In this paper we will work with probability defined on intuitionistic fuzzy sets (shortly IF-sets). First we will describe the structures which we will work with. Then the relations between the different structures and their properties will be showed. Let start with the definition of the important term in probability theory - state. Consider a measurable space $(\Omega, \mathcal{S})$ with a σ-algebra $\mathcal{S}$. Let $\mathcal{J}$ be the family of all measurable functions $f : \Omega \to [0, 1]$. To define the state on $\mathcal{J}$ we need two binary operations. In this paper we will use the Lukasiewicz operations

$$f \oplus g = \min(f + g, 1_\Omega),$$

$$f \odot g = \max(f + g - 1_\Omega, 0_\Omega).$$

These binary operations play the same role as the union and the intersection in the set theory.

A state on $\mathcal{J}$ is a mapping $m_{\mathcal{J}} : \mathcal{J} \to [0, 1]$ satisfying the following conditions

1. $m_{\mathcal{J}}(0_\Omega) = 0, m_{\mathcal{J}}(1_\Omega) = 1$.

2. If $f \odot g = 0_\Omega$ then $m_{\mathcal{J}}(f \oplus g) = m_{\mathcal{J}}(f) + m_{\mathcal{J}}(g)$.

3. If $f_n \nearrow f$ then $m_{\mathcal{J}}(f_n) \nearrow m_{\mathcal{J}}(f)$.

Since $\mathcal{J}$ contains the functions $f : \Omega \to [0, 1]$ then $\mathcal{J}$ represent the family of fuzzy sets. In the fuzzy set theory f is called membership function. Let us now look at the more general structure - intuitionistic fuzzy sets. The Atanassov IF-set [1, 2] is a pair $A = (\mu_A, \nu_A)$ of fuzzy sets $\mu_A : \Omega \to [0, 1], \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1_\Omega$. Similarly as in the fuzzy set theory the function μ_A is called the membership function. The second function, ν_A is called the non-membership function. Denote by $\mathcal{F}$ the family of all IF-sets such that μ_A, ν_A are $\mathcal{S}$-measurable. On the set $\mathcal{F}$ there is defined the following ordering

$$A = (\mu_A, \nu_A) \leq (\mu_B, \nu_B) = B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

By this ordering for each $A \in \mathcal{F}$ it holds

$$(0_\Omega, 1_\Omega) \leq A \leq (1_\Omega, 0_\Omega).$$

The family $\mathcal{J}$ can be considered as a subset of $\mathcal{F}$ if for $f \in \mathcal{J}$ we put $A = (f, 1 - f)$. Then $\mu_A + \nu_A = 1$.

For $A, B \in \mathcal{F}$ we could define the Lukasiewicz operations

$$A \oplus B = (\min(\mu_A + \mu_B, 1_\Omega), \max(\nu_A + \nu_B - 1_\Omega, 0_\Omega)),$$

$$A \odot B = (\max(\mu_A + \mu_B - 1_\Omega, 0_\Omega), \min(\nu_A + \nu_B, 1_\Omega)).$$

A state on $\mathcal{F}$ is a mapping $m : \mathcal{F} \to [0, 1]$ satisfying the following conditions [6]

1. $m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1$,

2. $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B) = m(A) + m(B)$,

3. $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$.

In this contribution we will also work with MV-algebras ([19,21,22]). For the first recall the definition of a lattice ordered group (ℓ-group).

Definition 1. *By an ℓ-group G we consider an algebraic system $(G, +, \leq)$ such that*

1. *$(G, +)$ is commutative group,*

2. *$(G, \leq)$ is lattice,*

3. *$a \leq b \Rightarrow a + c \leq b + c$.*

We will use the form of MV-algebra which is generated by fuzzy sets ([25], [26], [27]). Let $u \in G$, $u > 0$. Then an MV-algebra $\mathcal{M}$ is a set $M = [0, u]$ for which it holds

$$a \oplus b = (a + b) \wedge u,$$

$$a \odot b = (a + b - u) \wedge 0.$$

The element u is supposed to be a strong unit of G. Then an MV-algebra state ([13, 15, 16, 20]) is a mapping $\bar{m} : \mathcal{M} \to [0, 1]$ satisfying the following conditions:

1. $\bar{m}(0) = 0, \bar{m}(u) = 1.$
2. $a \odot b = 0 \Rightarrow \bar{m}(a \oplus b) = \bar{m}(a) + \bar{m}(b).$
3. $a_n \nearrow a \Rightarrow \bar{m}(a_n) \nearrow \bar{m}(a).$

2 Representation of states by probabilities

Let us look at the representation of states from two points of view. The first will be general representation, the second local representation.

