11th Int. Workshop on IFSs, Banská Bystrica, Slovakia, 30 Oct. 2015
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 5, 16–19

On the Poincaré recurrence theorem on IF-sets

Jaroslav Považan

Faculty of Natural Sciences, Matej Bel University
Tajovského 40, Banská Bystrica, Slovakia
e-mail: jaroslav.povazan@umb.sk

Abstract: The Recurrence theorem by Poincaré is one of basic results of the standard ergodic theory. In classical sense the main structure is a σ-algebra of sets and the measure-preserving maps are represented by preimages of classical maps. In this article we change the σ-algebra by a family $\mathscr{F}$ of Intuitionistic Fuzzy Sets (IF-sets), which were introduced by Krassimir T. Atanassov, and the probability by an IF-state.

Keywords: recurrence theorem, IF-sets, s-preserving mappings, IF-state.
AMS Classification: 03E72.

1 Introduction

In a probability space $(\Omega, \mathscr{S}, P)$, we say that $T : \Omega \to \Omega$ is P-preserving, iff it holds

$$(\forall A \in \mathscr{S}) \left(P\big(T^{-1}(A)\big) = P(A) \right).$$

Let $(\Omega, \mathscr{S}, P)$ be a probability space, $T : \Omega \to \Omega$ a measure preserving map. Consider $A \in \mathscr{S}$. The classical Poincaré recurrence theorem tells that almost every point x of A returns to A infinitely many times. Hence to any $k \in \mathbb{N}$ there exists $n \geq k$ such that $T^n(x) \in A$, i.e. $x \in T^{-n}(A)$. Hence we have for any $k \in \mathbb{N}$

$$P\left(A \setminus \bigcup_{n=k}^{\infty} T^{-n}(A) \right) = 0.$$

In the article we are going to use IF-sets instead of sets, IF-states instead of a probability, and s-preserving maps instead of measure preserving maps.

IF-sets are ordered pairs of fuzzy sets. Elements of pair can be represented as measure of agreement and disagreement with some statement. Concept of IF-sets was introduced by K.T.

Atanassov in the monograph [1]. It is a good model for those statements or events, for which we cannot exactly set measure of truth.

I was inspired by Beloslav Riečan, who has shown and proved some variations of the recurrence theorem on some other structures in the article [9].

2 IF-sets

Definition 1. *Let Ω be a nonempty set. A mapping $A : \Omega \to [0,1]^2$ is called an IF-subset of the set Ω iff:*

$$(\forall x \in \Omega)\,(A(x) = (f(x), g(x)) \Rightarrow f(x) + g(x) \leq 1)\,.$$

On the system $\mathbb{IF}$ of all IF-subsets of the set Ω we define the partial ordering $\leq_{IF}$ by

$$(f,g) \leq_{IF} (h,k) \Leftrightarrow f \leq h \;\&\; g \geq k$$

and the operations $\oplus, \odot, \neg$ by

$$\begin{aligned}(f,g) \oplus (h,k) &= (\min(f+h, 1_\Omega), \max(g+k-1_\Omega, 0_\Omega))\\ (f,g) \odot (h,k) &= (\max(f+h-1_\Omega, 0_\Omega), \min(g+k, 1_\Omega))\\ \neg(f,g) &= (g,f)\,.\end{aligned}$$

Definition 2. *Let $\mathscr{F}$ be a system of IF-subsets of a nonempty set Ω closed under the operations $\oplus, \odot, \neg$. A mapping $m : \mathscr{F} \to [0,1]$ is called an IF-state iff:*

- $(\forall\,(0_\Omega, g) \in \mathscr{F})\,(m((0_\Omega, g)) = 0)\,,$
- $(\forall A, B \in \mathscr{F})\,(A \leq_{IF} B \Rightarrow m(A) \leq m(B))\,,$
- $(m(A_i \odot A_j) = 0 \text{ for } i \neq j) \Rightarrow \left((\forall n \in \mathbb{N})\left(\sum_{i=1}^{n} m(A_i) \leq 1\right)\right)$

Definition 3. *Let $\mathscr{F}$ be a system of IF-subsets of a nonempty set Ω closed under the operations $\oplus, \odot, \neg$. Let $m : \mathscr{F} \to \mathbb{R}$ be a function on $\mathscr{F}$. Let $T : \mathscr{F} \to \mathscr{F}$ be a mapping. Then T is called $m-$preserving iff:*

- $(\forall\,(0_\Omega, g) \in \mathscr{F})\,(T((0_\Omega, g)) = (0_\Omega, g))\,,$
- $(\forall A, B \in \mathscr{F})\,(A \leq_{IF} B \Rightarrow T(A) \leq_{IF} T(B))\,,$
- $(\forall A, B \in \mathscr{F})\,(T(A \odot B) = T(A) \odot T(B))\,,$
- $(\forall A \in \mathscr{F})\,(m(T(A)) = m(A))$

