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Laws of large numbers for M-observables

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Abstract

In the paper M-observables are considered, their independence and law of large numbers. The proof is based on a representation of a sequence of M-observables by random variables. This method has been presented in [2].

Keywords: M-observable,M-state, convergence

1 Introduction

Let us consider a measurable space $(\Omega, \mathcal{S})$, where $\mathcal{S}$ is a σ - algebra of subset of Ω, $\mathcal{F} = \{A = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \longrightarrow [0,1]; \mu_A, \nu_A$ are $\mathcal{S}$- measurable; $\mu_A + \nu_A \leq 1\}$. The ordering on $\mathcal{F}$ is defined

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \ \nu_A \geq \nu_B.$$

We shall use the following max-min connectives for $A, B \in \mathcal{F}$:

$$A \vee B = (\mu_A \vee \mu_B, \ \nu_A \wedge \nu_B)$$

$$A \wedge B = (\mu_A \wedge \mu_B, \ \nu_A \vee \nu_B).$$

Definition 1.1 *An M-state is a mapping $m : \mathcal{F} \longrightarrow [0,1]$ satisfying the following conditions:*
(i) $m((1_\Omega, 0_\Omega)) = 1, \ m((0_\Omega, 1_\Omega)) = 0$;
(ii) $m(A) + m(B) = m(A \vee B) + m(A \wedge B)$ *for any* $A, B \in \mathcal{F}$;
(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow m(A_n) \nearrow m(A), m(B_n) \searrow m(B)$.

Definition 1.2 *A mapping $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is called M-observable if the following properties are satisfied:*
(i) $x(\mathcal{R}) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega)$;
(ii) $x(A \cup B) = x(A) \vee x(B), \ x(A \cap B) = x(A) \wedge x(B)$;
(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow x(A_n) \nearrow x(A), x(B_n) \searrow x(B)$.

Proposition 1.3 *If $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is an M-observable and $m : \mathcal{F} \longrightarrow [0,1]$ is an M-state, then $m \circ x : \mathcal{B}(R) \longrightarrow [0,1]$ is a probability measure.*

Proof.
See [2] Proposition 1.4.

Definition 1.4 *Let $x_1, x_2, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be M-observables. The joint M-observable of $x_1, x_2, ...x_n$ is a mapping $h : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ satisfying the following conditions:*
(i) $h(R^n) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;
(ii) $h(A_1 \cap A_2 \cap ... \cap A_n) = h(A_1) \wedge h(A_2) \wedge ... \wedge h(A_n)$,
$h(A_1 \cup A_2 \cup ... \cup A_n) = h(A_1) \vee h(A_2) \vee ... \vee h(A_n)$
for any $A_1, A_2, ..., A_n \in \mathcal{B}(R^n)$;
(iii) if $A_k, B_k \in \mathcal{B}(R^n)$, $(k = 1, 2, ...)$ and $A_k \nearrow A, B_k \searrow B$
$\Longrightarrow h(A_k) \nearrow h(A), h(B_k) \searrow h(B)$;
(iv) for any $C_1, ..., C_n \in \mathcal{B}(R)$, $h(C_1 \times C_2 \times ... \times C_n) = x_1(C_1) \wedge x_2(C_2) \wedge ... \wedge x_n(C_n)$.

Proposition 1.5 *For any M-observables there exists their joint M-observable.*

Proof.
See [3] Theorem 2.2.2.

Proposition 1.6 *Let $x_1, x_2, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be M-observables. There exists exactly one n-dimensional M-observable $h : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$, for which*

$$h(C_1 \times ... \times C_n) = x_1(C_1).x_2(C_2).....x_n(C_n),$$

for any $C_1, ..., C_n \in \mathcal{B}(R)$

Proof.
Let us assume, that there exists two n-dimensional M-observables $g, h : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$, for which

$$h(C_1 \times ... \times C_n) = x_1(C_1).x_2(C_2).....x_n(C_n),$$

$$g(C_1 \times ... \times C_n) = x_1(C_1).x_2(C_2).....x_n(C_n),$$

Let $\mathcal{M} = \{A;\ A \in \mathcal{B}(R^n);\ h(A) = g(A)\}$
and
$\mathcal{D} = \{C_1 \times ... \times C_n;\ C_i \in \mathcal{B}(R)\}$.
Evidently $\mathcal{M} \supset \mathcal{D}$. Let $\mathcal{R}(\mathcal{D})$ be the ring generated by $\mathcal{D}$.
Then $\mathcal{R}(\mathcal{D}) = \{\bigcup_{i=1}^{n} E_i; E_i \in \mathcal{D}; n = 1, 2, ...\}$,

$$h(\bigcup_{i=1}^{n}(E_i)) = \bigvee_{i=1}^{n} h(E_i) = \bigvee_{i=1}^{n} g(E_i) = g(\bigcup_{i=1}^{n}(E_i)).$$

$\mathcal{M}$ is a monotone family, hence $\mathcal{M} \supset \mathcal{M}(R^n) = \sigma(R^n) = \mathcal{B}(R^n)$.
Therefore $h(A) = g(A)$ for any $A \in \mathcal{B}(R^n)$.

