Intuitionistic Fuzzy Interpretations of Conway's Game of Life

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In memory of our teacher in
informatics Prof. Peter Barnev

Abstract. Conway's Game of Life is a popular heuristic zero-player game, devised by John Horton Conway in 1970, and it is the best-known example of a cellular automaton. Its "universe" is an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, alive or dead. Every cell interacts with its eight neighbours, which are the cells that are directly horizontally, vertically, or diagonally adjacent. In a stepwise manner, the state of each cell in the grid preserves or alternates with respect to a given list of rules. Intuitionistic fuzzy sets (IFS) are an extension of Zadeh's fuzzy sets, which introduce a degree of membership and a degree of non-membership whose sum is equal to or less than 1 and the complement to 1 is called a degree of uncertainty. The article proposes an intuitionistic fuzzy estimation of the cells' state in a modified Game of Life. For each cell we can define its IF estimation as a pair consisting of the degrees l_p and l_a, namely degrees of presence and absence of life, where $l_p + l_a \leq 1$. In the classical Conway's Game of Life, the alive and dead states correspond to the elementary IF estimations $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. The article presents the formulas for calculating the IF state of liveliness of each cell, as functions of the current states of the cell's neighbours. Criteria of liveliness will be also determined in terms of IFS.

1 Introduction

Conway's Game of Life is devised by John Horton Conway in 1970, and already 40 years it is an object of research, software implementions and modifications. In [1] there is a list of many papers devoted to Conway's Game of Life. In 1976, the authors who were then students in Sofia University, also introduced one modification of this game. In the present paper another modification will be introduced. It is based on the idea of the intuitionistic fuzziness.

The standard Conway's Game of Life has a "universe" which is an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, alive or dead, or as we learned the game from the lectures of Prof. Barnev in the middle of 1970s, in the square there is an asterisk or not.

Every cell interacts with its eight neighbours, namely the cells that are directly adjacent either in horizontal, vertical, or diagonal direction. In a stepwise manner, the state of each cell in the grid preserves or alternates with respect to a given list of rules.

Here, we will discuss some versions of the game in which we will keep the condition for the necessary number of existing neighbours asterisks for birth or dying of an asterisk in some square. For our aims, we will use elements of intuitionistic fuzzy set theory (see, [2]).

2 Remarks on intuitionistic fuzzy sets and logic

Let us have some set of propositions S. To every proposition p from this set there are assigned real numbers $\mu(p)$ and $\nu(p)$, such that $\mu(p), \nu(p) \in [0,1]$ and

$$\mu(p) + \nu(p) \leq 1.$$

These numbers correspond to the "truth degree" and to the "falsity degree" of p.

Let this assignment be provided by an evaluation function V, defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Everywhere below, we shall assume that for the two variables x and y the equalities $V(p) = \langle a, b\rangle$, $V(q) = \langle c, d\rangle$ $(a, b, c, d, a+b, c+d \in [0,1])$ hold.

Obviously, when V is an ordinary fuzzy truth-value estimation, then

$$b = 1 - a.$$

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see, [2]) by:

$$p \text{ is an IFT if and only if } a \geq b,$$

while p will be a tautology iff $a = 1$ and $b = 0$.

In some definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

When values $V(p)$ and $V(q)$ of the propositional forms p and q are known, the evaluation function V can be extended also for the operations "conjunction" (two forms), "disjunction" (two forms), "implication" (about 140 different forms), "negation" (34 different forms) and others. Here, we will only need operation "negation" $(\neg)$, that for proposition p will have the forms given in Table 1.