2.1 General representation of states

Very interesting is the relation between probability measure and state. In [3] there was proved the Butnaria-Klement theorem:

Theorem 2. *To any state $m_{\mathcal{J}} : \mathcal{J} \to [0, 1]$ there exists a probability measure $P : \mathcal{S} \to [0, 1]$ such that*

$$m_{\mathcal{J}}(f) = \int_{\Omega} f dP$$

for any $f \in \mathcal{J}$.

This theorem has been generalized also for Atanassov IF-sets. The generalization of the Butnaria-Klement theorem have the form:

Theorem 3. *If $m : \mathcal{F} \to [0, 1]$ is a state then there exists a probability measure $P : \mathcal{S} \to [0, 1]$ and $\alpha \in [0, 1]$ such that*

$$m(A) = \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} (\mu_A + \nu_A) dP \right)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

This theorem is called Representation theorem for probabilities on IF-states and it was proved in [4–6, 10, 18]. The Butnaria-Klement theorem can be obtained as a consequence if $\nu_A = 1 - \mu_A$.

2.2 Local representation of states

In this section we will look at the relations between the observables and random variables. Let start with definitions of these structures [21].

Definition 4. *An observable is a mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions*

1. $x(R) = (1_\Omega, 0_\Omega)$,
2. $A \cap B = \emptyset \Rightarrow x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$,
3. $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$.

where $A, A_n, B \in \mathcal{F}$ *for* $n = 1, 2, \ldots$.

Definition 5. *A random variable is such function $\xi : \Omega \to R$ that for any $A \in \mathcal{B}(R)$ it holds $\xi^{-1}(A) \in \mathcal{F}$.*

It is easy to prove (see [21]) that:

Proposition 6. *For any random variable $\xi : \Omega \to R$ it holds*

1. $\xi^{-1}(R) = \Omega$,
2. $\xi^{-1}(A \cup B) = \xi^{-1}(A) \cup \xi^{-1}(B)$,
3. $\xi^{-1}(A_n) \nearrow \xi^{-1}(A)$.

Proposition 7. *Let $\Omega = R^N$. Define the function $\xi_n : R^N \to R$ by the formula $\xi_n((t_i)_{i=1}^{\infty}) = t_n$. Then ξ is the random variable.*

Denote by $\mathcal{C}$ the system of all sets of the form

$$A = \{(t_i)_{i=1}^{\infty}; t_1 \in A_1, \ldots t_n \in A_n\}$$

where $n \in N$ and $A_1, \ldots, A_n \in \mathcal{B}(R)$ are Borel sets. To any sequence $(x_n)_n$ of observables there exist the probability space $(R^N, \sigma(\mathcal{C}), P)$ such that convergences of $(x_n)_n$ and $(\xi_n)_n$ are in correlation. Of course, the space $(R^N, \sigma(\mathcal{C}), P)$ depends on a concrete sequence $(x_n)_n$. For different sequences various spaces can be obtain. We could speak about following types of convergences (for the proof see [19, 21]).

Proposition 8. *Let $(x_n)_n$ be a sequence of observables, $(\xi_n)_n$ be the sequence of corresponding random variables. Then*

1. $(x_n)_n$ *converges to* $F : R \to R$ *in distribution if and only if* $(\xi_n)_n$ *converges to* F*;*
2. $(x_n)_n$ *converges to* 0 *in state* $m : \mathcal{F} \to [0, 1]$ *if and only if* $(\xi_n)_n$ *converges to* 0 *in measure* $P : \mathcal{S} \to [0, 1]$*;*
3. *if* $(\xi_n)_n$ *converges* P*-almost everywhere to* 0 *then* $(x_n)_n$ *converges* m*-almost everywhere to* 0.

