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On some issues related to the distances between the Atanassov intuitionistic fuzzy sets described on universe with weights

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Abstract

This paper is a continuation of our previous works on the concepts and properties of distances between the Atanassov intuitionistic fuzzy sets (A-IFSs, for short). We remind the necessity of taking into account all three terms (membership, non-membership and hesitation margin) describing A-IFSs while considering the distances that provides a foundation of our works. Next, we show that the considered three term continuous Hamming distance is the counterpart of the discrete Hamming distance, and is a metric.

Keywords: Intuitionistic fuzzy sets, distances.

1 Introduction

The concept of a distance in the context of fuzzy sets (Zadeh [28]), or some generalization – intuitionistic fuzzy sets, or A-IFSs for short (Atanassov [1], [2]) – is of utmost importance, for the theory and applications, notably in similarity related issues in pattern recognition, classifications, group decisions, soft consensus measures, etc.

There are well-known formulas for measuring distances between fuzzy sets using e.g. the Minkowski r-metrics (e.g. the Hamming distances for $r = 1$, the Euclidean distances for $r = 2$, the dominance metric for $r = \infty$), or the Hausdorff metric.

The situation is quite different for A-IFSs for which there are two ways of measuring distances which is a result of two possible lines of reasoning. Some researchers use two terms only (the memberships and non-memberships) in the formulas whereas the others use all three terms (the membership, non-membership and hesitation margin) characterizing A-IFSs. Both methods are correct in the *Minkowski r-metrics* as all necessary and sufficient conditions are fulfilled for a distance in spite of the formulas used (with two or with three parameters) - cf. Szmidt and Kacprzyk [11], [20].

One could say that if both methods follow (in the Minkowski r-metrics) all mathematical assumptions, the problem does not exist – both methods are correct and can be used interchangeably. Though this may be true from a lomited formal view, unfortunately, the fact which method we use does influence the final results as the results of calculations differ not only in the values (what is obvious) but also give qualitatively quite different answers! See, for instance Szmidt and Kacprzyk [11], [20], [13], [23], [24].

In his paper we remind briefly why the three term A-IFSs representation (for discrete cases) is appealing from the theoretical point of view. However, instead of a purely formal analyzes, we also relate them to intuition which is of a crucial importance for applications. Next, we discuss a continuous distance, the Hamming distance, which turns out to be the counterpart of its discrete form (the distances converge), and clearly fulfills the metric conditions.

2 A brief introduction to A-IFSs

One of the possible generalizations of a fuzzy set in X (Zadeh [28]), given by

$$A' = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0, 1]$ is the membership function of the fuzzy set A', is an A-IFS, i.e. Atanassov's intuitionistic fuzzy set, (Atanassov [1], [2]) A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following A-IFS

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{4}$$

For each A-IFS in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

an *intuitionistic fuzzy index* (or a *hesitation margin*) of $x \in A$, and it expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [2]). It is obvious that $0 \le \pi_A(x) \le 1$, for each $x \in X$.

The hesitation margin turns out to be important while considering the distances (Szmidt and Kacprzyk [8], [11], [20], entropy (Szmidt and Kacprzyk [13], [23]), similarity (Szmidt and Kacprzyk [24]) for the A-IFSs, etc. i.e., the measures that play a crucial role in virtually all information processing tasks.

The use of A-IFSs instead of fuzzy sets implies the introduction of another degree of freedom (non-memberships) into the set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imperfect knowledge which leads to describing many real problems in a more adequate way. Applications of intuitionistic fuzzy sets to group decision making, negotiations, voting and other situations are presented in Szmidt and Kacprzyk [7], [9], [10], [12], [14], [15], [16], [16], [22], Szmidt and Kukier [25], [26].

3 Distances between A-IFSs

Distances between A-IFSs are calculated in the literature in two ways, using two parameters only or all three parameters describing elements belonging to the sets. Both ways are proper from the point of view of pure mathematical conditions concerning distances (all properties are fulfilled in both cases). Unfortunately one cannot say that both ways are equal when assessing the results obtained by the two approaches. Now we will present some arguments why in our opinion all three parameters should be used in the respective formulas, and what additional qualities their inclusion can give (cf. Szmidt and Kacprzyk [11], [20], Szmidt and Baldwin [5], [6]).

3.1 Distances between discrete A-IFSs

Employing all three terms (membership, non-membership, and hesitation margin), we can calculate distances between any two A-IFSs A and B in $X = \{x_{1,}x_2, \ldots, x_n\}$ (Szmidt and Kacprzyk [11], [20], Szmidt and Baldwin [5], [6]), e.g.

- the normalized Hamming distance:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+ \\ &+ |\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|) \end{aligned} \tag{6}$$

- the normalized Euclidean distance:

$$\begin{aligned} e_{IFS}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2 + \\ + (\nu_A(x_i)-\nu_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2)^{\frac{1}{2}} \end{aligned} \tag{7}$$

Both distances are from the interval [0,1].

It is easy to notice in an analytical way why all three parameters should be used when calculating distances.