Definition 1.7 *Let $m : \mathcal{F} \longrightarrow [0,1]$ be an M-state and $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an M-observable. Then x is said integrable if the expectation (mean value)*

$$E(x) = \int_R t \; dm_x(t)$$

exists.

We say that x is square integrable if the dispersion (variance)

$$\sigma^2(x) = \int_R (t - E(x))^2 dm_x(t) = \int_R t^2 dm_x(t) - E(x)^2$$

exists.

2 Independence of M-observables

Definition 2.1 *Let $x_1, x_2, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be M-observables and $m : \mathcal{F} \longrightarrow [0,1]$ be an M-state. We say that the M-observables $x_1, x_2, ...x_n$ are independent if there exists an n-dimensional M-observable $h : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ such that*

$$m(h(C_1 \times ... \times C_n)) = m(x_1(C_1)).....m(x_n(C_n)) = m_{x_1}(C_1).....m_{x_n}(C_n)$$

for all $C_1, C_2, ..., C_n \in \mathcal{B}(R)$.

Theorem 2.2 *Let $x_1, x_2, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be M-observables and $m : \mathcal{F} \longrightarrow [0,1]$ be an M-state. Then $x_1, x_2, ..., x_n$ are independent if and only if*

$$m(x_1(C_1) \wedge x_2(C_2) \wedge ... \wedge x_n(C_n)) = m_{x_1}(C_1).....m_{x_n}(C_n)$$

Proof.
The proof is an immediate consequence of Definition 2.1, together with Propositions 1.5 and 1.6.

3 Convergence of M-observables

Definition 3.1 *Let $y_1, y_2, ...$ be a sequence of M-observables, $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$, for $n = 1, 2, ...$ and $m : \mathcal{F} \longrightarrow [0,1]$ be an M-state.*

1. *The sequence is said to converge in distribution to a function $F : R \longrightarrow [0,1]$ if for each $t \in R$*

$$\lim_{n \longrightarrow \infty} m(y_n((-\infty, t))) = F(t).$$

2. *The sequence is said to converge in measure to* $(0_\Omega, 1_\Omega)$ *if for each* $\varepsilon > 0$

$$\lim_{n\longrightarrow\infty} m(y_n((-\varepsilon,\varepsilon))) = 1.$$

3. *The sequence converges to* $(0_\Omega, 1_\Omega)$ *almost everywhere, if*

$$\lim_{p\longrightarrow\infty}\lim_{k\longrightarrow\infty}\lim_{i\longrightarrow\infty} m(\bigwedge_{n=k}^{k+i} y_n((-\frac{1}{p},\frac{1}{p}))) = 1.$$

Definition 3.2 *Let* $x_1, x_2, ..., x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be M-observables,* $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ *be their joint M-observable,* $g_n : R^n \longrightarrow R$ *be a Borel function (i.e.* $A \in \mathcal{B}(R) \Longrightarrow g_n^{-1}(A) \in \mathcal{B}(R^n)$*). Then we define* $y_n = g_n(x_1, x_2, ..., x_n)$ *by the formula*

$$y_n = g_n(x_1, x_2, ..., x_n) = h_n \circ g_n^{-1}.$$

Remark 3.3 It is easy to see that the mapping $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *is an M-observable.*

Theorem 3.4 *Let* $x_1, x_2, ...$ *be a sequence of M-observable,* $x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$, $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ *be the joint M-observable of* $x_1, x_2, ..., x_n$ *and* $g_n : R^n \longrightarrow R$ *be a Borel function,* $y_n = g_n(x_1, x_2, ..., x_n)$*, for* $n = 1, 2, ...$.
Then there exists a probability space (X,S,P) and a sequence (ξ_n) *of random variables,* $\xi_n : X \longrightarrow R$ *such that if* $\eta_n = g_n(\xi_1, ..., \xi_n)$, $(n = 1, 2, ...)$*, then*

1. *the sequence* $y_1, y_2, ...$ *converges in distribution to a function F if and only if so does the sequence* $\eta_1, \eta_2, ...$ *;*

2. *the sequence* $y_1, y_2, ...$ *converges to* $(0_\Omega, 1_\Omega)$ *in measure m if and only if* $\eta_1, \eta_2, ...$ *converges to 0 in measure P;*

3. *if* $\eta_1, \eta_2, ...$ *converges P-almost everywhere to 0, then* $y_1, y_2, ...$ *converges m-almost everywhere to* $(0_\Omega, 1_\Omega)$.

Proof.
See [2] Theorem 2.3.

