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A "Family" of Similarity Measures for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory

Eulalia Szmidt * and Jim F. Baldwin **

* Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01–447 Warsaw, Poland
E-mail: szmidt@ibspan.waw.pl

** Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, England
E-mail: Jim.Baldwin@bristol.ac.uk

We remind some similarities/parallels between two of the theories dealing with widely understood (not only as randomness) uncertainty – mass assignment theory and intuitionistic fuzzy set theory. Mass assignment theory is well known tool for dealing with both probabilistic and fuzzy uncertainties whereas intuitionistic fuzzy set theory is an extension of fuzzy set theory which makes it possible to describe imprecise information. Next, we recall the measures of similarity which we have proposed for both theories. The proposed measures take into account not only a pure distance between compared elements but answer the questions if the considered elements/objects are more similar or more dissimilar (the measures take into account and compare two types of distances). It is shown that even if a distance between compared objects is small, it can happen that the objects are completely dissimilar. The disadvantage of the measures is the range of their values – not consistent with the tradition as far as the similarity measures are concerned. In this paper we propose the whole array of the new similarity measures preserving the advantages of the previously proposed similarity measures and the same time following the commonly assumed number values.

1 Introduction

For several centuries (starting from the mid-seventeenth century) uncertainty was identified and expressed in terms of probability theory only. Uncertainty was a synonym of randomness. This situation was challenged in the 1960s by other theories, distinct from probability theory, characterizing different aspects of uncertain situations. Uncertainty started to be perceived as a multidimensional concept manifesting one of its dimensions as randomness. Other dimensions turned out equally important from the point of view of representing and processing information. The most visible of the theories, dealing with different aspects of uncertainty, are the theory of fuzzy sets (Zadeh [52]), theory of evidence (Dempster-Shafer theory [19, 25]), possibility theory (Zadeh [53]), theory of fuzzy measures (Sugeno [26]). In this paper we explore two of the theories dealing with widely understood uncertainty – intuitionistic fuzzy set theory (Atanassov [1]) which is a generalization of fuzzy sets, and mass assignment theory (Baldwin [8, 7]) related to the theory of evidence.

We show some similarities/parallels between mass assignment theory (Baldwin [8], Baldwin et al. [11, 12]) and intuitionistic fuzzy set theory [Atanassov [2, 5]]. The similarities we point out do not mean that one of the theories is better or could replace the other. Opposite – the similarities we show seem to be important as far as further development of both theories is concerned.

First, we notice that the voting interpretation is common for both theories. This observation made it possible to realize the counterparts of the parameters in both theories.

The counterparts of the parameters in mass assignment theory and intuitionistic fuzzy set theory made it possible to propose a common (or, to be more precise – parallel) geometrical representation useful when introducing and discussing some new similarity measures with their advantages in comparison to the commonly used similarity measures being a dual concept to a (single) distace.

Similarity assessment plays a fundamental role in inference and approximate reasoning in virtually all applications of fuzzy logic. For different purposes different measures of similarity are to be used. Importance of the problem motivates researchers to compare and examine the effectiveness and properties of the different measures of similarity for fuzzy sets (e.g. Zwick at al. [54], Pappis and Karacapilidis [24], Chen at al. [17], Wang at al. [47], Yager [49, 50, 51], Bouchon-Meunier at al. [16], Cross and Sudkamp [18]).

We are aware that the proposed here measures of similarity does not solve all problems one meets when assessing similarity. Similarity is a complex problem and even the terminology used by different researchers is not the same. For example, some researchers assume dissimilarity to be the inverse of similarity (Zwick at al. [54]) whereas others call it non-similarity, and specify the dissimilarity between two fuzzy sets by a similarity measure between the complements of the two fuzzy sets (Dubois and Prade [21]). Even assumptions concerning properties of similarity measures are different – typically it is assumed that similarity should be symmetric. But the axiom of symmetry is relaxed in similarity measures used in psychological studies (Cross and Sudkamp [18]).

The similarity measures we recall (Szmidt and Baldwin [29]) are not the standard similarity measures in the sense that they are not a dual concept to a (general) distance (Tversky [46]). In commonly used similarity measures dissimilarity behaves like a distance function. Such a standard approach – formulated for objects as crisp values was later extended and used to assess similarity of fuzzy sets (Cross and Sudkamp [18]). Distances were also proposed to measure similarity between intuitionistic fuzzy sets (Dengfeng and Chuntian [20]). The measures we proposed were not that kind of similarity – they did not measure just a distance between the compared elements/objects. The measures answered the question if the compared elements/objects were more similar or more dissimilar. The measures took into account two kinds of distances – one to an object to be compared, and one to its complement. We infered about the similarity on the basis of a ratio of the two types of distances. In a special case – when we assess similarity of any element/object and a crisp element/object, the first proposed measure is equivalent to the Jaccard index. For more details we refer an interested reader to (Szmidt and Kacprzyk [44]). In the recalled measures we used a concept of a complement element and showed that a distance between considered elements/objects X and F could be small (so classical measures of similarity would indicate that the objects are similar) whereas the distance between element X and the complement of F could be even smaller what was not taken into account by commonly used similarity measures. We show examples when the distance between objects is small but the same time they are more dissimilar than similar. In other words – inferring without taking into account a distance to a complement of an object can be misleading. The only disadvantage of the proposed measures is that they do not follow the range of the usually assumed values for the similarity measures. So we constructed a whole array of the new similarity measures preserving the advantages of the previously proposed ones, and which numerical values are consistent with the common scientific tradition.

2 Intuitionistic Fuzzy Set Theory

Let us start with basic concepts related to fuzzy sets.

Definition 1 *A fuzzy set $A^{'}$ in $X = \{x\}$ is given by (Zadeh [52]):*

$$A^{'} = \{< x, \mu_A(x) > | x \in X\} \tag{1}$$

where $\mu_A : X \to [0, 1]$ is the membership function of the fuzzy set $A^{'}$; $\mu_A \in [0, 1]$.

The intuitionistic fuzzy set (IFS) theory is based both on extensions of corresponding definitions of fuzzy sets objects and definitions of new objects and their properties (Atanassov [1, 2, 3, 4, 5]).

Definition 2 *An intuitionistic fuzzy set A in X is given by (Atanassov [1, 5]):*

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where

$$\mu_A : X \to [0, 1]$$

$$\nu_A : X \to [0, 1]$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X$$

The numbers $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote the degree of membership and non-membership of x to A, respectively.

Obviously, each fuzzy set A' corresponds to the following intuitionistic fuzzy set:

$$A = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in X\} \tag{3}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{4}$$

the intuitionistic fuzzy index (or a hesitation margin) of x in A and, it expresses a lack of knowledge of whether x belongs to A or not (Atanassov [2, 3, 4, 5]).

