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Intuitionistic Fuzzy Bi-Matrix Games

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Abstract

When handling different problems related with management or engineering problems associated with optimal alternative selection a researcher often deals with not sufficiently accurate data. The alternatives are usually associated by applying several different criteria.

The basic aim of this paper is to an application of IFS to a single non-cooperative bi-matrix games (a pair of payoff matrices) and attempt is made to conceptualize at first the meaning of an Nash equilibrium solution for such games. The intuitionistic fuzzy could be in terms of intuitionistic fuzzy payoffs. A method takes advantage of the relationship between IFS and matrix game theories can be offered for decision making problems. Practical investigations have already been discussed for selecting the alternative in selling a book.

Keywords : Bi-matrix game, Intuitionistic fuzzy, Non-cooperative game, Nash equilibrium

1 Introduction

Fuzzy set theory proposed by Zadeh [12], have been researched and applied widely and highly in many fields and various modification methods and generalization theories have appeared. Out of several higher order fuzzy sets Atasunov's [1] intuitionistic fuzzy sets (IFSs) have been found to be highly useful to deal with vagueness and a very important development of fuzzy set. It contains not only the degree of membership of an element $x \in X$ to a given set $U \subset X$, but also the degree of non-membership such that they can be arbitrary in $[0, 1]$ and the sum of both values less than or equal to 1.

Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences (or utilities) of those players, none of which

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might have been intended by any of them. This theory plays a significant role in the field of decision making problems. A two person game is a simplest case of game theory. Here the two players are defined as decision makers. Recently, much more attention has been focused on bi-matrix games with fuzzy payoff, this means that elements of the payoff matrix are fuzzy numbers, where it is assumed that there membership functions are known. The bi-matrix game theory, takes care of problems that involve vagueness. Bustine and Burillo [5] pointed out that the notion of vague sets is same as that of IFSs by Atanassov [1]. There are situations where due to insufficiency in the information available, the evaluation of membership values is not possible up to our satisfaction. Due to the same reason evaluation of non-membership values is not always possible and consequently there remains an indeterministic part on which hesitation survives. Certainly fuzzy set theory is not appropriate to deal with such problems. In such situations, IFSs theory proposed by Atanassov [1], serve better to model the game as a game with IFSs payoffs.

In this paper, we consider IFSs matrix games, namely two person bi-matrix games with IFSs payoffs. Different solution concepts have been proposed for nonzero sum matrix games by Nishizaki and Sakawa [11] who defined the concept of equilibrium solution in bi-matrix games with a pair of matrices and presented methods for obtaining them. Based on inequality relations on IFSs, we define the concept of equilibrium strategy for such intuitionistic fuzzy bi-matrix games (IFBG). We shall show that this equilibrium strategy is characterized as Nash equilibrium strategy [10]. The computational procedure is illustrated via application to a optimization problem.

2 Fuzzy bi-matrix game

The theory of bi-matrix games is used alongside the theory of fuzzy sets, which offers the possibility to take into account the phenomenon known as fuzziness. Fuzzy bi-matrix games are described by Nishizaki and Sakawa [11]. Let I, II denote two decision makers (DM) and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for DM I, II respectively. By $\{\langle a_{ij}, \mu_{ij}^A\rangle, \langle b_{ij}, \mu_{ij}^B\rangle\}$ we denote the pay-offs that the DM I and II receive when DM I plays the pure strategy i and DM II plays the pure strategy j, where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a fuzzy number. Use is generally made of the symbolic notation $\Gamma = \langle\{I, II\}, \hat{A}, \hat{B}\rangle$.

In this paper, we consider two person IFBG, namely, the games where the number of players are two and IFS payoffs, which one DM receives are not necessarily equal to the IFS payoffs which the other DM.

3 Intuitionistic Fuzzy Sets

Before we proceed further in defining the intuitionistic fuzzy bi-matrix game (IFBG), we introduce first the necessary notations. In this section, we recapitulate some relevant basic preliminaries, notations and definitions of IFSs in particular, the works of Atanassov [2, 3].

