Fourth International Workshop on IFSs
Banska Bystrica, Slovakia, 13 Oct. 2008
NIFS 14 (2008), 4, 11-14

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Weak law of large numbers in P-probability theory

Karol Samuelčík

Faculty of Natural Sciences, Matej Bel University
Department of Mathematics
Tajovského 40
974 01 Banská Bystrica, Slovakia
e-mail:
ksamuelcik@fpv.umb.sk

Abstract. The weak law of large numbers in P-probability theory is proved for a sequence of independent, equally distributed, integrable P-observables . The notion of P-states , P-observables and joint P-observables is introduced .

0. Introduction

An axiomatic IF- probability theory is based on the Lukasiewicz connectives

$a \oplus b = \min(a + b, 1),$

$a \odot b = \max(a + b - 1, 0).$

Our criterion for good IF-probability theory is the existence of the joint observable (corresponding to the notion of random vector in the Kolmogorov theory).

In this paper weak law of large numbers is proved in P-probability theory based on product connectives

$a \oplus b = a + b - ab, \quad a \odot b = ab.$

1. P-state , P-observable and Joint P-observable

We shall work with a probability space $(\Omega, \mathcal{S}, P)$ and with the family $\mathcal{F}$ of pairs $A = (\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1$. In this family $\mathcal{F}$ we introduce a partial ordering $\leq$ by the prescription

$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$

and two binary operations on $\mathcal{F}$:

$$A \bar{\oplus} B = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B),$$

$$A\bar{\odot}B = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B).$$

In the next text we shall ofently assume that $A\bar{\odot}B = (0,1)$. It means

$$\mu_A\mu_B = 0, \nu_A + \nu_B - \nu_A\nu_B = 1$$

The second equality can be rewritten as

$$(1-\nu_A)(1-\nu_B) = 0.$$

1.1 Definition. A mapping $m : \mathcal{F} \longrightarrow [0,1]$ is called a P-state, if the following conditions are satisfied:

(i) $m((1,0)) = 1, m((0,1)) = 0$;

(ii) $A\bar{\odot}B = (0,1) \Longrightarrow m(A\bar{\oplus}B) = m(A) + m(B)$;

(iii) $A_n \nearrow A \Longrightarrow m(A_n \nearrow m(A)$.

1.2. Definition. A mapping $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is a P-observable if the following conditions are satisfied:

(i) $x(R) = (1,0), x(\emptyset) = (0,1)$:

(ii) if $A \cap B = \emptyset$, then $x(A)\bar{\odot}x(B) = (0,1)$, and
$x(A \cup B) = x(A)\bar{\oplus}x(B)$.

(iii) $A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A)$.

1.3. Theorem. Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be a P-observable, $m : \mathcal{F} \longrightarrow [0,1]$ be a P-state. Then the function

$$m \circ x : \mathcal{B}(R) \longrightarrow [0,1],$$

is a probability measure.

Proof. See [2].

1.4. Definition. Let $x, y : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be P-observables. By the joint P-observable h of x, y we understand a mapping $h : \mathcal{B}(R^2) \longrightarrow \mathcal{F}$ satisfying the following conditions:

(i) $h(R^2) = (1,0), h(\emptyset) = (0,1)$;

(ii) if $A \cap B = \emptyset$, then $h(A)\bar{\odot}h(B) = (0,1)$, and $h(A \cup B) = h(A)\bar{\oplus}h(B)$;

(iii) $A_n \nearrow A \Longrightarrow h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C)\bar{\odot}y(D)$ for any $C, D \in \mathcal{B}(R)$.

1.5. Theorem. To any P-observables $x, y : \mathcal{B}(R) \longrightarrow \mathcal{F}$ there exists their joint P-observable $h : \mathcal{B}(R) \longrightarrow \mathcal{F}$.

Proof. See [2].

1.6. Definition. Let $(x_n)_{n=1}^{\infty}$ be a sequence of P-observables, $h_n : \mathcal{B}(R_n) \longrightarrow \mathcal{F}$ be a sequence of the joint P-observables of $x_1, x_2, ..., x_n (n = 1, 2, 3, ...), m : \mathcal{F} \longrightarrow [0, 1]$ be a P-state. The sequence (x_n) is independent, if for any n and any $C_1, ..., C_n \in \mathcal{B}(R)$ there holds

$$m(h_n(C_1 \times ... \times C_n)) = m(x_1(C_1)). \; ... \; .m(x(C_n)).$$

1.7. Definition. A sequence (x_n) of P-observables is identically distributed, if

$$m_{x_n} = m_{x_1}$$

for any $n \in N$.

1.8. Definition. For any P-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ we define

$$E(x) = \int_R t dm_x(t),$$

assuming that the integral exists.

1.9. Definition. Let $g_n : R^n \longrightarrow R$ be a Borel function, $x_1, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be P-observables, $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ their joint P-observable. Then we define the P-observable $y_n = g_n(x_1, ..., x_n) : \mathcal{B}(R) \longrightarrow \mathcal{F}$ by the prescription

$$y_n(A) = h_n(g_n^{-1}(A)).$$

1.10. Theorem. Let (x_n) be a sequence of independent, equally distributed, integrable P-observables,

$$E(x_n) = a,$$

Then for any $\epsilon > 0$

$$\lim_{n \to \infty} m \circ y_n(-\epsilon, \epsilon) = 1,$$

where

$$y_n = \frac{1}{n} \sum_{i=1}^{n} x_i - a$$

Proof. Consider the family $\mathcal{F}$ with its ordering $\leq$, and the greatest element (1,0) and the least element (0,1).

Define a partial binary operation $A \oplus B$ by the following way. $A \oplus B$ is defined if and only if $A \bar{\odot} B = (0, 1)$, i.e. $\mu_A.\mu_B = 0, \nu_A + \nu_B - \nu_A.\nu_B = 1$. In this case

$$A \oplus B = (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

The structure $(\mathcal{F}, \bar{\oplus}, \leq, (0, 1), (1, 0))$ is the B-structure ([3]). By Prop. 5.2 of ([3])there exists a probability space $(\Omega, \mathcal{S}, p)$ and random variables $\xi_1, \xi_2, \xi_3, ...$ such that

$$\lim_{n \to \infty} p(\{\omega; g_n(\xi_1(\omega), ...\xi_n(\omega)) < \epsilon\} = 1$$

if and only if

$$\lim_{n\to\infty} m(g_n(x_1, ..., x_n)(-\epsilon, \epsilon)) = 1.$$

Define $g_n : R^n \to R$ by the formula

$$g_n(u_1, ..., u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i - a.$$

Then the weak law of large numbers states that

$$\lim_{n\to\infty} p(\{\omega; \mid \bar{\xi_n}(\omega) - a \mid < \epsilon) = 1$$

and therefore

$$\lim_{n\to\infty} m((y_n - a)(-\epsilon, \epsilon)) = 1$$

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