It is obvious, that

$$0 \leq \pi_A(x) \leq 1 \quad for \quad each \quad x \in X$$

For each fuzzy set A' in X, evidently,

$$\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0, \quad for \quad each \quad x \in X$$

In our further considerations we will use the notion of the complement elements, which definition is a simple consequence of a complement set A^C

$$A^C = \{< x, \nu_A, \mu_A > | x \in X\} \tag{5}$$

The application of intuitionistic fuzzy sets instead of fuzzy sets means the introduction of another degree of freedom into a set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imperfect knowledge what leads to describing many real problems in a more adequate way.

Intuitionistic fuzzy sets based models may be adequate mainly in the situations when we face human testimonies, opinions, etc. involving answers of three types:

- yes,
- no,
- abstaining, i.e. which can not be classified (because of different reasons, eg. "I do not know", "I am not sure", "I do not want to answer", "I am not satisfied with any of the options" etc.).

Voting can be a good example as the human voters may be divided into three groups of those who: vote for, vote against, abstain or giving invalid votes. Applications of intuitionistic fuzzy sets to group decision making, negotiations and other real situations are presented in (Szmidt and Kacprzyk [32, 33, 35, 39, 41, 42, 43, 45]).

The methods of assigning the membership and non-membership values for IFSs are proposed by Szmidt and Baldwin [31].

It is important that employing of intuitionistic fuzzy sets just forces an individual to consider both advantages (memberships) and disadvantages (non-memberships) of a considered solution. Next, the imprecise area is taken into account as well. The importance of such an approach lies in the fact that most people concentrate usually on one or two “most visible” aspects of a problem. They do not try to find out the contrary arguments or to consider uncertain (in wide sense, i.e. not restricted to randomness) aspects of a situation (cf. Sutherland [27]). Intuitionistic fuzzy sets with their structure make us consider a situation/problem more properly. We refer again an interested reader to (Szmidt and Kacprzyk [32, 33, 35, 39, 41, 42, 43, 45]) where we exploit this fact - using intuitionistic fuzzy sets to group decision making. In short, the problem boils down to selecting an option or a set of options which are best accepted by most of the individuals. The options are considered in pairs. Employing intuitionistic fuzzy sets forces each individual to look at each pair (i,j) of the options considering: advantages of the first option over the second one (membership function), disadvantages of the first option over the second one (non-membership function), and taking into account lack of knowledge (intuitionistic fuzzy index) as far as the two options are concerned. In other words, intuitionistic fuzzy sets force a user to explore a problem from different points of view – including all important aspects which should be taken into account but, unfortunately, are often omitted by people making decisions. This fact, strongly connected with a phenomenon called by the Nobel Prize winner Kahneman (cf. Kahneman [22]) ”bounded rationality”, caused among others by framing effect (explained in terms of salience and anchoring playing a central role in treatments of judgements and choice) places intuitionistic fuzzy sets among the up-to-date means of knowledge representation and processing.

Example 1 Let us assume that we have a set X of n individuals who vote for/against building of nuclear power plant (judges voting for/against acquittal, electors voting for/against a given candidate or his opponent, consumers expressing/not expressing interest in buying a product). Let us assume that each individual x_i belongs to

- a set of individuals (judges, electors) voting for – to the extent $\mu(x_i)$
- a set of individuals voting against – to the extent $\nu(x_i)$

It is worth noticing that by means of the fuzzy set theory we cannot consider the situation in more details. By means of intuitionistic fuzzy set theory we can also point out

- a set of individuals who did not answer neither “yes” nor “no”– to the extent $\pi(x_i)$ whereas: $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$; $\pi(x_i)$ – an intuitionistic fuzzy index.

From the point of view of e.g. market analysts (election committees) it could be tempting to assess the above data in terms of the possible final results of voting giving intervals containing

- probability of voting for

$$Pr_{for} \in [\mu, \mu + \pi]$$

where:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} \mu\left(x_i\right)$$

$$\pi = \frac{1}{n} \sum_{i=1}^{n} \pi\left(x_i\right)$$

- probability of voting against

$$Pr_{against} \in [\nu, \nu + \pi]$$

where:

$$\nu = \frac{1}{n} \sum_{i=1}^{n} \nu(x_i)$$

with the condition $Pr_{for} + Pr_{against} = 1$.

In terms of mass assignment (see Section 3) we could say that necessary support for is equal to μ, necessary support against is equal to ν, whereas possible support for (the best possible result) is equal to $\mu + \pi$, possible support against (the worst possible result) is equal to $\nu + \pi$.

Remark

In the above example we made a simplifying assumption assigning a sing of equality to probabilities and memberships/non-memberships. This assumption is valid under the condition that each value of membership/non-membership occurs with the same probability for each x_i. In this paper, for the sake of simpler notation, we follow this assumption. However, in general, probabilities for intuitionistic fuzzy sets are calculated in the following way (Szmidt and Kacprzyk [36, 37]):

Definition 3 *Let us assign to every element of an intuitionistic fuzzy event $A \subset E = \{x_1, ..., x_n\}$ (where E is the elementary event space) its probability of occurrence, i.e. $p(x_1), ..., p(x_n)$.*
Minimal probability $p_{\min}(A)$ of an intuitionistic fuzzy event A is equal to

$$p_{\min}(A) = \sum_{i=1}^{n} p(x_i)\mu(x_i)$$

Maximal probability of an intuitionistic fuzzy event A is equal to

$$p_{\max}(A) = p_{min}(A) + \sum_{i=1}^{n} p(x_i)\pi(x_i)$$

so probability of an event A is a number from the interval $[p_{min}(A), p_{max}(A)]$, or

$$p(A) \in [\sum_{i=1}^{n} p_A(x_i)\mu_A(x_i), \sum_{i=1}^{n} p_A(x_i)\mu_A(x_i) + \sum_{i=1}^{n} p_A(x_i)\pi_A(x_i)] \tag{6}$$

and probability of a complement event A^C is a number from the interval $[p_{min}(A^C), p_{max}(A^C)]$, or

$$p(A^C) \in [\sum_{i=1}^{n} p_A(x_i)\nu_A(x_i), \sum_{i=1}^{n} p_A(x_i)\nu_A(x_i) + \sum_{i=1}^{n} p_A(x_i)\pi_A(x_i)] \tag{7}$$

3 Mass Assignment Theory

The theory of mass assignment has been developed by Baldwin (Baldwin [8], Baldwin et al. [11, 12]) to provide a formal framework for manipulating both probabilistic and fuzzy uncertainty.

A fuzzy set can be converted into a mass assignment (Baldwin [6]). This mass assignment represents a family of probability distributions.