An IFSs A in a given universe of discourse U is characterized by a degree membership function $\mu_A(x)$ and the degree of non-membership function $\nu_A(x)$. Here the degree of association of defined elements is determined by an association function that must come within the scope of the particular mathematical definitions, axioms or operational rules. When U is discrete, an IFSs A in U is the set of all ordered pair:

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in U\}$$

where $\mu_A : U \to [0, 1]$ is the lower bound on the degree of association of x with the IFSs A derived from the evidence of u and $\nu_A : U \to [0, 1]$ is the lower bound on the negation of u derived from the evidence against u, satisfying the following conditions :

(i) The degree of acceptance $\mu_A(x)$ and degree of rejection $\nu_A(x)$ is restricted on a closed interval and satisfying the intuitionistic condition (IC) : $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in U$.

(ii) $\exists$ an unique real number x, such that $\mu_A(x) = 1$ and $\nu_A(x) = 0$.

(iii) Both $\mu_A(x)$ and $\nu_A(x)$ are quasi-concave and upper semi-continuous.

(iv) For all $A \in IFSs(U)$, the indeterministic part of x, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of uncertainty of the IFS A satisfying $0 \leq \pi_A(x) \leq 1$ for all $x \in U$.

The set of all intuitionistic fuzzy set on U is denoted by $IFSs(U)$. Thus the numbers $\mu_A(x)$ and $\nu_A(x)$ reflect respectively the extent of acceptance and the degrees of rejection of the element x to the set A, and the numbers $\pi_A(x)$ is the extent of indeterminacy between both. In sone cases, IFSs can deal with problems more effectively.

Result 1 *When $\pi_A(x) = 0$ for every $x \in U$ then $\mu_A(x) + \nu_A(x) = 1$ and so*

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in U\} = \{< x, \mu_A(x) >: x \in U\}.$$

As a matter of fact, IFSs are the generalized version of fuzzy sets.

For a fuzzy decision, the association function $\mu_A(x)$ indicates the degree to which each element x will satisfy the respective requirements. An element $x \in A$ means an optimum fuzzy decision if x possess the maximum degree of the association with A. Various concepts can be used to determine the association function. For constituting the association function, the totality of $\mu_A(x)$ for all elements x from U should be taken into account. Here discrete values for the association function are considered.

3.1 Operations on IFSs

For the sake of completeness, we first recall the definitions of some operations. Let A and B be two IFSs of the set U, then their operations are defined by membership functions

(*i*) $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle | x \in U\}$

(*ii*) $A.B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle | x \in U\}$

(*iii*) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in U\}$

(*iv*) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in U\}$

(*v*)$\overline{A} = \{< x, \nu_A(x), \mu_A(x) >: x \in U\}$.

3.2 Inequality relations on IFSs

It is difficult to compare the sets directly due to intuitionistic nature. An extensive research and wide convergence on IFSs arithmetics and its applications can be found in Atanassov [2, 3]. Li and Cheng [8] introduced a useful way to measure the degree of similarity between IFSs. Measures of similarity between IFSs is an important content in fuzzy mathematics have gained attention from researchers for their wide applications in real world.

However, in a recent work, Zhizhen and Pengfei [13] showed that there exists a set of pairs of IFSs for which the definition of Li and Cheng [8] do not hold. They proposed a simple and efficient definition for comparing two IFSs. We can define a fuzzy preference or objective using the theory of IFSs, introduced by Atanassov [2, 3]. He suggested an order relation of IFSs as :

$$(i)\ \ A \preceq B \text{ iff } \mu_A(x) \leq \mu_B(x) \quad \text{and} \quad \nu_A(x) \geq \nu_B(x); \forall x \in U$$

$$(ii)\ \ A = B \text{ iff } \mu_A(x) = \mu_B(x) \ \text{ and } \ \nu_A(x) = \nu_B(x); \forall x \in U.$$

These processes help us to automate the selection of fuzzy strategies chosen by the players and their corresponding fuzzy preferences needed to play a IFBG.

4 Intuitionistic Fuzzy Bi-matrix Game

The fuzzy bi-matrix games described in Section 2 of this paper will now be considered in terms of the theory of IFSs. The IFBG is a non-cooperative two person, in general, a non-zero sum (in the sense of IFS theory) game. It can be considered as a natural extension of classical game to cover situations in which the outcome of a decision process does not necessarily dictate the verdict that what one player gains and the other has to lose. The elements of the game theory are affected by various sources of fuzziness.

