19[th] Int. Workshop on IFSs, Burgas, 4–6 June 2015
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 21, 2015, No. 2, 43–50

Intuitionistic fuzzy semigroup

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Abstract: In this work, we generalize the definition of a intuitionistic fuzzy strongly continuous semi-group and its generator. We establish some of their properties and some results about the existence and uniqueness of solutions for intuitionistic fuzzy equation.
Keywords: Intuitionistic fuzzy semigroup, Intuitionistic fuzzy strongly continuous semi-group.
AMS Classification: 03E72.

1 Introduction

The initial value crisp problem

$$\begin{cases} x(t) = Ax(t) + f(t, x(t)) & t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{1.1}$$

has a unique mild solution under assumption some conditions, if A is the generator of a C_0-semigroup, $(S(t))_{t\geq 0}$ on a Banach space X, the system (1.1) has a unique mild solution $x \in \mathcal{C}([0, T])$. In [7], C. G. Gal and S. G. Gal studied, with more details, fuzzy linear and semi-linear (additive and positive homogeneous) operators theory, introduced semigroups of operators of fuzzy-number-valued functions, and gave various applications to fuzzy differential equation.

In this work, we study the existence and uniqueness of mild solution for fractional differential equation with intuitionistic fuzzy data of the following form:

$$\begin{cases} x(t) = Ax(t) + f(t, x(t)) & t \in [0, T] \\ x(0) = x_0 \in IF_1 \end{cases} \tag{1.2}$$

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $X = \mathbb{R}$ and we denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

Definition 2.1. *We denote*

$$IF_1 = \Big\{(u,v) : \mathbb{R} \to [0,1]^2 \ |\forall x \in \mathbb{R} \ /0 \le u(x) + v(x) \le 1\Big\}$$

where

1. *(u,v) is normal i.e there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.*
2. *u is fuzzy convex and v is fuzzy concave.*
3. *u is upper semicontinuous and v is lower semicontinuous*
4. *$supp(u,v) = cl(\{x \in \mathbb{R} : v(x) < 1\})$ is bounded.*

For $\alpha \in [0,1]$ and $(u,v) \in IF_1$, we define

$$\Big[(u,v)\Big]^\alpha = \{x \in \mathbb{R} \mid v(x) \le 1-\alpha\} \quad \textit{and} \quad \Big[(u,v)\Big]_\alpha = \{x \in \mathbb{R} \mid u(x) \ge \alpha\}$$

Remark 2.1. *We can consider $\Big[(u,v)\Big]_\alpha$ as $[u]^\alpha$ and $\Big[(u,v)\Big]^\alpha$ as $[1-v]^\alpha$ in the fuzzy case.*

Definition 2.2. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1,0) & x = 0 \\ (0,1) & x \neq 0 \end{cases}$$

Definition 2.3. *Let (u,v), $(u',v') \in IF_1$ and $\lambda \in \mathbb{R}$, we define the addition by :*

$$\Big((u,v) \oplus (u',v')\Big)(z) = \Big(\sup_{z=x+y} \min\Big(u(x), u'(y)\Big); \inf_{z=x+y} \max\Big(v(x), v'(y)\Big)\Big)$$

$$\lambda(u,v) = \begin{cases} (\lambda u, \lambda v) & \textit{if } \lambda \neq 0 \\ 0_{(0,1)} & \textit{if } \lambda = 0 \end{cases}$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space IF_1 as follows :

$$\Big[(u,v) \oplus (z,w)\Big]^\alpha = \Big[(u,v)\Big]^\alpha + \Big[(z,w)\Big]^\alpha, \qquad \Big[\lambda(u,v)\Big]^\alpha = \lambda\Big[(u,v)\Big]^\alpha$$
$$\Big[(u,v) \oplus (z,w)\Big]_\alpha = \Big[(u,v)\Big]_\alpha + \Big[(z,w)\Big]_\alpha, \qquad \Big[\lambda(u,v)\Big]_\alpha = \lambda\Big[(u,v)\Big]_\alpha$$

where (u,v), $(z,w) \in IF_1$ and $\lambda \in \mathbb{R}$.
We denote

$$\Big[(u,v)\Big]_l^+(\alpha) = \inf\Big\{x \in \mathbb{R} \mid u(x) \ge \alpha\Big\}, \qquad \Big[(u,v)\Big]_r^+(\alpha) = \sup\Big\{x \in \mathbb{R} \mid u(x) \ge \alpha\Big\}$$
$$\Big[(u,v)\Big]_l^-(\alpha) = \inf\Big\{x \in \mathbb{R} \mid v(x) \le 1-\alpha\Big\}, \qquad \Big[(u,v)\Big]_r^-(\alpha) = \sup\Big\{x \in \mathbb{R} \mid v(x) \le 1-\alpha\Big\}$$