Definition 4 *Let $A^{'}$ be a normalized fuzzy set in $X = \{x\}$ such that*

$$A^{'} = \sum_{x_i \in X} x_i / \mu(x_i)$$

$$\mu(x_1) = 1,\ \mu(x_i) \leq \mu(x_j) \quad for \quad i > j$$

where $\mu(x)$ is the membership function.
The mass assignment associated with $A^{'}$ is (Baldwin [9])

$$\{x_1\} : 1 - \mu(x_2), \quad \{x_1, ..., x_i\} : \mu(x_i) - \mu(x_{i+1}) \quad for \quad i = 2, ...; \tag{8}$$

with $\mu(x_k) = 0 \quad for \quad x_k \notin X$

Example 2 (Baldwin [9])
Let $X = \{x_1,\ x_2,\ x_3,\ x_4\}$
If $A^{'} = x_1/1 + x_2/0.7 + x_3/0.4 + x_4/0.3$
then the associated mass assignment is
$m_f = x_1 : 0.3, \quad \{x_1,\ x_2\} = 0.3, \quad \{x_1,\ x_2,\ x_3\} = 0.1, \quad \{x_1,\ x_2,\ x_3,\ x_4\} = 0.3$

Support Pairs (the basic representation of uncertainty in the language FRIL [Baldwin at al. [11, 15]) are associated with mass assignments and represent an interval containing an unknown probability. Support Pairs are used to characterize uncertainty in facts and conditional probabilities in rules. A Support Pair (n, p) comprises a necessary and possible support and can be interpreted as an interval in which the unknown probability lies. A voting interpretation is also useful (Baldwin and Pilsworth [10]): the lower (necessary) support n represents the proportions of a sample population voting in favour of a proposition, whereas $(1 - p)$ represents the proportion voting against; $(p - n)$ represents the proportion abstaining.

For intuitionistic fuzzy sets (cf. Section 2) we have

- the proportion of a sample population voting in favour of a proposition is equal to μ (membership function),
- the proportion voting against is equal to ν (non-membership function),
- π represents the proportion abstaining.

Table 1: Equality of the the parameters for Baldwin's voting model and IFS voting model

	Baldwin's voting model	IFS voting model
voting in favour	n	μ
voting against	$1-p$	ν
abstaining	$p-n$	π

In Table 1 equality of parameters from Baldwin's voting model and from intuitionistic fuzzy set (IFS) voting model is presented.

So we can represent a Support Pair (n, p) using notation of intuitionistic fuzzy sets in the following way

$$(n, p) = (n, n + p - n) = (\mu, \mu + \pi) \tag{9}$$

i.e.: a Support Pair in Baldwin's voting model can be expressed by using notation of intuitionistic fuzzy sets.

It should be noted as well that the necessary support for the statement not being true is one minus the possibility of the support for the statement being true, i.e. $1 - p$. Similarly, the possible support for the statement being not true is one minus the necessary support for the statement being true i.e. $1 - n$. Taking into account the counterparts of the parameters, we can express this fact using notation of intuitionistic fuzzy sets as

$$(1 - p, 1 - n) = (\nu, \nu + \pi)$$

Let us look at three Support Pairs (n, p) of special interests (Baldwin and Pilsworth [10])

- (1, 1) which represents total support for the associated statement,
- (0, 0) which represents total support against and
- (0, 1) which characterizes complete uncertainty in the support.

Of course the above Support Pairs have exactly the same meaning in intuitionistic fuzzy set model (under the assumption that we consider probabilities for intuitionistic fuzzy memberships/non-memberships as it was explained in Section 2):

- (1, 1) means that $\mu = 1$ and $\pi = 0$, i.e. total support,
- (0, 0) means $\mu = 0$ and $\pi = 0$ what involves $\nu = 1$, i.e. total support against,
- (0, 1) means $\mu = 0$ and $\pi = 1$ i.e.: complete uncertainty in the support.

In other words both Support Pairs and intuitionistic fuzzy set models give the same intervals containing the probability of the fact being true, and the difference between the upper and lower values of intervals is a measure of the uncertainty associated with the fact.

The mass assignment structure is best used to represent knowledge that is statistically based such that the values can be measured, even if the measurements themselves are approximate or uncertain (Baldwin [13]).

The above considerations pointed out some parallels between mass assignment theory and intuitionistic fuzzy set theory. The parallels make it possible to propose a common geometrical representation useful when explaining advantages of the new similarity measures.

4 Geometrical representation

Having in mind that the parameters characteristic for intuitionistic fuzzy sets add up to one, i.e.

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$$

and the same for their counterparts for mass assignment theory, i.e.

$$n + (1 - p) + (p - n) = 1$$

and each of the parameters is from interval [0, 1], we can imagine a unit cube (Figure 1 inside which there is ABD triangle where the above equations are fulfilled. In other words, ABD triangle represents a surface where coordinates of any element belonging to an intuitionistic fuzzy set or representing any Support Pair can be represented. Each point belonging to ABD triangle is described via three coordinates: $(\mu, \nu, \pi) = (n,\ 1-p,\ p-n)$ – respectively for intuitionistic fuzzy set theory and mass assignment theory. Points A and B represent crisp elements. Point $A(1, 0, 0)$ – represents elements fully belonging to an intuitionistic fuzzy set as $\mu = 1$ or equivalently, 100% population voting for (as $n = 1$). Point $B(0, 1, 0)$ represents elements fully not belonging to an intuitionistic fuzzy set as $\nu = 1$ or equivalently, 100% population voting against (as $1 - p = 1$). Point $D(0, 0, 1)$ represents elements about which we are not able to say if they belong or not belong to an intuitionistic fuzzy set (intuitionistic fuzzy index $\pi = 1$) or equivalently, the proportion abstaining $p - n = 1$. Segment AB (where $\pi = 0$) represents elements belonging to classical fuzzy sets ($\mu + \nu = 1$), or the situation when ($p - n = 0$) what means in terms of Mass Assignment that there is not uncertainty in the voting model.

Employing the above geometrical representation, we can calculate distances between any two intuitionistic fuzzy sets A and B containing n elements (see Szmidt [28], Szmidt and Kacprzyk [38]), e.g.

