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Operation "concatenation" over intuitionistic fuzzy index matrices

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Abstract: In this paper, a new operation, concatenation, denoted by $\circledwedge$, is introduced over extended intuitionistic fuzzy index matrices and over intuitionistic fuzzy index matrices. Some of its properties are discussed.
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1 Introduction

Here, as a continuation of [1, 3, 4, 5, 6, 7], we discuss the concepts of an Extended Intuitionistic Fuzzy Index Matrix (EIFIM), that generalizes the concept of an Intuitionistic Fuzzy Index Matrix (IFIM). These concepts are described in detail in [7].

In [8], a new operation was introduced over standard IM.

Here, we introduce it for the case of EIFIMs and in particular case – to IFIMs.

2 Basic definition

Firstly, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [2, 5]) and especially, of their partial case, Intuitionistic Fuzzy Pairs (IFPs; see [9]). The IFP is an object in the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and which conponents (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

First, we define two examples for definitions of operations "conjunction" and "disjunction":

$$\begin{gathered} x \vee_1 y = \langle \max(a, c)), \min(b, d)\rangle, \\ x \wedge_1 y = \langle \min(a, c), \max(b, d)\rangle, \\ x \vee_2 y = \langle a + c - a.c, b.d\rangle, \\ x \wedge_2 y = \langle a.c, b + d - b.d\rangle. \end{gathered}$$

Second, following [7], the definition of an Extended IM is proposed.

Let $\mathcal{I}$ be again a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \underset{1\leq n\leq\infty}{\cup} \mathcal{I}^n.$$

Let everywhere below $\mathcal{X}$ be a fixed set of some objects. In particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be given.

We call the object $[K, L, \{a_{k_i,l_j}\}]$ with index sets K and L $(K, L \subset \mathcal{I}^*)$ and elements from set $\mathcal{X}$, "Extended IM" (EIM) if it is defined in the form (see, [7]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_j} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1} & \dots & a_{k_i,l_j} & \dots & a_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_j} & \dots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{X}$.

When elements a_{k_i,l_j} are some variables, propositions or formulas, we obtain an EIM with elements from the respective type. Then, we can define the evaluation function V that juxtaposes to this IM a new one with elements – Intuitionistic Fuzzy pairs (IFPs) $\langle \mu, \nu\rangle$, where $\mu, \nu, \mu +$

$\nu \in [0, 1]$. The new IM, called Intuitionistic Fuzzy IM (IFIM), contains the evaluations of the variables, propositions, etc., i.e., it has the form

$$V([K, L, \{a_{k_i,l_j}\}]) = [K, L, \{V(a_{k_i,l_j})\}] = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$= \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$: $V(a_{k_i,l_j}) = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$ and $0 \le \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \le 1$.

3 Main results

The Extended IFIM (EIFIM) is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$:

$$0 \le \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1],$$

$$\alpha_i^k, \beta_i^k, \alpha_i^k + \beta_i^k \in [0, 1],$$

$$\alpha_j^l, \beta_j^l, \alpha_j^l + \beta_j^l \in [0, 1],$$

and here and below

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \le i \le m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \le j \le n\}.$$

Let us have, as above, the two IMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$. Then, following [8] we define the operation "concatenation" over two IMs:

$$A \circledS B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} \langle a_{k_i,l_j}, \perp\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ \langle \perp, b_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ \langle a_{k_i,l_j}, b_{p_r,q_s}\rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle \perp, \perp\rangle, & \text{otherwise} \end{cases}$$

Now, this definition will be transformed for the case of EIFIMs.

Let $(\circ, *) \in \{(\max, \min), (\min, \max), ...\}$.

Now, for the EIFIMs $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$ and $B = [P^*, Q^*, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle\}]$, where K^* and L^* are as above, we define the operation "concatenation", or $\circledwedge$, but now, in the form $\circledwedge_{(\circ_1,*_1);(\circ_2,*_2)}$, by:

$$A \circledwedge_{(\circ,*)} B = [T^*, V^*, \{\langle c_{t_u,v_w}, d_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cup Q\},$$

$$\langle \alpha_u^t, \beta_u^t\rangle = \begin{cases} \langle \alpha_i^k, \beta_i^k\rangle, & \text{if } t_u = k_i \in K - P \\ \\ \langle \alpha_r^p, \beta_r^p\rangle, & \text{if } t_u = p_r \in P - K \\ \\ \langle \circ(\alpha_i^k, \alpha_r^p), *(\beta_i^k, \beta_r^p)\rangle & \text{if } t_u = k_i = p_r \in K \cap P \end{cases},$$