Remark 2.2.

$$\Big[(u,v)\Big]_\alpha = \bigg[\Big[(u,v)\Big]_l^+(\alpha), \Big[(u,v)\Big]_r^+(\alpha)\bigg], \qquad \Big[(u,v)\Big]^\alpha = \bigg[\Big[(u,v)\Big]_l^-(\alpha), \Big[(u,v)\Big]_r^-(\alpha)\bigg]$$

Theorem 2.1. *[5]. let $\mathcal{M} = \{M_\alpha,\ M^\alpha\ :\ \alpha \in [0,1]\}$ be a family of subsets in $\mathbb{R}$ satisfying Conditions $(i)-(iv)$*

i) $\alpha \leq \beta \Rightarrow M_\beta \subset M_\alpha$ *and* $M^\beta \subset M^\alpha$

ii) M_α *and* M^α *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha \in [0,1]$.

iii) *for any nondecreasing sequence* $\alpha_i \to \alpha$ *on* $[0,1]$*, we have* $M_\alpha = \bigcap_i M_{\alpha_i}$ *and* $M^\alpha = \bigcap_i M^{\alpha_i}$.

iv) *For each* $\alpha \in [0,1]$, $M_\alpha \subset M^\alpha$ *and define* u *and* v, *by*

$$u(x) = \begin{cases} 0 & \text{if } x \notin M_0 \\ \sup\Big\{\alpha \in [0,1]\ :\ x \in M_\alpha\Big\} & \text{if } x \in M_0 \end{cases}$$

$$v(x) = \begin{cases} 1 & \text{if } x \notin M^0 \\ 1 - \sup\Big\{\alpha \in [0,1]\ :\ x \in M^\alpha\Big\} & \text{if } x \in M^0 \end{cases}$$

Then $(u,v) \in IF_1$.

The space IF_1 is metrizable by the distance of the following form :
For $p \in [1,\infty)$

$$d_p\Big((u,v),(z,w)\Big) = \bigg(\frac{1}{4}\int_0^1 \Big|[(u,v)]_r^+(\alpha) - [(z,w)]_r^+(\alpha)\Big|^p d\alpha + \frac{1}{4}\int_0^1 \Big|[(u,v)]_l^+(\alpha) - [(z,w)]_l^+(\alpha)\Big|^p d\alpha + \frac{1}{4}\int_0^1 \Big|[(u,v)]_r^-(\alpha) - [(z,w)]_r^-(\alpha)\Big|^p d\alpha + \frac{1}{4}\int_0^1 \Big|[(u,v)]_l^-(\alpha) - [(z,w)]_l^-(\alpha)\Big|^p d\alpha\bigg)^{\frac{1}{p}}$$

and

$$d_\infty\Big((u,v),(z,w)\Big) = \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(u,v)]_r^+(\alpha) - [(z,w)]_r^+(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(u,v)]_l^+(\alpha) - [(z,w)]_l^+(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(u,v)]_r^-(\alpha) - [(z,w)]_r^-(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(u,v)]_l^-(\alpha) - [(z,w)]_l^-(\alpha)\Big|$$

Theorem 2.2. *[5] For* $p \in [1,\infty)$
$\Big(IF_1, d_p\Big)$ *is a complete and separable metric space, but* $\Big(IF_1, d_\infty\Big)$ *is a complete and not separable metric space.*

Remark 2.3. *We can see that* $d_1 \leq d_\infty$

Definition 2.4. *we say that a mapping $F : [a,b] \to IF_1$ is strongly measurable if for all $\alpha \in [0,1]$, the set-valued mapping $F_\alpha : [a,b] \to P_k(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]_\alpha$ and $F^\alpha : [a,b] \to P_k(\mathbb{R})$ defined by $F^\alpha(t) = [F(t)]^\alpha$ are (Lebesgue) measurable, when $P_k(\mathbb{R})$ is endowed with the topology generated the Hausdorff metric d_H*