- the normalized Hamming distance:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + \\ &+ |\pi_A(x_i) - \pi_B(x_i)|). \end{aligned} \tag{10}$$

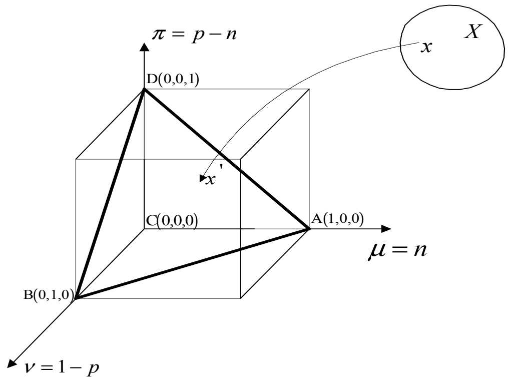

Figure 1: Geometrical representation

- the normalized Euclidean distance:

$$e_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}((\mu_A(x_i)-\mu_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2 + \\ + (\pi_A(x_i)-\pi_B(x_i))^2)^{\frac{1}{2}} \quad (11)$$

Both distances are from interval [0,1].

It is easy to give analogical formulas for m fuzzy sets or more complicated situations (e.g., m experts comparing n options in pairs – see [Szmidt and Kacprzyk [43, 45]).

It is easy to notice why all three parameters should be used when calculating distances. As the geometrical represntation shows (Figure 1), each side of the considered triangle is of the same length, i.e. $AB=BD=AD$. But when using two parameters only when calculating distances i.e., using the following formula

$$l_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|) \quad (12)$$

we obtain

$$l_{IFS}(A,B) = \frac{1}{2}(|1-0|+|0-1|) = 1 \quad (13)$$

$$l_{IFS}(A,D) = \frac{1}{2}(|1-0|+|0-0|) = \frac{1}{2} \quad (14)$$

$$l_{IFS}(B,D) = \frac{1}{2}(|0-0|+|1-0|) = \frac{1}{2} \quad (15)$$

so

$$l_{IFS}(A,B) \neq l_{IFS}(A,D) \text{ and } l_{IFS}(A,B) \neq l_{IFS}(B,D) \quad (16)$$

i.e., applying formula (12) means using two different scales: one scale for measuring distances for fuzzy sets (segment AB), and another one for "pure" intuitionistic fuzzy sets (for which intuitionistic fuzzy index is greater than zero - the whole area of the trangle

ABD over the segment AB). Only when using formula (10) with all three parameters we obtain

$$l_{IFS}(A,B) = \frac{1}{2}(|1-0| + |0-1| + |0-0|) = 1 \tag{17}$$

$$l_{IFS}(A,D) = \frac{1}{2}(|1-0| + |0-0| + |0-1|) = 1 \tag{18}$$

$$l_{IFS}(B,D) = \frac{1}{2}(|0-0| + |1-0| + |0-1|) = 1 \tag{19}$$

which means that the condition $l_{IFS}(A,D)$=$l_{IFS}(A,B)$=$l_{IFS}(B,D)$ is fulfilled, i.e. the distances for fuzzy sets and intuitionistic fuzzy sets are measured using the same scale.

In other words, when taking into account two parameters only, for elements from classical fuzzy sets (which are a special case of intuitionistic fuzzy sets – segment AB in Figure 1) we obtain distances from a different interval than for elements belonging to intuitionistic fuzzy sets. It practically makes it impossible to consider by the same formula the two types of sets as two different scales are used for measurements. For a deeper discussion of the problem we refer an interested reader to (Szmidt [28]), (Szmidt and Kacprzyk [38]).

Analogical explanations are valid in a case of calculating distances for Support Pairs. In effect distances are

- the normalized Hamming distance for two facts x_1, x_2 supported by respective support pairs: supported by respective support pairs:

$$\begin{aligned} l_{MASS}(x_1,x_2) &= |n(x_1) - n(x_2)| + |(1-p(x_1)) - (1-p(x_2))| + \\ &+ |(p - n(x_1)) - (p - n(x_2))| \end{aligned} \tag{20}$$

- the normalized Hamming distance for two sets of facts A and B expressed via support pairs:

$$\begin{aligned} l_{MASS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|n_A(x_i) - n_B(x_i)| + |(1-p_A(x_i)) - (1-p_B(x_i))| + \\ &+ |(p_A - n_A(x_i)) - (p_B - n_B(x_i))| \end{aligned} \tag{21}$$

- the normalized Euclidean distance for two sets of facts A and B expressed via support pairs:

$$\begin{aligned} q_{MASS}(A,B) &= (\frac{1}{2n}\sum_{i=1}^{n}((n_A(x_i) - n_B(x_i))^2 + ((1-p_A(x_i)) - (1-p_B(x_i)))^2 + \\ &+ ((p_A - n_A(x_i)) - (p_B - n_B(x_i)))^2)^{\frac{1}{2}} \end{aligned} \tag{22}$$

Example 3 We compare two pieces of equipment A and B taking into account three attributes: $x(1)$ – fault in software, $x(2)$ – fault in hardware, $x(3)$ – difficulties with repairs. The attributes are supported with the following support pairs (n, p)

$$\begin{array}{lll} x_A(1) = (0.6 \ \ 0.7) & x_A(2) = (0.9 \ \ 1) & x_A(3) = (0.3 \ \ 0.9) \\ x_B(1) = (0.8 \ \ 0.9) & x_B(2) = (0.5 \ \ 0.7) & x_B(3) = (0.7 \ \ 1) \end{array}$$

The simplest way to compare A and B is to calculate the distance between them. From (17) we have

$$\begin{aligned} l_{MASS}(A, B) &= \frac{1}{6}(|0.6-0.8| + |0.3-0.1| + |0.1-0.1| + |0.9-0.5| + |0-0.3| + |0.1-0.2| \\ &+ |0.3-0.7| + |0.1-0| + |0.6-0.3|) = \frac{1}{3} \end{aligned}$$

But comparing objects via pure distances between them seems not enough. Now we will give a new similarity measure and show its advantages.

5 Similarity measures

Making use of the geometrical representation (Section 4) we presented (Szmidt and Baldwin [29]) a similarity measures which could be used both in the frame of intuitionistic fuzzy set theory and mass assignment theory (when data is given in terms of Support Pairs).

We have stressed that

- the geometrical representation is based on the definition of an intuitionistic fuzzy set introduced by Atanassov [2], [5], and it does not introduce any additional assumptions; the same concerns mass assignment theory,
- any combination of the values of the membership, non-membership, and hesitation function $[(\mu, \nu, \pi)]$ which characterize an intuitionistic fuzzy element can be represented inside the triangle ABD (Figure 2); the same concerns combination of any possible values characterizing support pairs.

We proposed [29] a similarity measure (Definition 5) for intuitionistic fuzzy sets and their counterpart (Definition 6) for mass assignment theory. In the presented measures we made use of a concept of a complement element, and considered

- a distance between considered elements/objects X and F, and
- a distance between considered element/object X and the complement F^C of the considered element/object F

We have showed that to conclude about similarity of X and F, both types of distances have to be taken into account. In Section 5.1 we remind the similarity measures comparing the above two kinds of distances by their ratio. In Section 5.2 we discuss the whole family of the similarity measures comparing the above two kinds of distances by different functions.