Table 1

$\neg_1$	$\langle x, b, a\rangle$
$\neg_2$	$\langle x, \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle$
$\neg_3$	$\langle x, b, a.b + a^2\rangle$
$\neg_4$	$\langle x, b, 1 - b\rangle$
$\neg_5$	$\langle x, \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(1 - b)\rangle$
$\neg_6$	$\langle x, \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a)\rangle$
$\neg_7$	$\langle x, \overline{\mathrm{sg}}(1 - b), a\rangle$
$\neg_8$	$\langle x, 1 - a, a\rangle$
$\neg_9$	$\langle x, \overline{\mathrm{sg}}(a), a\rangle$
$\neg_{10}$	$\langle x, \overline{\mathrm{sg}}(1 - b), 1 - b\rangle$
$\neg_{11}$	$\langle x, \mathrm{sg}(b), \overline{\mathrm{sg}}(b)\rangle$
$\neg_{12}$	$\langle x, b.(b + a), a.(b^2 + a + b.a)\rangle$
$\neg_{13}$	$\langle x, \mathrm{sg}(1 - a), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{14}$	$\langle x, \mathrm{sg}(b), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{15}$	$\langle x, \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{16}$	$\langle x, \overline{\mathrm{sg}}(a), \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{17}$	$\langle x, \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(b)\rangle$
$\neg_{18}$	$\langle x, b.\mathrm{sg}(a), a.\mathrm{sg}(b)\rangle$
$\neg_{19}$	$\langle x, b.\mathrm{sg}(a), 0\rangle$
$\neg_{20}$	$\langle x, b, 0\rangle$
$\neg_{21}$	$\langle x, \min(1 - a, \mathrm{sg}(a)), \min(a, \mathrm{sg}(1 - a))\rangle$
$\neg_{22}$	$\langle x, \min(1 - a, \mathrm{sg}(a)), 0\rangle$
$\neg_{23}$	$\langle x, 1 - a, 0\rangle$
$\neg_{24}$	$\langle x, \min(b, \mathrm{sg}(1 - b)), \min(1 - b, \mathrm{sg}(b))\rangle$
$\neg_{25}$	$\langle x, \min(b, \mathrm{sg}(1 - b)), 0\rangle$
$\neg_{26}$	$\langle x, b, a.b + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{27}$	$\langle x, 1 - a, a.(1 - a) + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{28}$	$\langle x, b, (1 - b).b + \overline{\mathrm{sg}}(b)\rangle$
$\neg_{29}$	$\langle x, a.b + \overline{\mathrm{sg}}(1 - b), a.(a.b + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{30}$	$\langle x, a.b, a.(a.b + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{31}$	$\langle x, (1 - a).a + \overline{\mathrm{sg}}(a), a.((1 - a).a + \overline{\mathrm{sg}}(a)) + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{32}$	$\langle x, (1 - a).a, a.((1 - a).a + \overline{\mathrm{sg}}(a)) + \overline{\mathrm{sg}}(1 - a)\rangle$
$\neg_{33}$	$\langle x, b.(1 - b) + \overline{\mathrm{sg}}(1 - b), (1 - b).(b.(1 - b) + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(b))\rangle$
$\neg_{34}$	$\langle x, b.(1 - b), (1 - b).(b.(1 - b) + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(b)\rangle$

3 Intuitionistic fuzzy criteria of existence, birth and death of an asterisk

Let us have a plane tesselated with squares. Let in some of these squares there be symbols "*", meaning that the squares are "alive". Now we will extend this construction of the Game of Life to some new forms.

Let us assume that the square $\langle i, j\rangle$ is assigned a pair of real numbers $\langle \mu_{i,j}, \nu_{i,j}\rangle$, so that $\mu_{i,j} + \nu_{i,j} \leq 1$. We can call the numbers $\mu_{i,j}$ and $\nu_{i,j}$ degree of existence and degree of non-existence of symbol "*" in square $\langle i, j\rangle$. Therefore, $\pi(i, j) = 1 - \mu_{i,j} - \nu_{i,j} \leq 1$ will correspond to the degree of uncertainty, e.g., lack of information about existence of an asterisk in the respective square.

Below we will formulate a series of different criteria for correctness of the intuitionistic fuzzy interpretations that will include as a particular case the standard game.

3.1 Six criteria of existence of an asterisk

We will suppose that there exists an asterisk in square $\langle i, j\rangle$ if:

- (**1.1**) $\mu_{i,j} > 0.5$. Therefore $\nu_{i,j} < 0.5$. In the particular case, when $\mu_{i,j} = 1 > 0.5$ we obtain $\nu_{i,j} = 0 < 0.5$, i.e., the standard existence of the asterisk.
- (**1.2**) $\mu_{i,j} \geq 0.5$. Therefore $\nu_{i,j} \leq 0.5$. Obviously, if case (1.1) is valid, then case (1.2) also will be valid.
- (**1.3**) $\mu_{i,j} > \nu_{i,j}$. Obviously, case (1.1) is particular case of the present one, but case (1.2) is not included in the currently discussed case for $\mu_{i,j} = 0.5 = \nu_{i,j}$.
- (**1.4**) $\mu_{i,j} \geq \nu_{i,j}$. Obviously, cases (1.1), (1.2) and (1.3) are particular cases of the present one.
- (**1.5**) $\mu_{i,j} > 0$. Obviously, cases (1.1), (1.2) and (1.3) are particular cases of the present one, but case (1.4) is not included in the currently discussed case for $\mu_{i,j} = 0.0 = \nu_{i,j}$.
- (**1.6**) $\nu_{i,j} < 1$. Obviously, cases (1.1), (1.2) and (1.3) are particular cases of the present one, but case (1.5) is not included in the currently discussed case for $\mu_{i,j} = 0.0$.