4.1 Payoff matrix

Let I, J denote two decision makers (DM) and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for DM I, J respectively. By the expression $\{\langle a_{ij}, \mu_{ij}^A, \nu_{ij}^A\rangle, \langle b_{ij}, \mu_{ij}^B, \nu_{ij}^B\rangle\}$ we mean the pay-off that the DM I and J receive when DM I plays the row pure strategy i and DM J plays the column pure strategy j. Then we have the following payoff matrices $\hat{A}$ and $\hat{B}$ whose (i, j) element is $\{\langle a_{ij}, \mu_{ij}^A, \nu_{ij}^A\rangle, \langle b_{ij}, \mu_{ij}^B, \nu_{ij}^B\rangle\}$. A two person non-zero sum bi-matrix game, comprised of two $m \times n$ dimensional matrices $(\hat{A}, \hat{B})$ can be written as

$$\hat{A} = \begin{pmatrix} \langle a_{11}, \mu_{11}^A, \nu_{11}^A\rangle & \cdots & \langle a_{1n}, \mu_{1n}^A, \nu_{1n}^A\rangle \\ \langle a_{21}, \mu_{21}^A, \nu_{21}^A\rangle & \cdots & \langle a_{2n}, \mu_{2n}^A, \nu_{2n}^A\rangle \\ \vdots & \vdots & \vdots \\ \langle a_{m1}, \mu_{m1}^A, \nu_{m1}^A\rangle & \cdots & \langle a_{mn}, \mu_{mn}^A, \nu_{mn}^A\rangle \end{pmatrix}; \ \hat{B} = \begin{pmatrix} \langle b_{11}, \mu_{11}^B, \nu_{11}^B\rangle & \cdots & \langle b_{1n}, \mu_{1n}^B, \nu_{1n}^B\rangle \\ \langle b_{21}, \mu_{21}^B, \nu_{21}^B\rangle & \cdots & \langle b_{2n}, \mu_{2n}^B, \nu_{2n}^B\rangle \\ \vdots & \vdots & \vdots \\ \langle b_{m1}, \mu_{m1}^B, \nu_{m1}^B\rangle & \cdots & \langle b_{mn}, \mu_{mn}^B, \nu_{mn}^B\rangle \end{pmatrix}.$$

where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a fuzzy number. Use is generally made of the symbolic notation $\Gamma = \langle\{I, J\}, \hat{A}, \hat{B}\rangle$. Each pair of entries $\{\langle a_{ij}, \mu_{ij}^A, \nu_{ij}^A\rangle, \langle b_{ij}, \mu_{ij}^B, \nu_{ij}^B\rangle\}$ denotes the outcome of the game corresponding to a particular pair of decisions made by the players. In this case, however, since the expected payoff of the game should be fuzzy-valued, there is no concept of equilibrium strategy to be accepted widely. So it is important task to define the concepts of equilibrium strategies.

4.2 Example

Problems in several areas are taking on dimensions that no longer allow satisfactory solution by currently employed methods. These are complex and interrelated problems. The solution of which depends on the goals pursued by different interested parties. Attempts to interpret such problems as conflict situations which can be addressed by Game theory.

To illustrate this idea, we consider the following situation in the context of selling a book. A book 'Numerical Analysis' is written by two authors say A and B. The framework for such games is as follows

(i) There are two authors A and B. The author A writes the same book in two forms theory base (Th) and problem base (Pr); while B writes the same book in two forms algorithm base (Al) and programm base ($Prog$)

(ii) Although they write these books for sale the number of demands of these books are unknown to both the authors. Thus each author has a choice of two strategies (alternatives): $Th, Pr; Al, Prog$. The four elements of the set U are described by $U = \{Th, Pr; Al, Prog\}$, which is the universe of discourse associated with decision space for all players.

(iii) The play of game consists of a single move : A and B simultaneously and independently choose one of the two alternatives available to each of them. Based on demanding of these books, they are allowed to use the different capacities of these alternatives. Thus there are four possible situations.

(a) Writer A considers the following two possible choices : write theory based book for sale; write problem based book for sale, $P1 = \{Th, Pr\}$.

(b) Writer B considers of the following two possible choices : write algorithm based book for sale; write programm based book for sale $P2 = \{Al, Prog\}$.