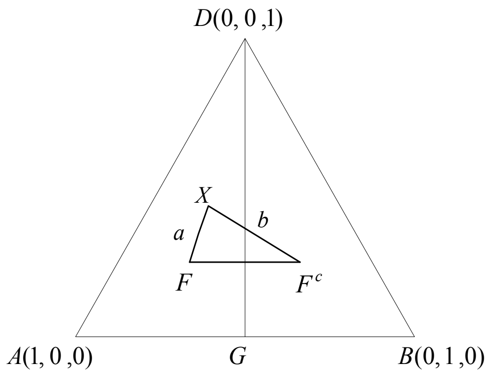

Figure 2: Triangle ABD explaining a ratio-based measure of similarity

5.1 The similarity measures presented in [29]

In the simplest situations we assess similarity of any two elements belonging to an intuitionistic fuzzy set (or sets) which are geometrically represented by points X and F (Figure 2) belonging to triangle ABD. The proposed measures indicate if X is more similar to F or to F^C, where F^C is a complement of F. In other words, the proposed measures answer the question if X is more similar or more dissimilar to F (Figure 2).

Definition 5

$$Sim(X,F) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F^C)} \tag{23}$$

where: $l_{IFS}(X,F)$ *is a distance from* $X(\mu_X,\nu_X,\pi_X)$ *to* $F(\mu_F,\nu_F,\pi_F)$,
$l_{IFS}(X,F^C)$ *is a distance from* $X(\mu_X,\nu_X,\pi_X)$ *to* $F^C(\nu_F,\mu_F,\pi_F)$,
F^C *is a complement of* F*, distances* $l_{IFS}(X,F)$ *and* $l_{IFS}(X,F^C)$ *are calculated from (10).*

For (23) we have

$$0 \leq Sim(X,F) \leq \infty \tag{24}$$

and

$$Sim(X,F) = Sim(F,X)$$

The similarity has typically been assumed to be symmetric. Tversky [46], however, has provided some empirical evidence that the similarity should not always be treated as a symmetric relation. We stress this to show that a similarity measure (23) may have some features which can be useful in some situations but are not welcome in others (see Cross and Sudkamp [18]), Wang and Kerre [47]).

It is obvious that the formula (23) can also be stated as

$$\begin{aligned} Sim(X,F) &= \frac{l_{IFS}(X,F)}{l_{IFS}(X,F^C)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X,F^C)} = \\ &= \frac{l_{IFS}(X,F)}{l_{IFS}(X^C,F)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X^C,F)} \end{aligned} \tag{25}$$

It is worth noticing that

- $Sim(X, F) = 0$ means the identity of X and F.
- $Sim(X, F) = 1$ means that X is to the same extent similar to F and F^C (i.e., values bigger than 1 mean in fact a closer similarity of X and F^C to X and F).
- When $X = F^C$ (or $X^C = F$), i.e. $l_{IFS}(X, F^C)$=$l_{IFS}(X^C, F)$= 0 means the complete dissimilarity of X and F (or in other words, the identity of X and F^C), and then $Sim(X, F) \to \infty$.
- When $X = F = F^C$ means the highest possible entropy (see [40]) for both elements F and X i.e. the highest "fuzziness" – not too constructive a case when looking for compatibility (both similarity and dissimilarity).

In other words, when applying measure (23) to analyse the similarity of two objects, one should be interested in the values $0 \leq Sim(X, F) < 1$.

The proposed measure (23) was constructed for selecting objects which are more similar than dissimilar [and well-defined in the sense of possessing (or not) attributes we are interested in].

Now we will show that a measure of similarity defined as mentioned above, (23), between $X(\mu_X, \nu_X, \pi_X)$ and $F(\mu_F, \nu_F, \pi_F)$ is more powerful then a simple distance between them.

Example 4 Let X and F be the geometrical representation of two elements belonging to an intuitionistic fuzzy set (with the coordinates (μ, ν, π)),

$$X = (0.5, 0.4, 0.1)$$

$$F = (0.4, 0.5, 0.1)$$

so

$$F^C = (0.5, 0.4, 0.1)$$

and from (10) we have

$$l_{IFS}(X, F) = \frac{1}{2}(|0.5 - 0.4| + |0.4 - 0.5| + |0.1 - 0.1|) = 0.1 \qquad (26)$$

what means that the distance is small, and just taking this into account, we would say that X and F are similar. However

$$l_{IFS}(X, F^C) = \frac{1}{2}(|0.5 - 0.5| + |0.4 - 0.4| + |0.1 - 0.1|) = 0 \qquad (27)$$

which means that X is just the same as F^C. We cannot speak at all about the similarity of X and F despite the fact that the distance between them is small. $\square$

Having in mind the parallels between intuitionistic fuzzy sets and mass assignment theory, the counterpart formula for similarity of two facts/elements supported by support pairs is

Definition 6

$$Sim(X,F) = \frac{l_{MASS}(X,F)}{l_{MASS}(X,F^C)} \tag{28}$$

where: $l_{MASS}(X,F)$ *is a distance from* $X(n,p)$ *to* $F(n,p)$*,*
$l_{MASS}(X,F^C)$ *is a distance from* $X(n,p)$ *to* $F^C(1-p,\ n)$*,*
F^C *is a complement of* F*, distances* $l_{MASS}(X,F)$ *and* $l_{MASS}(X,F^C)$ *are calculated from (20).*

Now we will show that the proposed measure of similarity (28) between $X(n,p)$ and $F(n,p)$ is more powerful then a simple distance between them.

Example 5 Suppose that we want to find out if the preference of individual k^1 given as support pair (n,p) $X = (0.2, 0.8)$ is more similar to the preference of individual k^2 equal to $F^1 = (0.3, 0.6)$, or to the preference of individual k^3 given as $F^2 = (0.1, 0.4)$. The distances (21) between preferences are equal to

$$l_{MASS}(X,F^1) \;=\; \frac{1}{2}(|0.2-0.3| + |0.2-0.4| + |0.6-0.3|) = 0.3$$

$$l_{MASS}(X,F^2) \;=\; \frac{1}{2}(|0.2-0.1| + |0.2-0.6| + |0.6-0.3|) = 0.4$$

As $l_{MASS}(X,F^1)$ is less than $l_{MASS}(X,F^2)$ we could come to the conclusion that individuals k^1 and k^2 agree more than k^1 and k^3 But let us calculate similarity (28) between the pairs of preferences. As

$$F^{1,C} = (0.4, 0.3)$$
$$F^{2,C} = (0.6, 0.1)$$

so

$$l_{MASS}(X,F^{1,C}) = 0.3$$
$$l_{MASS}(X,F^{2,C}) = 0.4$$

and from (28) we have

$$Sim(X,F^1) = l_{MASS}(X,F^1)/l_{MASS}(X,F^{1,C}) = 1$$

$$Sim(X,F^2) = l_{MASS}(X,F^2)/l_{MASS}(X,F^{2,C}) = 1$$

so in fact in both cases similarity is the same, very weak in fact (agreement between k^1 and k^2 is very weak as X is similar to the same extent to F^1 and $F^{1,C}$; the same concerns k^1 and k^3.