From these criteria it follows that if one is valid – let it be the s-th criterion $(1 \leq s \leq 6)$ then we can assert that the asterisk exists with respect to the s-th criterion and, therefore, it will exist with respect to all other criteria, whose validity follows from the validity of the s-th criterion.

On the other hand, if s-th criterion is not valid, then we will say that the asterisk does not exist with respect to s-th criterion. It is very important that in this case the square may not be absolutely empty. It is appropriate to tell that the square $\langle i, j\rangle$ is totally empty, if its degrees of existence and non-existence are $\langle 0, 1\rangle$.

It is suitable to tell that the square is s-full if it contains an asterisk with respect to the s-th criterion and that the same square is s-empty if it does not satisfy the s-th criterion.

For the aims of the game-method for modelling, it will be suitable to use (with respect to the type of the concrete model) one of the first four criteria for existence of an asterisk. Let us say for each fixed square $\langle i, j\rangle$ that therein is an asterisk by s-th criterion for $1 \leq s \leq 4$, if this criterion confirms the existence of an asterisk.

3.2 Four criteria for the birth of an asterisk

In the standard game, the rule for birth of a new asterisk is: the (empty) square has exactly 2 or 3 neighbouring squares containing asterisks. Now we will formulate a series of different rules that will include as a particular case the standard rule.

- **2.1** (extended standard rule): The s-empty square has exactly 2 or 3 neighbouring s-full squares. Obviously, this rule for birth is a direct extension of the standard rule.
- **2.2** (pessimistic rule): For the natural number $s \geq 2$, the s-empty square has exactly 2 or 3 neighbouring $(s-1)$-full squares.
- **2.3** (optimistic rule): For the natural number $s \leq 5$, the s-empty square has exactly 2 or 3 neighbouring $(s+1)$-full squares.
- **2.4** (average rule): Let $M_{i,j}$ and $N_{i,j}$ be, respectively, the sums of the μ-degrees and of the ν-degrees of all neighbours of the s-empty square. Then the inequality

  $$\frac{1}{4}.N_{i,j} \leq M_{i,j} \leq \frac{3}{8}.N_{i,j}$$

  holds.

3.3 Four criteria for the death of an asterisk

In the standard game the rule for the death of an existing asterisk is: the (full) square has exactly 2 or 3 neighboard squares containing asterisks. Now we will formulate a series of different rules that will include as a particular case the standard rule.

- **3.1** (extended standard rule): The s-full square has less than 2 or more than 3 neighboring s-full squares. Obviously, this rule for dying is a direct extension of the standard rule.
- **3.2** (pessimistic rule): For the natural number $s \geq 2$, the s-full square has less than 2 or more than 3 neighboring $(s-1)$-full squares.
- **3.3** (optimistic rule): For the natural number $s \leq 5$, the s-full square has less than 2 or more than 3 neighboring $(s+1)$-full squares.
- **3.4** (average rule): Let $M_{i,j}$ and $N_{i,j}$ be, respectively, the sums of the μ-degrees and of the ν-degrees of all neighbours of the s-full square. Then one of the inequalities

  $$\frac{1}{4}.N_{i,j} > M_{i,j} \quad \text{or} \quad M_{i,j} > \frac{3}{8}.N_{i,j}$$

  holds.

4 Intuitionistic fuzzy rules for changing of the game-field

In the standard game the game-field is changed by the above mentioned rules for birth and death of the asterisks. Now, we will discuss some intuitionistic fuzzy rules for changing of the game-field. They use the separate forms of operation "negation".

Let us suppose that in a fixed square there is an asterisk if and only if the square is s-full. Therefore, we tell that in the square there is no asterisk if and only if the square is not s-full. In this case we can call this square s-empty.

As we saw above, the difference between standard and intuitionistic fuzzy form of the game is the existence of values corresponding to the separate squares. In the standard case they are 1 or 0, or "there exists an asterisk", "there is no asterisk". In the intuitionistic fuzzy form of the game we have pairs of real numbers as in the case when the asterisk exists, as well as in the opposite case. In the classical case, the change of the status of the square is obvious. In the intuitionistic fuzzy we can construct different rules. They are of two types.

The first type contains two modifications of the standard rule:

- **4.1** (extended standard rule): If an s-full square $\langle i, j\rangle$ must be changed, then we can use negation $\neg_1$ for pair $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and in a result we will obtain pair $\langle \nu_{i,j}, \mu_{i,j}\rangle$.
- **4.2** (non-standard, or intuitionistic fuzzy rule): If an s-full square $\langle i, j\rangle$ must be changed, then we can use any of the other negations $\neg_m$ from Table 1 ($2 \leq m \leq 34$).