$$\langle \alpha_w^v, \beta_w^v\rangle = \begin{cases} \langle \alpha_j^l, \beta_j^l\rangle, & \text{if } v_w = l_j \in L - Q \\ \\ \langle \alpha_s^q, \beta_s^q\rangle, & \text{if } v_w = q_s \in Q - L \\ \\ \langle \circ(\alpha_j^l, \alpha_s^q), *(\beta_j^l, \beta_s^q)\rangle, & \text{if } v_w = l_j = q_s \in L \cap Q \end{cases},$$

and

$$\langle c_{t_u,v_w}, d_{t_u,v_w}\rangle = \begin{cases} \langle\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, \perp\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ \langle \perp, \langle \mu_{p_r,q_s}, \nu_{p_r,q_s}\rangle\rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ \langle\langle \mu_{k_i,l_j}, \mu_{p_r,q_s}\rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ \langle \nu_{k_i,l_j}, \nu_{p_r,q_s}\rangle\rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle \perp, \perp\rangle, & \text{otherwise} \end{cases}$$

In the partial case, when A and B are IFIM, $A \oslash_{\perp} B = [T, V, \{\langle c_{t_u,v_w}, d_{t_u,v_w}\rangle\}]$, where $T = K \cup P$, $V = L \cup Q$, and

$$\langle c_{t_u,v_w}, d_{t_u,v_w}\rangle = \begin{cases} \langle\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, \perp\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ \langle \perp, \langle \mu_{p_r,q_s}, \nu_{p_r,q_s}\rangle\rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ \langle\langle \mu_{k_i,l_j}, \mu_{p_r,q_s}\rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ \langle \nu_{k_i,l_j}, \nu_{p_r,q_s}\rangle\rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle \perp, \perp\rangle, & \text{otherwise} \end{cases}.$$

For brevity, we can write $A \oslash B = A \oslash_{(\perp)} B$.

Now, we can check that operation $\oslash$ over EIFIMs is not idempotent and commutative, but we can prove the following theorem.

Theorem 1. Operation “concatenation” ($\oslash$) over EIFIMs is associative.

Let $\mathcal{M}$ be the set of all EIFIMs. Let

$$I_{\emptyset} = [\emptyset, \emptyset, \perp],$$

where symbol “$\perp$” here denotes the lack of IM-elements.

Let for every natural number $n \geq 2$ and for every n EIFIMs $A_1, ..., A_n$ so that for s $(1 \leq s \leq n)$:

$$A_s = \begin{array}{l|ccc} & l_{s,1}, \langle \alpha^l_{s,1}, \beta^l_{s,1}\rangle & \dots & l_{s,n_s}, \langle \alpha^l_{s,n_s}, \beta^l_{s,n_s}\rangle \\ \hline k_{s,1}, \langle \alpha^k_{s,1}, \beta^k_{s,1}\rangle & \langle \mu_{k_{s,1},l_{s,1}}, \nu_{k_{s,1},l_{s,1}}\rangle & \dots & \langle \mu_{k_{s,1},l_{s,n_s}}, \nu_{k_{s,1},l_{s,n_s}}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_{s,i}, \langle \alpha^k_{s,i}, \beta^k_{s,i}\rangle & \langle \mu_{k_{s,i},l_{s,1}}, \nu_{k_{s,i},l_{s,1}}\rangle & \dots & \langle \mu_{k_{s,i},l_{s,n_s}}, \nu_{k_{s,i},l_{s,n_s}}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ k_{s,m_s}, \langle \alpha^k_{s,m_s}, \beta^k_{s,m_s}\rangle & \langle \mu_{k_{s,m_s},l_{s,1}}, \nu_{k_{s,m_s},l_{s,1}}\rangle & \dots & \langle \mu_{k_{s,m_s},l_{s,n_s}}, \nu_{k_{s,m_s},l_{s,n_s}}\rangle \end{array},$$

it is valid

$$\mathop{\oslash}_{i=1}^{2} A_i = A_1 \oslash A_2,$$

$$\mathop{\oslash}_{i=1}^{n} A_i = \left(\mathop{\oslash}_{i=1}^{n-1} A_i \right) \oslash A_n.$$

Finally, we mention that from the above definitions it follows the validity of the following theorem.

Theorem 2. $\langle \mathcal{M}, \oslash, I_{\emptyset}\rangle$ is a monoid.

4 Conclusion

The EIFEM constructed in this manner can also be used in graph theory, e.g., for representation of intuitionistic fuzzy graphs, introduced by two of the authors in [10] with weighted vertices and arc, in multi-criteria multi-person decision making procedures, in which the experts (persons) have provided their own scores as estimates, in intercriteria analysis, in which the different criteria have intuitionistic fuzzy weights, and in other applications.

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