Let us summarize the properties of the proposed measures (23) and (28). Their main advantage (as compared to other measures taking into account only the distance between the compared elements/objects) consists in the deeper insight into the nature of similarity – here similarity and dissimilarity are compared and a user is informed if the objects are more similar or more dissimilar. If the objects are similar, the values of (23) and (28) are in interval $[0, 1]$. For dissimilar objects both measures increase quickly what may be quite a transparent tip that we are outside the region of similarity (even if the distance between compared objects is small - see Examples 4 and 6 above). On the other hand, we constructed both measures in such a way that for identical elements/objects the measures are equal 0, and for dissimilar objects the measures are equal to 1 whereas the tradition is that a similarity measure is equal 1 for identical objects and 0 for dissimilar objects. In other words, the idea behind the proposed measures (23) and (28) is fully justified but to be in agreement with the values usually reflecting similarity as 1 and dissimilarity as 0, we present in Section 5.2 other similarity measures which preserve the advantages of (23) and (28) but the same time their values are normalized and consistent with the common scientific tradition.

5.2 The normalized similarity measures

Our main idea when constructing the new similarity measures is to use the same two kinds of distances as in (23) (i.e., $l_{IFS}(X, F)$, $l_{IFS}(X, F^C)$) and (28) (i.e., $l_{MASS}(X, F)$, $l_{MASS}(X, F^C)$). But now we look for such a function of the ("weighted") two types of distances which values are from interval $[0, 1]$. The following functions fulfill our demands

$$f(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F) + l_{IFS}(X,F^C)} \tag{29}$$

$$f(l_{MASS}(X,F), l_{MASS}(X,F^C)) = \frac{l_{MASS}(X,F)}{l_{MASS}(X,F) + l_{MASS}(X,F^C)} \tag{30}$$

for intuitionistic fuzzy sets and mass assignments, respectively.

The above functions are constructed under condition that we exlude from our considerations the case when $X = F = F^C$. Albeit such a situation is theoretically possible, practically it is not interesting. The assumption $X = F = F^C$ means that we try to compare an element X about we know nothing to another element about we know nothing $F = F^C$ (in terms of the geometrical representation – Figure 2 – it means that X, F and F^C are at the same point on DG segment). So we exclude from our considerations the cases: $l_{IFS}(X, F) = l_{IFS}(X, F^C) = 0$ or $l_{MASS}(X, F) = l_{MASS}(X, F^C) = 0$. Other cases are presented in Table 2.

This way we have constructed the functions which take into account the same two distances like the previous measures (23) and (28) but now the new measures are normalized (their values are in $[0, 1]$). It is obvious (see Table 2) that (29) and (30) are dual concepts to a similarity measure (if (29) and (30) are equal to zero then similarity is equal to 1; if (29) and (30) are equal to 1 then similarity is equal to zero, and so on). In other words, we may use (29) and (30) to construct a similarity measure. As

$$0 \leq f(l_{IFS}(X,F), l_{IFS}(X,F^C)) \leq 1 \tag{31}$$

we would like to find such a monotone decreasing function g that:

$$g(1) \leq g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) \leq g(0) \tag{32}$$

Table 2: the possible values of (29) and (30), a,b$\in (0,1)$

$l_{IFS}(X,F)$		$l_{IFS}(X,F^C)$	f
0		1	0
1		0	1
1		1	0.5
a	less than	b	a/(a+b)<0.5
a	bigger than	b	b/(a+b)>0.5
a	equal to	b	0.5

which means

$$0 \leq g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1) \leq g(0) - g(1) \tag{33}$$

$$0 \leq \frac{g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \leq 1 \tag{34}$$

This way we obtained a function having the properties of a similarity measure in a sense that it is monotone decreasing function of (29) and (30).

Definition 7

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \tag{35}$$

where $(f(l_{IFS}(X,F), l_{IFS}(X,F^C))$ *is given by (29)*

and

Definition 8

$$Sim(l_{MASS}(X,F), l_{MASS}(X,F^C)) = \frac{g(f(l_{MASS}(X,F), l_{MASS}(X,F^C))) - g(1)}{g(0) - g(1)} \tag{36}$$

where $(f(l_{MASS}(X,F), l_{MASS}(X,F^C))$ *is given by (30)*

The simplest function g which may be applied in both definitions is

$$g(x) = 1 - x \tag{37}$$

which gives from (35)

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = 1 - f(l_{IFS}(X,F), l_{IFS}(X,F^C)) \tag{38}$$

and from (8)

$$Sim(l_{MASS}(X,F), l_{MASS}(X,F^C)) = 1 - f(l_{MASS}(X,F), l_{MASS}(X,F^C)) \tag{39}$$

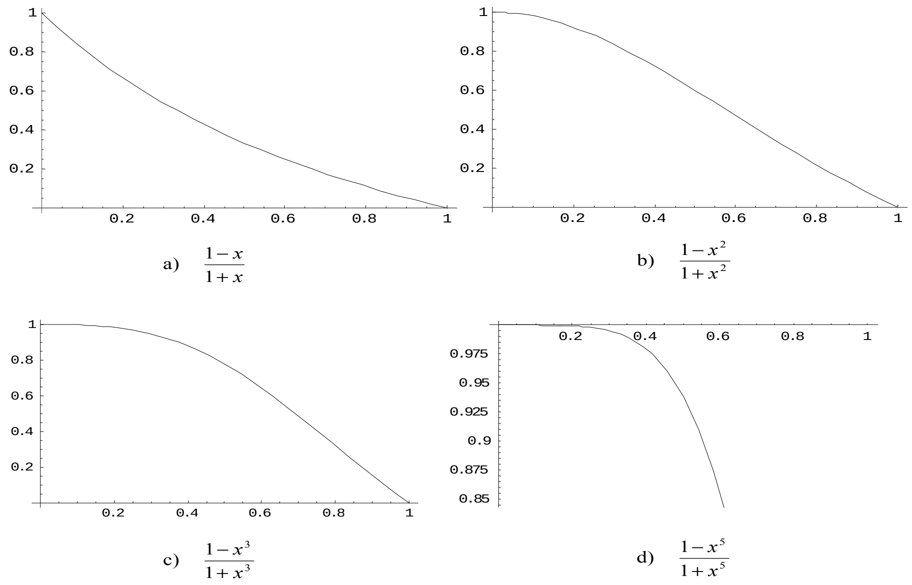

Figure 3: The shapes of $\frac{1-x^n}{1+x^n}$

Another function could be

$$g(x) = \frac{1}{1+x} \tag{40}$$

giving respectively

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{1 - f(l_{IFS}(X,F), l_{IFS}(X,F^C))}{1 + f(l_{IFS}(X,F), l_{IFS}(X,F^C))} \tag{41}$$

$$Sim(l_{MASS}(X,F), l_{MASS}(X,F^C)) = \frac{1 - f(l_{MASS}(X,F), l_{MASS}(X,F^C))}{1 + f(l_{MASS}(X,F), l_{MASS}(X,F^C))} \tag{42}$$