Lemma 2.1. *[5] Let $F : [a,b] \to IF_1$ be strongly measurable and denote $F_\alpha(t) = [\lambda_\alpha(t), \lambda^\alpha(t)]$, $F^\alpha(t) = [\mu_\alpha(t), \mu^\alpha(t)]$ for $\alpha \in [0,1]$. Then $\lambda_\alpha, \lambda^\alpha, \mu_\alpha, \mu^\alpha$ are measurable.*

Definition 2.5. *Suppose $A = [a,b]$, $F : [a,b] \to IF_1$ is integrably bounded and strongly measurable for each $\alpha \in (0,1]$,*

$$\Big[\int_A F(t)dt\Big]_\alpha = \int_A \Big[F(t)\Big]_\alpha dt = \Big\{\int_A f dt | f : A \to \mathbb{R} \text{ is a measurable selection for } F_\alpha\Big\}$$

$$\left[\int_A F(t)dt\right]^\alpha = \int_A [F(t)]^\alpha \, dt = \left\{\int_A f dt | f : A \to \mathbb{R} \text{ is a measurable selection for } F^\alpha\right\}.$$

if there exist $(u,v) \in IF_1$ such that $\Big[(u,v)\Big]^\alpha = \left[\int_A F(t)dt\right]^\alpha$ and $\Big[(u,v)\Big]_\alpha = \left[\int_A F(t)dt\right]_\alpha$ $\forall \alpha \in (0,1]$.
Then F is called integrable on A, write $(u,v) = \int_A F(t)dt$.

Remark 2.4. 1. *If $F(t) = (u_t, v_t)$ is integrable, with theorem 2.1 we have $\int \Big(u_t, v_t\Big) = \Big(\int u_t, \int v_t\Big)$*

2. *If $F : [a,b] \to IF_1$ is integrable then in view of Lemma (2.1) $\int F$ is obtained by integrating the α-level curves, that is*

$$\left[\int F\right]_\alpha = \left[\int \lambda_\alpha, \int \lambda^\alpha\right] \text{ and } \left[\int F\right]^\alpha = \left[\int \mu_\alpha, \int \mu^\alpha\right], \alpha \in [0,1]$$

$F_\alpha(t) = \Big[F(t)\Big]_\alpha$ *and* $F^\alpha(t) = \Big[F(t)\Big]^\alpha$.

3 The embedding theorem

Definition 3.1. *A intuitionistic fuzzy set (u,v) is called intuitionistic fuzzy convex set if and only if u is convex fuzzy set and v is concave fuzzy set.*

IF_1 with addition and multiplication by scalar laws is not a vector space. In order to extend the Radstrom embedding theorem to IF_1, we need to define a linear structure is defined in IF_1 by

1. $\Big((u,v) \oplus (u',v')\Big)(x) = (\beta, 1-\beta)$ where $\beta = \sup\left\{\alpha \in [0,1], x \in \Big[(u,v)\Big]^\alpha + \Big[(u',v')\Big]^\alpha\right\}$

2.

$$\Big(\lambda(u,v)(x)\Big) = \begin{cases} \Big(u(x/_{\lambda}), v(x/_{\lambda})\Big) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \; x \neq 0 \\ \Big(\sup_{y\in X} u(y), 1 - \sup_{y\in X} v(y)\Big) & \text{if } \lambda = 0 \;,\; x = 0 \end{cases}$$

Theorem 3.1. *There exists a normed space $\mathcal{X}$ and a function $j : IF_1 \to \mathcal{X}$ with properties*

1. *j is an isometry i.e.* $\Big\|j\Big((u,v)\Big) - j\Big((u',v')\Big)\Big\| = d_1\Big((u,v);(u',v')\Big) \leq d_\infty\Big((u,v);(u',v')\Big)$
2. $j\Big((u,v) \oplus (u',v')\Big) = j\Big((u,v)\Big) + j\Big((u',v')\Big)$
3. $j\Big(\lambda(u,v)\Big) = \lambda j\Big((u,v)\Big)\ \ \lambda \geq 0$