Let us verify if we can discard the values π from the formula (6). Taking into account (5) we have

$$\begin{aligned} |\pi_A(x_i)-\pi_B(x_i)| &= |1-\mu_A(x_i)-\nu_A(x_i)-1+\mu_B(x_i)+\nu_B(x_i)| \leq \\ &\leq |\mu_B(x_i)-\mu_A(x_i)|+|\nu_B(x_i)-\nu_A(x_i)| \end{aligned} \tag{8}$$

Inequality (8) means that the third parameter in (6) should not be omitted as it was in the case of fuzzy sets for which taking into account the second parameter would only result in the multiplication by a constant value. For A-IFSs omitting the third parameter has an influence on the results.

A similar situation occurs for the Euclidean distance. Let us verify the effect of omitting the third parameter (π) in (7). Taking into account (5), we have

$$\begin{aligned}(\pi_A(x_i) - \pi_B(x_i))^2 &= (1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i))^2 = \\ &= (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + \\ &+ 2(\mu_A(x_i) - \mu_B(x_i))(\nu_A(x_i) - \nu_B(x_i)) \end{aligned} \tag{9}$$

which means that taking into account the third parameter π while calculating the Euclidean distance for the A-IFSs does have an influence on the final result. For a deeper discussion of the problem of distances, especially on the connections between geometrical representations of A-IFSs we refer an interested reader to Szmidt and Kacprzyk [11], [20], Tasseva et al. [27], Atanassov et al. [3].

So far we have considered why the formulas with all three parameters should be used when calculating distances for the discrete cases only. Now we will consider the continuous counterparts of the above discrete cases.

3.2 Distances between continuous A-IFSs

In our further considerations we will consider the continuous counterpart of the normalized Hamming distance (12) between fuzzy sets A_n, B_n in $X_n = \{x_{1,}^{(n)}, \ldots, x_n^n\}$ Szmidt and Baldwin [5], [6], Szmidt and Kacprzyk [11], [20]:

$$\begin{aligned} l_{IFS}(A_n, B_n) &= \frac{1}{2n}\sum_{i=1}^{n}(\left|\mu_A(x_i^{(n)}) - \mu_B(x_i^{(n)})\right| + |\nu_A(x_i^{(n)}) - \\ &+ \nu_B(x_i^{(n)})| + \left|\pi_A(x_i^{(n)}) - \pi_B(x_i^{(n)})\right|) \end{aligned} \tag{10}$$

Let

$$\begin{aligned} l_{IFS}^1(A_n, B_n) &= \frac{1}{2}\sum_{i=1}^{n} g(x_i^{(n)})(\left|\mu_A(x_i^{(n)}) - \mu_B(x_i^{(n)})\right| + |\nu_A(x_i^{(n)}) - \\ &+ \nu_B(x_i^{(n)})| + \left|\pi_A(x_i^{(n)}) - \pi_B(x_i^{(n)})\right|) \end{aligned} \tag{11}$$

Where $g : X_n \rightarrow [0, 1]$ and $\sum_{i=1}^{n} g(x_i^{(n)}) \leq 1$. When $g(x_i^{(n)}) = \frac{1}{n}$ we have $l_{IFS}^1(A_n, B_n) \equiv l_{IFS}(A_n, B_n)$

Let

$$l^2_{IFS}(A_n,B_n) = \frac{1}{2n}\sum_{i=1}^{n} g(x_i^{(n)})(\left|\mu_A(x_i^{(n)})-\mu_B(x_i^{(n)})\right| + |\nu_A(x_i^{(n)}) - $$
$$+ \nu_B(x_i^{(n)})| + \left|\pi_A(x_i^{(n)})-\pi_B(x_i^{(n)})\right|) \quad (12)$$

Where $g : X_n \to [0,1]$. When $g(x_i^{(n)}) = 1$ we have $l^2_{IFS}(A_n,B_n) \equiv l_{IFS}(A_n,B_n)$
Let

$$l'_{IFS}(A,B) = \frac{1}{2(b-a)}\int_a^b g(x)(|\mu_A(x)-\mu_B(x)| + |\nu_A(x) - $$
$$+ \nu_B(x)| + |\pi_A(x)-\pi_B(x)|)dx \quad (13)$$

where $0 \le g(x) \le 1$

Proposition

Let $g(x)$, $\mu_A(x)$ and $\nu_A(x)$ are Riemann's integrable. There exist A_n and B_n for which $\lim_{n\to\infty} l^2_{IFS}(A_n,B_n) = l'_{IFS}(A,B)$

Proof

Let $X = [a,b]$ For every $n \in N \setminus 0$, where $y_i = a + i\frac{b-a}{n}$ for each $i \in N_n = \{1,...,n\}$, and $y_0 = a$ by convention. For convention, let

$$\Delta_n = \frac{b-a}{n}$$

so that $y_i = a + i\Delta_n$.
Let $x_i^{(n)} \in [y_{i-1}, y_{i-1}]$, $X_n = \{x_i^{(n)}, i = 1...n\}$ and

$$S_n = \sum_{i=1}^{n}\frac{1}{2} g(x_i^{(n)})(\left|\mu_A(x_i^{(n)})-\mu_B(x_i^{(n)})\right| + |\nu_A(x_i^{(n)}) - $$
$$+ \nu_B(x_i^{(n)})| + \left|\pi_A(x_i^{(n)})-\pi_B(x_i^{(n)})\right|)\Delta_n \quad (14)$$