Function

$$g(x) = \frac{1}{1+x^2} \tag{43}$$

gives

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{1 - f(l_{IFS}(X,F), l_{IFS}(X,F^C))^2}{1 + f(l_{IFS}(X,F), l_{IFS}(X,F^C))^2} \tag{44}$$

We could theoretically use as well $g(x) = \frac{1}{1+x^n}$ where $n = 3, 4, \ldots, k$ but the counterpart similarity measures $(\frac{1-x^n}{1+x^n})$ give the values which are less convenient to be compared for small values of x. This fact is illustarted in Figure 3 – the bigger n the flater the similarity measures $(\frac{1-x^n}{1+x^n})$ for smaller values of x. It means that formal fulfilling of some mathematical assumptions is necessary but may be not enough condition to use a measure.

Also the exponential function may be used (cf. Pal and Pal [23])

$$g(x) = e^{-x} \tag{45}$$

giving for our functions (29) and (30), respectively

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{e^{-f(l_{IFS}(X,F), l_{IFS}(X,F^C))} - e^{-1}}{1 - e^{-1}} \tag{46}$$

and

$$Sim(l_{MASS}(X,F), l_{MASS}(X,F^C)) = \frac{e^{-f(l_{MASS}(X,F), l_{MASS}(X,F^C))} - e^{-1}}{1 - e^{-1}} \tag{47}$$

It is obvious that one could continue generating more complicated functions g (being the decreasing functions of f) but it would not give any additional insight as far as similarity is concerned.

All the similarity measures introduced in this Section assess similarity of any two elements (X and F) belonging to an intuitionistic fuzzy set (or sets). A counterpart similarity measures for intuitionistic fuzzy sets A and B containing n elements each, are just the sum of the respective measures constructed for separate elements

$$Sim(A,B) = \frac{1}{n}\sum_{i=1}^{n} Sim(l_{MASS}(X_i,F_i), l_{MASS}(X_i,F_i^C)) \tag{48}$$

$$Sim(A,B) = \frac{1}{n}\sum_{i=1}^{n} Sim(l_{IFS}(X_i,F_i), l_{IFS}(X_i,F_i^C)) \tag{49}$$

Albeit in the formulas presented in this Section we used the normalized Hamming distance, it is possible to replace it by other kinds of distances.

6 Conclusions

Some similarities between mass assignment theory and intuitionistic fuzzy set theory have been reminded. Next, the similarity measures useful from the point of view of both theories were recalled. The measures take into account two kinds of distances - to the compared element/object and to its complement. It was shown on the examples that neglecting of the distance to the complement of an element/object leads to wrong conclusions about similarity. The measures of similarity have some connections to the Jaccard coefficient (cf. Szmidt and Kacprzyk [44]). The only weakness of the similarity measures is that they values do not follow the commonly assumed values for the similarity measures.

To avoid the above mentioned drawback, we introduced the whole array of the functions which preserve the advantages of the previously proposed similarity measures and the same time their values are as usually expected for the similarity measures.

References

[1] Atanassov K. (1983), Intuitionistic Fuzzy Sets. VII ITKR Session. Sofia (Deposed in Centr. Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] Atanassov K. (1986), Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 20, 87–96.

[3] Atanassov K. (1989), More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33, 37–46.

[4] Atanassov K. (1994), New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61, 137–142.

[5] Atanassov K. (1999), Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[6] Baldwin. J.F. (1991), Combining Evidences for Evidential Reasoning. International Journal of Intelligent Systems, 6, 569–616.

[7] Baldwin J.F. (1992a), A Calculus for mass Assignments in Evidential Reasoning. In Advances in the Dempster-Shafer Theory of Evidence. (ed. M. Fedrizzi, J. Kacprzyk, R.R. Yager), John Wiley, 513– 531.

[8] Baldwin J.F. (1992b), The Management of Fuzzy and Probabilistic Uncertainties for Knowledge Based Systems. In Encyclopaedia of AI (ed. S.A. Shapiro), John Wiley (2nd ed.) 528–537.

[9] Baldwin J.F. (1994), Mass assignments and fuzzy sets for fuzzy databases. In. Advances in the Dempster-Shafer theory of evidence. Ed. R. Yager at al. John Wiley, 577–594.

[10] Baldwin J.F., Pilsworth B.W. (1990), Semantic Unification with Fuzzy Concepts in Fril. IPMU'90, Paris.

[11] Baldwin J.F., T.P. Martin, B.W. Pilsworth (1995) FRIL – Fuzzy and Evidential Reasoning in Artificial Intelligence. John Wiley.

[12] Baldwin J.F., Lawry J., Martin T.P.(1995a), A Mass Assignment Theory of the Probability of Fuzzy Events. ITRC Report 229, University of Bristol, UK.

[13] Baldwin J.F., Coyne M.R., Martin T.P.(1995b), Intelligent Reasoning Using General Knowledge to Update Specific Information: A Database Approach. Journal of Intelligent Information Systems, 4, 281–304.

[14] Baldwin J.F., T.P. Martin T.P. (1995c), Extracting Knowledge from Incomplete Databases using the Fril Data Browser. EUFIT'95, Aachen, 111–115.

[15] Baldwin J.F., T.P. Martin (1996), FRIL as an Implementation Language for Fuzzy Information Systems. IPMU'96, Granada, 289–294.

[16] Bouchon-Meunier B., Rifgi M., and Bothorel S. (1996). General measures of comparison of objects. Fuzzy Sets and Systems, Vol. 84, No. 2, 143–153.

[17] Chen S., Yeh M. and Hsiao P. (1995). A comparison of similarity measures of fuzzy values. Fuzzy Sets and Systems, Vol. 72, No. 1, 79–89.

[18] Cross V. and Sudkamp T. (2002) Similarity and Compatibility in Fuzzy Set Theory. Assessment and Applications. (Series: Studies in Fuzziness and Soft Computing). Physica-Verlag.