Proof. Define an equivalence relation in $IF_1 \times IF_1$ by

$$\Big((u,v);(u',v')\Big)\mathcal{R}\Big((z,w);(z',w')\Big) \Leftrightarrow (u,v) \oplus (z',w') = (u',v') \oplus (z,w)$$

The space of equivalence class $\overline{\Big((u,v);(u',v')\Big)}$ of pairs $\Big((u,v);(u',v')\Big)$ is denoted by $\mathcal{X}$. The norm in $\mathcal{X}$ is define by $\Big\|\overline{\Big((u,v);(u',v')\Big)}\Big\| = d_1\Big((u,v);(u',v')\Big)$. It is easy to check that $j : IF_1 \to \mathcal{X}$ defined by

$$j\Big((u,v)\Big) = \overline{\Big((u,v);O_{(0,1)}\Big)}$$

is an isometry, and properties (2), (3) follow the definition. □

Proposition 3.1. *[8] $\mathcal{X}$ is a Banach real space.*

4 Intuitionistic fuzzy strongly continuous semi-group

In this section we give the approach of the concept of intuitionistic fuzzy Semi-group.

Definition 4.1. *We called an intuitionistic fuzzy $\mathcal{C}^0$-Semi-group (one parameter, strongly continuous, nonlinear) the whole family $\{T(t), t \geq 0\}$ of operators from IF_1 into itself satisfying the following conditions*

- *$T(0) = i$, the identity mapping on IF_1.*
- *$T(t+s) = T(t)T(s)$, $\forall t,\ s \geq 0$.*
- *The function $g : [0, +\infty[\to IF_1$, defined by $g(t) = T(t)(u,v)$ is continuous at $t = 0$ for all $(u,v) \in IF_1$ i.e*

$$\lim_{t\to 0^+} T(t)(u,v) = (u,v)$$

- *There exist two constants $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$d_\infty\Big(T(t)(u,v), T(t)(u',v')\Big) \leq Me^{\omega t} d_\infty\Big((u,v),(u',v')\Big),$$

for $t \geq 0,\ \Big((u,v),(u',v')\Big) \in IF_1^2$

In particular, if $M = 1$ and $\omega = 0$, we say that $\{T(t); t \geq 0\}$ is a contraction intuitionistic fuzzy semigroup.

Remark 4.1. *The continuity of g at 0, implies the continuity of g at $t_0 \geq 0$.*

Definition 4.2. *Let $\{T(t), t \geq 0\}$ be an intuitionistic fuzzy $\mathcal{C}^0$-semigroup on IF_1 and $(u,v) \in IF_1$, if for $h > 0$ sufficiently small, the Hukuhara difference $T(h)(u,v) -_H (u,v)$ exists, we define*

$$\lim_{h\to 0} d_\infty\left(\frac{T(h)(u,v) -_H (u,v)}{h}, A\Big((u,v)\Big)\right) = 0$$

whenever this limit exists in the metric space $\Big(IF_1, d_\infty\Big)$. Then the operator $A : (u,v) \to A(u,v)$ defined on

$$D(A) = \left\{(u,v) \in IF_1, \lim_{h\to 0}\left\{\frac{T(h)(u,v) -_H (u,v)}{h}\right\} \mathit{exists}\right\}$$

is called the infinitesimal generator of the intuitionistic fuzzy semigroup $\{T(t), t \geq 0\}$.

Proposition 4.1. *Let $A : IF_1 \to IF_1$ and $A_1 = jAj^{-1} : \mathcal{X} \to \mathcal{X}$ two operators. A is the infinitesimal generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on IF_1 if and only if A_1 is the infinitesimal generator of the semigroup $\{T_1(t); t \geq 0\}$ defined on $\mathcal{X}$ by $T_1(t) = jT(t)j^{-1}$ for $t \geq 0$.*

Proof. We assume that A is the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on IF_1, then we have for all $(u,v) \in j^{-1}(D(A))$

$$\begin{aligned}
\lim_{h\to 0^+} \frac{T_1(h)(u,v) - (u,v)}{h} &= \lim_{h\to 0^+} \frac{jT(h)j^{-1}(u,v) - jj^{-1}(u,v)}{h} \\
&= j \lim_{h\to 0^+} \frac{T(h)j^{-1}(u,v) -_H j^{-1}(u,v)}{h} \\
&= jAj^{-1}(u,v) = A_1(u,v)
\end{aligned}$$