Our first step in modelling this situation as a game is to represent it in terms of utility functions. It should be noted that it is often quite difficult to obtain precise information. Now it can represent this entire situation on a matrix, the strategic form of the game:

$$\hat{A} = \overset{P_1}{\begin{array}{|c|c|}\hline \langle a_{11}, \mu_{11}^A, \nu_{11}^A\rangle & \langle a_{12}, \mu_{12}^A, \nu_{12}^A\rangle \\ \hline \langle a_{21}, \mu_{21}^A, \nu_{21}^A\rangle & \langle a_{22}, \mu_{22}^A, \nu_{22}^A\rangle \\ \hline \end{array}} P_2 \qquad \hat{B} = \overset{P_2}{\begin{array}{|c|c|}\hline \langle b_{11}, \mu_{11}^B, \nu_{11}^B\rangle & \langle b_{12}, \mu_{12}^B, \nu_{12}^B\rangle \\ \hline \langle b_{21}, \mu_{21}^B, \nu_{21}^B\rangle & \langle b_{22}, \mu_{22}^B, \nu_{22}^B\rangle \\ \hline \end{array}} P_1$$

where we assume that each of the two writers chooses a strategy, a payoff for each of them is represented as in IFSs. Here

a_{11} = The demand of $A's$ Th base book in the market as the sell proceeds.

μ_{11}^A = The membership value of the demand of the Th base book by the people who possibly accept this book due to the presentation of the book nicely in the theoretical standpoint.

ν_{11}^{A} = The membership value of the rejection of the Th base book by the peoples who possibly reject this book due to not appropriate illustrations, algorithms, programmes, etc.

Similarly for others. For each choice of the writer A and the corresponding choice of the writer B we can compute fuzzy objective using the theory of IFSs. If the author A includes the algorithms and programming in his/her respective books and the author B includes theories and problems in his/her respective books, then resolve the problem in a fair way. Such a situation, however, does not carry the game-theoretic meaning.

5 Nash Equilibrium

Now we calculate two possible actions by comparing the payoffs in each column, as this shows which of A actions is preference for each possible action by B. What we referred to as its 'solution' is the Nash equilibrium (NE) of the game.

Basar and Olsder [4] defined the concept of Nash equilibrium solutions in bi-matrix games for single pair of payoff matrices and presented methodology for obtaining them. No studies, however, have been made for Nash equilibrium solutions for bi-matrix payoff matrices in IFSs environment.

Let I, II denote two decision makers (DMs) and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all pure strategies available for DM I, II respectively. A Nash solution represents an equilibrium point when each players reacts to the other by choosing the option that given him/her the largest value preference [12].

Definition 1 *A pair of strategies $\{rowr, columns\}$ is said to constitute a Nash equilibrium solution to a bi-matrix game $\hat{A} = \langle a_{ij}, \mu_{ij}^{A}, \nu_{ij}^{A} \rangle, \hat{B} = \langle b_{ij}, \mu_{ij}^{B}, \nu_{ij}^{B} \rangle$ if the following pair of inequalities is satisfied for all $i = 1, 2, ..., m$ and for all $j = 1, 2, ..., n$:*

$$a_{rs} \leq a_{is}; \quad \mu_{rs}^{A} \leq \mu_{is}^{A}; \quad \nu_{rs}^{A} \geq \nu_{is}^{A}$$

$$b_{rs} \leq b_{rj}; \quad \mu_{rs}^{B} \leq \mu_{rj}^{B}; \quad \nu_{rs}^{B} \geq \nu_{rj}^{B}.$$

Since the strategy sets are finite, the expressions always exist and so every IFBG admits a Nash equilibrium solution for pure strategy. The pair $(\langle a_{ij}, \mu_{ij}^{A}, \nu_{ij}^{A} \rangle, \langle b_{ij}, \mu_{ij}^{B}, \nu_{ij}^{B} \rangle)$ is known as a Nash equilibrium outcome of the IFBG in pure strategies. A IFBG can admit more than one Nash equilibrium solution, with the equilibrium outcomes being different in each case.