[19] Dempster A.P. (1968) A generalization of Bayesian inference. J. of the Royal Statistical Society, Series B, 30, 205–247.

[20] Dengfeng L. and Chuntian C. (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognition Letters, Vol. 23, 221–225.

[21] Dubois D. and Prade H. (1982) A unifying view of comparison indices in a fuzzy set-theoretical framework. In R. Yager,editor, Fuzzy Set and Possibility Theory Recent Developments. Pergamon Press, New York, pp. 3–13.

[22] Kahneman D. (2002) Maps of bounded rationality: a perspective on intuitive judgment and choice. Nobel Prize Lecture, December 8, 2002.

[23] Pal N.R. and Pal S.K. (1991) Entropy: a new definition and its applications. IEEE Trans. on Systems, Man, and Cybernetics, Vol. 21, No. 5, 1260–1270.

[24] Pappis C., and Karacapilidis N. (1993). A comparative assessment of measures of similarity of fuzzy values. Fuzzy Sets and Systems, Vol. 56, 171–174.

[25] Shafer G. (1976) A Mathematical Theory of Evidence. Princeton Univ. Press.

[26] Sugeno M. (1974) Theory of Fuzzy Integrals and its Applications. Tokyo Institute of Technology, Japan.

[27] Sutherland S. (1994) Irrationality. The Enemy Within. Penguin Books.

[28] Szmidt E. (2000) Applications of Intuitionistic Fuzzy Sets in Decision Making. (D.Sc. dissertation) Techn. Univ., Sofia, 2000.

[29] Szmidt E. and Baldwin J. (2003) New Similarity Measure for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. Notes on IFSs, Vol. 9, No. 3, 60–76.

[30] Szmidt E. and Baldwin J. (2004) Entropy for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. Notes on IFSs, Vol. 10, No. 3, 15–28.

[31] Szmidt E. and Baldwin J. (2006) Intuitionistic Fuzzy Set Functions, Mass Assignment Theory, Possibility Theory and Histograms. 2006 IEEE World Congress on Computational Intelligence, 237–243.

[32] Szmidt E. and Kacprzyk J. (1996a) Intuitionistic fuzzy sets in group decision making. Notes on IFS, Vol. 2, 15–32.

[33] Szmidt E. and Kacprzyk J. (1996c) Remarks on some applications of intuitionistic fuzzy sets in decision making. Notes on IFS, Vol.2, No.3, 22–31.

[34] E. Szmidt and J. Kacprzyk (1998a) An intuitionistic fuzzy set corresponding to a fuzzy set. Int.Journal of Uncertainty Fuzziness and Knowledge Based Systems, Vol. 6, No. 5, 427–435.

[35] Szmidt E. and Kacprzyk J. (1998a) Group Decision Making under Intuitionistic Fuzzy Preference Relations. Proc. IPMU'98, 172–178.

[36] Szmidt E. and Kacprzyk J. (1999). Probability of Intuitionistic Fuzzy Events and their Applications in Decision Making. Proc. of EUSFLAT-ESTYLF Conf. 1999. Palma de Mallorca, 457–460.

[37] Szmidt E. and Kacprzyk J. (1999b) - A Concept of a Probability of an Intuitionistic Fuzzy Event. Proc. of FUZZ-IEEE'99 - 8th IEEE International Conference on Fuzzy Systems, August 22-25, 1999 Seoul, Korea, III 1346–1349.

[38] Szmidt E. and Kacprzyk J. (2000) Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 114, No. 3, 505–518.

[39] E.Szmidt and J. Kacprzyk (2000b) On Measures of Consensus Under Intuitionistic Fuzzy relations. Proc. IPMU'2000, Madrid, July 3-7, 641–647.

[40] E. Szmidt and J. Kacprzyk (2001) – Szmidt E., Kacprzyk J.: Entropy for intuitionistic fuzzy sets. Fuzzy Sets and Systems, vol. 118, No. 3, 2001, 467–477.

[41] E.Szmidt and J. Kacprzyk (2002a) – Analysis of Agreement in a Group of Experts via Distances Between Intuitionistic Fuzzy Preferences. Proc. IPMU'2002, Annecy, France, 1-5 July, 1859–1865.

[42] E.Szmidt and J. Kacprzyk (2002b) An Intuitionistic Fuzzy Set Based Approach to Intelligent Data Analysis (an application to medical diagnosis). In A. Abraham, L. Jain, J. Kacprzyk (Eds.): Recent Advances in Intelligent Paradigms and Applications. Springer-Verlag, 57–70.

[43] E.Szmidt and J. Kacprzyk (2002c) Evaluation of Agreement in a Group of Experts via Distances Between Intuitionistic Fuzzy Sets. Proc. IS'2002 – Int. IEEE Symposium: Intelligent Systems, Varna, Bulgaria, IEEE Catalog Number 02EX499, 166–170.

[44] Szmidt E. and Kacprzyk J. (2004) Similarity of intuitionistic fuzzy sets and the Jaccard coefficient. Proc. 10th Int. Conf. IPMU 2004, Perugia, Italy, 1405–1412.

[45] Szmidt E. and Kacprzyk J. (2005) A New Concept of a Similarity Measure for Intuitionistic Fuzzy Sets and its Use in Group Decision Making. In V. Torra, Y. Narukawa, S. Miyamoto (Eds.): Modelling Decisions for Artificial Intelligence. LNAI 3558, Springer 2005, 272–282.

[46] Tversky A. (1977) Features of similarity. Psychol. Rev. Vol. 84, 327–352.

[47] Wang X., De Baets B., and Kerre E. (1995). A comparative study of similarity measures. Fuzzy Sets and Systems, Vol.73, No. 2, 259–268.

[48] Yager R.R. (1979) - On the measure of fuziness and negation. Part I: Membership in the unit interval. Internat. J. of Gen. Systems, Vol. 5, 189–200.

[49] Yager R.R. (1982) Measuring tranguillity and anxiety in decision making: An application of fuzzy sets. Inter. J. of Gen. Systems, 8, 139–146.

[50] Yager R.R. (1990) Ordinal measures of specifity. Inter. J. of Gen. Systems, 17, 57–72.

[51] Yager R.R. (1991) Similarity based measures of specifity. Inter. J. of Gen. Systems, 19, 91–106.

[52] Zadeh L.A. (1965) Fuzzy sets. Information and Control, 8, 338–353.

[53] Zadeh L.A. (1978) Fuzzy Sets as the Basis for a Theory of Possibility. Fuzzy Sets and Systems, 1, 3–28.

[54] Zwick R., Carlstein E., Budescu D. (1987). Measures of similarity among fuzzy concepts: A comparative analysis. Int. J. of Approx. Reasoning, Vol. 1, 221–242.