3 Embedding of fuzzy spaces to MV-algebra

Recall that IF-set can be geometrically regarded as a function $A : \Omega \to \Delta$ where

$$\Delta = \{(x, y); x \geq 0, y \geq 0, x + y \leq 1\}$$

(see *Figure 1*).

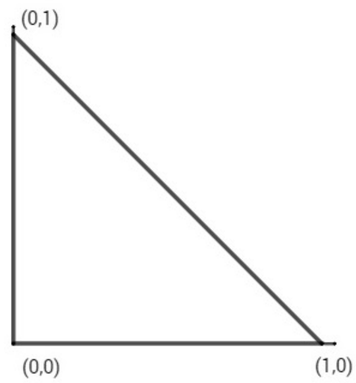

Figure 1. Elements of IF-sets

Let us take MV-algebra $\mathcal{M}$ such that $G = R^2$. Then the addition of two elements is defined as

$$C = A + B = (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega).$$

This addition represents the addition of two vectors with fixed point $(0, 1)$ (see *Figure 2*). Therefore there is a natural question about the possibility to embedding the IF-set to MV-algebra.

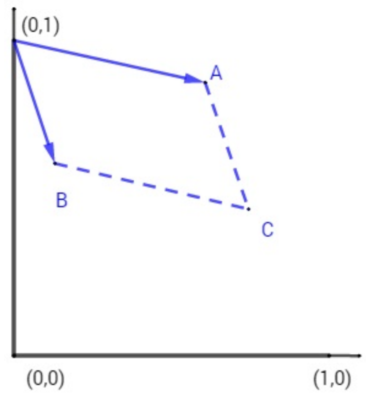

Figure 2. Addition in MV-algebra

For any point $A = (\mu_A, \nu_A) \in \mathcal{M}$ it holds (see *Figure 3*)

$$(\mu_A, \nu_A) + (0_\Omega, 1_\Omega - \nu_A) = (\mu_A, 0_\Omega).$$

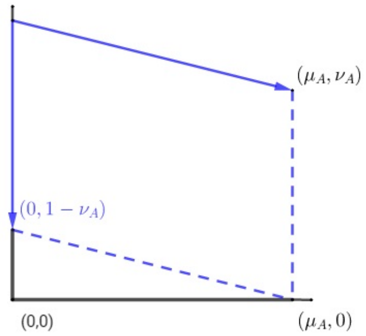

Figure 3. Embedding IF-set to MV-algebra

From this reason we can define the function $\bar{m} : \mathcal{M} \to [0, 1]$ by the formula

$$\bar{m}(\mu_A, \nu_A) = m(\mu_A, 0_\Omega) - m(0_\Omega, 1_\Omega - \nu_A)$$

This function represent the state on $\mathcal{M}$ generated by a state defined on $\mathcal{F}$.
Recall that there is defined another structure as generalization of fuzzy sets, called interval valued fuzzy set (shortly IV-set) which was defined by Zadeh ([27]). The IV-set is a pair of functions $A = (\mu_A, \nu_A); \mu_A : \Omega \to [0, 1], \nu_A : \Omega \to [0, 1]$ for which it holds $\nu_A \geq \mu_A$. The ordering on this set is given by

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B, \nu_A \leq \nu_B$$

and the operation $+$ is defined by

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B).$$

Also the family $\mathcal{K}$ of all IV-sets can be embedded to an MV-algebra. Here again $G = R^2$. Of course the ordering in G and also the operation $+$ are given by the same way as in the IV-sets.

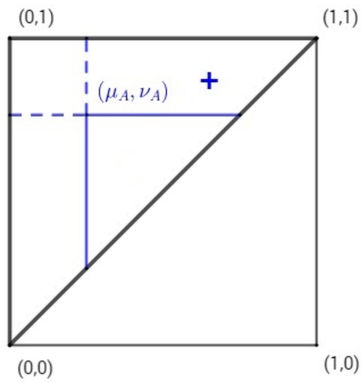

Figure 4. IV-set

4 Review of results

There are many IF-state applications. We will mention just some of them. In probability theory, there was proved for example central limit theorem [19, 21], law of large numbers [13], martingale convergence theorem [12, 24]. In IF-dynamical system it was proved Poincare recurrence theorem [20], individual ergodic theorem [19,21], invariant states [14], entropy of IF-dynamical system [7–9, 17, 23].

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