S_n is Riemann's integral sum. From definition Riemann's integral we have

$$\lim_{n\to\infty} S_n = \frac{1}{2}\int_a^b g(x)(|\mu_A(x)-\mu_B(x)| + |\nu_A(x) - $$
$$+ \nu_B(x)| + |\pi_A(x)-\pi_B(x)|)dx \quad (15)$$

Having in mind that

$$S_n = (b-a)l^2_{IFS}(A,B)$$

we obtain

$$\lim_{n\to\infty} l^2_{IFS}(A_n,B_n) = \frac{1}{2(b-a)}\int_a^b g(x)(|\mu_A(x)-\mu_B(x)| + |\nu_A(x) - $$
$$+ \nu_B(x)| + |\pi_A(x)-\pi_B(x)|)dx \quad (16)$$

which means that

$$\lim_{n\to\infty} l^2_{IFS}(A_n, B_n) \;=\; l'_{IFS}(A,B) \tag{17}$$

Now we will show that the continuous counterpart of the discrete Hamming distance between A-IFSs is a metric.

Let $X = \bigcup_{i=1}^{n} X_i$ where $X_i \bigcap X_j = \emptyset$.

$$\begin{aligned} l'_{IFS}(A,B) &= \frac{1}{2m(X)} \sum_{i=1}^{n} \int_{X_i} g(x)(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \\ &+ \nu_B(x)| + |\pi_A(x) - \pi_B(x)|)dm \end{aligned} \tag{18}$$

$$\begin{aligned} l'_{IFS}(i)(A,B) &= \frac{1}{2m(X_i)} \int_{X_i} g(x)(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \\ &+ \nu_B(x)| + |\pi_A(x) - \pi_B(x)|)dm \end{aligned} \tag{19}$$

Where $g : X \to (0,1]$

Proposition

Let E is set from A-IFSs with g , μ , ν continuous functions . X is union from closed intervals in R - real line. Integrals above are Riemann. Then $\left\langle E, l'_{IFS} \right\rangle$ is metric space.

Proof

1) We prove that

$$l'_{IFS}(A,B) > 0$$

if and only if $A \neq B$. Then it is obviously

$$l'_{IFS}(A,B) = 0$$

if and only if $A \equiv B$.

($\Leftarrow$) Let $A \neq B$.Then we have $x \in X$ where $\mu_A(x) \neq \mu_B(x)$ or $\nu_A(x) \neq \nu_B(x)$.

From

$l_{IFS}(A,B) = \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x_)| + |\pi_A(x) - \pi_B(x)|)$

distance when $X = \{x\}$ (x is above) we have $A \neq B$ if $l_{IFS}(A,B) > 0 \Rightarrow l'_{IFS}(i)(A,B) > 0 \Rightarrow l'_{IFS}(A,B) = \sum_{i=1}^{n} \frac{m(X_i)}{m(X)} l'_{IFS}(i)(A,B) > 0$ (These inequalities are results from the Riemann integral's property).

When $A = B$ then obviously $l'_{IFS}(A,B) = 0$.

($\Rightarrow$) We have $A \neq B$ or $A \equiv B$. If $A \neq B$ everything is ok. If $A \equiv B$ we have $l'_{IFS}(A,B) = 0$. But it is contradiction. So we have that $A \neq B$.

2) From definition $l'_{IFS}(A,B)$ we have that $l'_{IFS}(A,B) = l'_{IFS}(B,A)$.

3) From

$$l_{IFS}(A,B) = \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)|)$$

distance when $X = \{x\}$ we have
$l_{IFS}(A,B) \leq l_{IFS}(A,C) + l_{IFS}(C,B) \Rightarrow \frac{1}{2}\int\limits_{X_i} g(x)(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)|)dm \leq \frac{1}{2}\int\limits_{X_i} g(x)(|\mu_A(x) - \mu_C(x)| + |\nu_A(x) - \nu_C(x)| + |\pi_A(x) - \pi_C(x)|)dm + \frac{1}{2}\int\limits_{X_i} g(x)(|\mu_C(x) - \mu_B(x)| + |\nu_C(x) - + \nu_B(x)| + |\pi_C(x) - \pi_B(x)|)dm \Rightarrow l'_{IFS}(A,B) \leq l'_{IFS}(A,C) + l'_{IFS}(C,B)$,
and the proof is complete.

4 Concluding remarks

We considered the problem of a proper definition of distances between the A-IFSs. We recalled the arguments why the three term representation (taking into account the membership, non-membership and hesitation margin) of the A-IFSs seems proper while calculating distances between the discrete A-IFSs. Next, we showed that the considered Hamming distance in the continuous space is the counterpart of the Hamming distance in the discrete space, and it fulfills the requirements of metric. This is a result that is relevant for both the theory of the A-IFSs and their applications.

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