4 Laws of large numbers for M-observable

Theorem 4.1 *(Weak law of large numbers)*
Let $x_1, x_2, ...$ *be an independent sequence of integrable M-observables having the same probability distribution* $m_{x_1} = m_{x_2} = ...$ *. Let* $a = E(x_1) = E(x_2) = ...$ *. Then the sequence of M-observables*

$$\frac{x_1 + x_2 + ... + x_n}{n} - a \quad (n = 1, 2, ...)$$

converges in measure to $(0_\Omega, 1_\Omega)$.

Proof.
Let $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ be the joint M-observable of $x_1, x_2, ..., x_n, (n = 1, 2, ...)$, $g_n : R^n \longrightarrow R$ be a Borel function given by

$$g_n(w_1, ..., w_n) = \frac{w_1 + w_2 + ... + w_n}{n} - a$$

and $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$; $y_n = g_n(x_1, x_2, ..., x_n) = h_n \circ g_n^{-1}, n = 1, 2, ...$

Consider the probability space (X,S,P) and the sequence (ξ_n) of random variables $\xi_n : X \longrightarrow R$ by Theorem 3.4. Put $\eta_n = g_n(\xi_1, ..., \xi_n) = g_n \circ T_n$, where $T_n = (\xi_1, ..., \xi_n)$.

We have the identities

$$P \circ \xi_n^{-1} = P_{\xi_n} = m_{x_n} = m \circ x_n$$

$$P \circ T_n^{-1} = m_{x_1} \times ... \times m_{x_n} = m \circ h_n$$

Then the mean value of ξ_n is:

$$E(\xi_n) = \int_X \xi_n dP = \int_R t \; dP_{\xi_n}(t) = \int_R t \; dm_{x_n}(t) = E(x_n) = a.$$

Since the M-observables $x_1, x_2, ...$ are independent, then the random variables $\xi_1, \xi_2, ...$ are independent, too. For simplicity assume n=2, $T_2 = (\xi_1, \xi_2), C = A \times B$. Then

$$P(\xi_1^{-1}(A) \cap \xi_2^{-1}(B)) = P \circ T_2^{-1}(C) =$$

$$= m \circ h_2(C) = m \circ h_2(A \times B) = m_{x_1}(A).m_{x_2}(B) =$$

$$= P_{\xi_1}(A).P_{\xi_2}(B).$$

Hence for any $\varepsilon > 0$

$$\lim_{n \longrightarrow \infty} P(\{\omega \in R^N; |\frac{\xi_1(\omega) + ... + \xi_n(\omega)}{n} - a| < \epsilon\}) = 1$$

$$\lim_{n \longrightarrow \infty} P(\eta_n^{-1}((-\varepsilon, \varepsilon))) = \lim_{n \longrightarrow \infty} P(\{\omega \in R^N; |\eta_n(\omega) - 0| < \epsilon\}) =$$

$$= \lim_{n \longrightarrow \infty} P(\{\omega \in R^N; |\frac{\xi_1(\omega) + ... + \xi_n}{n}(\omega) - a| < \varepsilon\}) = 1.$$

Theorem 4.2 *(Strong law of large number)*
Let $x_1, x_2, ...$ be an independent sequence of square integrable M-observables such that $\sum_{n=1}^{\infty} \sigma^2(x_n)/n^2 < \infty$. Then the sequence of observables

$$\frac{x_1 - E(x_1) + x_2 - E(x_2) + ... + x_n - E(x_n)}{n} \quad (n = 1, 2, ...)$$

converges m-almost everywhere to $(0_\Omega, 1_\Omega)$.

Proof.
Let $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ be the joint M-observable of $x_1, x_2, ..., x_n, (n = 1, 2, ...)$, $g_n : R^n \longrightarrow R$ be the Borel function given by

$$g_n(w_1, ..., w_n) = \frac{w_1 - E(x_1) + w_2 - E(x_2) + ... + w_n - E(x_n)}{n}$$

and $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$; $y_n = g_n(x_1, x_2, ..., x_n) = h_n \circ g_n^{-1}, n = 1, 2, ...$

We can use similar methods as in Theorem 4.1.

Consider the probability space (X,S,P) and the sequence (ξ_n) of random variables $\xi_n : X \longrightarrow R$ and put $\eta_n = g_n(\xi_1, ..., \xi_n)$.

It holds $E(\xi_n) = E(x_n)$, $\xi_1, \xi_2, ...$ are independent random variables and the dispersion of ξ_n is :

$$\sigma^2(\xi_n) = \int_R (t - E(\xi_n))^2 dm_{\xi_n}(t) = \int_R (t - E(x_n))^2 dm_{x_n}(t) = \sigma^2(x_n).$$

Therefore the classical law of large numbers can be applied and the sequence $(\sum_{i=1}^{n}(\xi_i - E(\xi_i))/n)$ converges P-almost everywhere to 0. Hence $\eta_1, \eta_2, ...$ converges P-almost everywhere to 0 and the sequence $y_1, y_2, ...$ converges m-almost everywhere to $(0_\Omega, 1_\Omega)$.

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