Theorem 1. *Let $\mathscr{F}$ be the system of IF-subsets of a nonempty set Ω closed under the operations $\oplus, \odot, \neg$. Let $m : \mathscr{F} \to [0,1]$ be an IF-state. Let $T : \mathscr{F} \to \mathscr{F}$ be an $m-$preserving mapping. Let $A \in \mathscr{F}$. Then for any $k \in \mathbb{N}$*

$$m\left(A \odot \bigwedge_{j=k}^{\infty} \bigodot_{i=k}^{j} T^i(\neg A)\right) = 0.$$

Proof. First, we show that for arbitrary $C \in \mathscr{F}$ there holds

$$m(C \odot \neg C) = 0.$$

Let $C = (f, g)$. Put

$$D = (f, g) \odot \neg(f, g) = (f, g) \odot (g, f) = (\max(f + g - 1_\Omega, 0_\Omega), \min(g + f, 1_\Omega)) .$$

But $C \in \mathscr{F}$, hence

$$D = (0_\Omega, f + g) \in \mathscr{F}, \text{ hence } m(D) = 0.$$

Put $B = A \odot \bigwedge_{n=1}^{\infty} \bigodot_{i=1}^{n} T^i(\neg A)$. Hence

$$B \leq_{IF} T^n(\neg A) \text{ and also } T^n(B) \leq_{IF} T^n(A).$$

From that for all positive integer n, there is

$$B \odot T^n(B) \leq_{IF} T^n(\neg A) \odot T^n(A) = T((0_\Omega, h)) = (0_\Omega, h) .$$

Hence

$$m(B \odot T^n(B)) = 0 \text{ and also } m\big(T^i(B) \odot T^j(B)\big) = 0 \text{ for } i \neq j.$$

From weak additivity of the function m and from the fact that T is $m-$preserving, we get

$$nm(B) = \sum_{i=1}^{n} m\big(T^i(B)\big) \leq 1.$$

From that necessarily $m(B) = 0$.
If we change T by T^k, we get

$$m\left(A \odot \bigwedge_{n=1}^{\infty} \bigodot_{i=1}^{n} T^{ik}(\neg A) \right) = 0.$$

But

$$A \odot \bigwedge_{n=1}^{\infty} \bigodot_{i=1}^{n} T^{ik}(\neg A) \geq A \odot \bigwedge_{n=k}^{\infty} \bigodot_{i=k}^{n} T^{i}(\neg A).$$

Hence

$$0 \leq m\left(A \odot \bigwedge_{n=k}^{\infty} \bigodot_{i=k}^{n} T^{i}(\neg A) \right) \leq m\left(A \odot \bigwedge_{n=1}^{\infty} \bigodot_{i=1}^{n} T^{ik}(\neg A) \right) = 0.$$

Hence the proof. □

3 Conclusions

This article showed variation of recurrence theorem in language of IF-sets. In [9] there was proved the theorem for any MV algebra. Any family of IF sets can be embedded to an MV algebra, of course usually $\neg(f, g) = (1 - f, 1 - g)$ what is not an element of IF sets. Therefore our version of the Poincaré recurrence theorem does not follow from the version stated in [9].

Acknowledgments

The support of the grant VEGA 1/0120/14 is kindly announced.

References

[1] Atanassov, K.T. (1999) *Intuitionistic Fuzzy Sets: Theory and Applications.* Springer, Heidelberg.

[2] Bennett, M. K. & D. J. Foulis (1994) Effect algebras and unsharp quantum logics. *Foundations of Physics*, 24, 1331–1352.

[3] Dvurečenskij, A. (1976) On some properties of transformation of a logic. *Math. Slovaca*, 26, 131–137.

[4] Chovanec, F. & F. Kôpka (1994) D-posets. *Math. Slovaca*, 44, 21–34.

[5] Kôpka, F. (2008) Quasiproduct on Boolean D-posets. *Int. J. Theor. Physics*, 47, 26–35.

[6] Maličký, P. (2007) Category version of the Poincaré recurrence theorem. *Topology Appl.*, 154, 2709–2713.

[7] Mundici, D. (2011) *Advanced Lukasiewicz calculus and MV-algebras*. Springer, New York.

[8] Poincaré, H. (1889) *Les methodes nouvelles de la mechanique classique celeste*, Vol. 3, Gauthiers-Villars, Paris.

[9] Riečan, B. (2013) Variation on a Poincaré theorem. *Fuzzy Sets and Systems*, 232, 39–45.

[10] Riečan, B. & D. Mundici (2002) Probability on MV-algebras. In: *Handbook of Measure Theory* (E. Pap. ed.), Elsevier Science, Amsterdam, 869– 909.

[11] Riečan, B. & T. Nebrunn (1997) *Integral, Measure, and Ordering*. Kluwer, Dordrecht.