Conversely, if A_1 is the generator of an semigroup $\{T_1(t); t \geq 0\}$ on $\mathcal{X}$, then for all $(u,v) \in D(A)$

$$\begin{aligned}
\lim_{h\to 0^+} \frac{T(h)(u,v) -_H (u,v)}{h} &= \lim_{h\to 0^+} \frac{j^{-1}T_1(h)j(u,v) -_H j^{-1}j(u,v)}{h} \\
&= j^{-1} \lim_{h\to 0^+} \frac{T_1(h)j(u,v) - j(u,v)}{h} \\
&= j^{-1}A_1 j(u,v) = A(u,v)
\end{aligned}$$

□

Remark 4.2. *Since the infinitesimal generator A_1 of $\{T_1(t); t \geq 0\}$ is unique, we deduce that the infinitesimal generator A of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ is also unique.*

Lemma 4.1. *Let A be the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on IF_1, then for all $(u, v) \in IF_1$ such that $T(t)(u, v) \in D(A)$ for all $t \geq 0$, the mapping $t \to g(t) = T(t)(u, v)$ is differentiable and*

$$g'(t) = AT(t)(u, v)$$

Proof. Let $(u, v) \in IF_1$, for $t, h \geq 0$ we have

$$T(t + h)(u, v) = T(h)T(t)(u, v)$$

Since $T(t)(u, v) \in D(A)$ then

$$\begin{aligned}
\lim_{h \to 0^+} \frac{g(t+h) -_H g(t)}{h} &= \lim_{h \to 0^+} \frac{T(t+h)(u,v) -_H T(t)(u,v)}{h} \\
&= \lim_{h \to 0^+} \frac{T(h)T(t)(u,v) -_H T(t)(u,v)}{h} \\
&= AT(t)(u,v)
\end{aligned}$$

Denote $A_h = \frac{T(h)-i}{h}$, for $h > 0$. Using the continuity of g and the definition of A, we have

$$\begin{aligned}
\lim_{h \to 0^+} \frac{g(t-h) -_H g(t)}{-h} &= \lim_{h \to 0^+} \frac{T(t-h)(u,v) -_H T(t)(u,v)}{-h} \\
&= \lim_{h \to 0^+} \frac{T(h)T(t-h)(u,v) -_H T(t-h)(u,v)}{h} \\
&= \lim_{h \to 0^+} A_h T(t-h)(u,v) \\
&= AT(t)(u,v)
\end{aligned}$$

Hence, g is differentiable and $g'(t) = AT(t)(u, v)$, for all $t \geq 0$. □

References

[1] Atanassov, K. (1986) Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87–96.

[2] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets : Theory and Applications,* Springer Physica-Verlag, Heidelberg.

[3] Atanassov, K. T., Vassilev, P. M., & Tsvetkov, R. T. (2013) *Intuitionistic Fuzzy Sets, Measures and integrals* Bulgarian Academic Monographs (12), Professor Marin Drinov Academic Publishing House, Sofia.

[4] Bede, B., & Gal, S. G.(2005) Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151, 581–599.

[5] Melliani, S., Elomari, M., Chadli, L. S. & Ettoussi, R. (2015) Intuitionistic fuzzy metric space, *Notes on Intuitionistic Fuzzy Sets*, 21(1), 43–53.

[6] Melliani, S., Ettoussi, R., Elomari, M. & Chadli, L. S. Solution of intuitionistic fuzzy differential equations (submitted in *Notes on Intuitionistic Fuzzy Sets*).

[7] Gal, C. G., & Gal, S. G. (2013) Semigroups of Operators on Spaces of Fuzzy-Number-Valued Functions with Applications to Fuzzy Differential Equations, arXiv:1306.3928v1, 17 June 2013.

[8] Klement, E. P., Puri, M. L., & Ralescu, D. A. (1986) Limit Theorems for Fuzzy Random Variables, *Proc. R. Soc. Lond. A 407*, 171–182.