The second type contains three non-standard modifications. The standard rule and the above two rules for changing of the current content of the fixed square (existence or absence of an asterisk) are related only to this content. Now, we can include a new parameter, that conditionally can be called *"the influence of the environment"*.

- **5.1** (optimistic (s, m)-rule) If an s-full/empty square $\langle i, j\rangle$ must be changed, then we can use m-th negation $\neg_m$ to pair (before change) $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and to juxtapose to it the pair $\langle \mu^*_{i,j}, \nu^*_{i,j}\rangle$, so that

$$\mu^*_{i,j} = \max(\mu'_{i,j}, \overset{*}{\max_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle}} \mu_{u,v})$$

$$\nu^*_{i,j} = \min(\nu'_{i,j}, \overset{*}{\min_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle}} \nu_{u,v}),$$

where

$$\langle \mu'_{i,j}, \nu'_{i,j}\rangle = \neg_m\langle \mu_{i,j}, \nu_{i,j}\rangle$$

and $\max^*, \min^*$ mean that we use only values that are connected to s-empty/full squares.

- **5.2** (optimistic-average (s,m)-rule) If an s-full/empty square $\langle i,j\rangle$ must be changed, then we can use m-th negation $\neg_m$ to pair (before change) $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and to juxtapose to it the pair $\langle \mu^*_{i,j}, \nu^*_{i,j}\rangle$, so that

$$\mu^*_{i,j} = \max(\mu'_{i,j}, \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \mu_{u,v})$$

$$\nu^*_{i,j} = \min(\nu'_{i,j}, \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \nu_{u,v}),$$

where $\langle \mu'_{i,j}, \nu'_{i,j}\rangle$ is as above, $\sum^*$ mean that we use only values that are connected to s-empty/full squares and $t(i,j)$ is the number of these squares.
- **5.3** (average (s,m)-rule) If an s-full/empty square $\langle i,j\rangle$ must be changed, then we can use m-th negation $\neg_m$ to pair (before change) $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and to juxtapose to it the pair $\langle \mu^*_{i,j}, \nu^*_{i,j}\rangle$, so that

$$\mu^*_{i,j} = \frac{1}{2}(\mu'_{i,j} + \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \mu_{u,v})$$

$$\nu^*_{i,j} = \frac{1}{2}(\nu'_{i,j} + \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \nu_{u,v}),$$

where $\langle \mu'_{i,j}, \nu'_{i,j}\rangle$, $\sum^*$ and $t(i,j)$ are as in 5.1 and 5.2.
- **5.4** (pessimistic-average (s,m)-rule) If an s-full/empty square $\langle i,j\rangle$ must be changed, then we can use m-th negation $\neg_m$ to pair (before change) $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and to juxtapose to it the pair $\langle \mu^*_{i,j}, \nu^*_{i,j}\rangle$, so that

$$\mu^*_{i,j} = \min(\mu'_{i,j}, \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \mu_{u,v})$$

$$\nu^*_{i,j} = \max(\nu'_{i,j}, \frac{1}{t(i,j)} \sum^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \nu_{u,v}),$$

where $\langle \mu'_{i,j}, \nu'_{i,j}\rangle$, $\sum^*$ and $t(i,j)$ are as in 5.1 and 5.2.
- **5.5** (pessimistic (s,m)-rule) If an s-full/empty square $\langle i,j\rangle$ must be changed, then we can use m-th negation $\neg_m$ to pair (before change) $\langle \mu_{i,j}, \nu_{i,j}\rangle$ and to juxtapose to it the pair $\langle \mu^*_{i,j}, \nu^*_{i,j}\rangle$, so that

$$\mu^*_{i,j} = \min(\mu'_{i,j}, \min^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \mu_{u,v})$$

$$\nu^*_{i,j} = \max(\nu'_{i,j}, \max^*_{u\in\{i-1,i,i+1\};v\in\{j-1,j,j+1\};\langle u,v\rangle\neq\langle i,j\rangle} \nu_{u,v}),$$

where $\langle \mu'_{i,j}, \nu'_{i,j}\rangle$ and $\max^*, \min^*$ are as in case 5.1.

5 Conclusion

Here a series of modifications of the laws of the Conway's Game of Life functioning, based on intuitionistic fuzzy set theory, were introduced for the first time. In the next authors' research new modifications of this game will be described. We will continue in two directions.

First, we will modify the standard game using other elements of the intuitionistic fuzzy set theory, e.g, the modal, topological and level operators, defined in it.

Second: we will modify the rules of the game, as we already prepared this in our previous research, e.g. [3–5].

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