Theorem 1 *A pair $(rowr, columns)$ is an equilibrium solution of a bi-matrix game if and only if it is an equilibrium solution of IFBG.*

Proof : We shall first show that a pair $(rowr, columns)$ which is a equilibrium solution of bi-matrix game is also an equilibrium solution of IFBG. As $(rowr, columns)$ is an equilibrium solution of bi-matrix game, by definition we have

$$a_{rs} \leq a_{is}; \quad b_{rs} \leq b_{rj}$$

$$\Leftrightarrow \mu(a_{rs}) \leq \mu(a_{is}); \quad \mu(b_{rs}) \leq \mu(b_{rj})$$

$$\nu(a_{rs}) \geq \nu(a_{is}); \quad \nu(b_{rs}) \geq \nu(b_{rj})$$

$$\Leftrightarrow a_{rs} \leq a_{is}; \quad b_{rs} \leq b_{rj}$$

$$\mu(a_{rs}) \leq \mu(a_{is}); \quad \mu(b_{rs}) \leq \mu(b_{rj})$$

$$\nu(a_{rs}) \geq \nu(a_{is}); \quad \nu(b_{rs}) \geq \nu(b_{rj})$$

which shows that $(rowr, columns)$ is an equilibrium solution of IFBG. The converse of this theorem is also true, follows by just going backward. In particular, when $\nu_{ij} = 0$, then it becomes fuzzy matrix games. In this case, our approaches are more general than fuzzy matrix games. □

Theorem 2 *In any IFBG, a Nash equilibrium strategy exists.*

Proof : We know fuzzy sets are non-empty, compact, bounded and locally convex. In [7], we see that μ's are convex, continuous and bounded in $[0, 1]$. In the definition of ν we see that they are also convex, continuous and bounded in $[0, 1]$. Hence by Brouwer's fixed point theorem " if A and B are compact convex sets, then $A \times B$ is also compact set and it has at least one fixed point", the IFBG must have at least one Nash equilibrium solution. □

Result : When all the elements of $\hat{A}$ and $\hat{B}$ are not IFSs, these definition coincide with that of two person bi-matix games [4]. Therefore this definition is natural extension of equilibrium strategy of a two person bi-matrix games to IFSs matrix game.

5.1 Numerical example

Consider the following 2×2 bi-matrix game, where each of these terms is specified with a membership and non membership functions.

$$A = \overset{P_1}{\begin{array}{|c|c|}\hline \langle 3, 0.5, 0.4\rangle & \langle 2, 0.6, 0.3\rangle \\ \hline \langle 4, 0.7, 0.2\rangle & \langle 1, 0.4, 0.6\rangle \\ \hline \end{array}}\; P_2 \qquad B = \overset{P_2}{\begin{array}{|c|c|}\hline \langle 4, 0.5, 0.1\rangle & \langle 5, 0.7, 0.1\rangle \\ \hline \langle 3, 0.55, 0.1\rangle & \langle 2, 0.41, 0.52\rangle \\ \hline \end{array}}\; P_1.$$

These intuitionistic fuzzy matrices account inherently for the uncertainties associated with the lack of complete knowledge of the other writer actions. This problem admits two Nash equilibrium solutions as indicated :
(i) The first Nash equilibrium is $\{row1, column1\}$ and the corresponding equilibrium outcome is $\{\langle 3, 0.5, 0.4\rangle; \langle 4, 0.5, 0.1\rangle\}$. This Nash solution represents an equilibrium point when each writer reacts to the other by choosing the option that gives him/her the smallest membership function and largest non-membership function.
(ii) The second Nash equilibrium is $\{row2, column2\}$ and the corresponding equilibrium outcome is $\{\langle 1, 0.4, 0.6\rangle; \langle 2, 0.41, 0.52\rangle\}$. This Nash solution represents an equilibrium point when each writer reacts to the other by choosing the option that gives him/her the smallest membership function and largest non-membership function.

By comparison between their membership and non-membership of the writers preferences, it is clear that the Nash solution $\{\langle 1, 0.4, 0.6\rangle; \langle 2, 0.41, 0.52\rangle\}$ as the most favorable equilibrium solution than the Nash solution $\{\langle 3, 0.5, 0.4\rangle; \langle 4, 0.5, 0.1\rangle\}$. This solution indicates that, in future A, B will try to stop the Th and Al base book respectively for possible decrease of demands.

6 Concluding remarks

In this paper, we present a new approach to define Nash equilibrium solution for a IFBG. This process helped us to automate the solution by the players of the fuzzy strategies chosen and their corresponding fuzzy preferences needed to play a IFBG. Also, we proved the existence of at least one Nash equilibrium solution for a IFBG. Numerical example is presented to illustrate the methodology. Our proposed method is simple and more accurate to deal with the general two person of IFBG problems. This game approach is especially appropriate for the DM, whose preference of attributes and alternatives are both unknown.

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