Notes on Intuitionistic Fuzzy Sets
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IF topological vector spaces

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Abstract: In the present paper a notion of intuitionistic fuzzy topology on an intuitionistic fuzzy set has been developed. Also the concept of IF topological vector space has been introduced which is a combined structure of intuitionistic fuzzy vector space and intuitionistic fuzzy topology as defined by us. In this study, we generalized the action of a group on a set to intuitionistic fuzzy action. We obtained some basic results.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy vector space, Intuitionistic fuzzy topology on intuitionistic fuzzy set, IF topological vector space.
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1 Introduction

Allowing some kind of independence in the falsity value from the truth value, K. T. Atanassov [1–4] proposed the notion of intuitionistic fuzzy set (IFS) which is a generalization of Zadeh's fuzzy set [24]. IFS theory have successfully been applied in knowledge engineering, medical diagnosis, decision making, career determination etc. [13, 14, 23]. Much work have been done in developing various mathematical structures such as groups, rings, topological spaces, metric spaces, topological vector spaces etc. in IFS [6, 12, 15, 18–21]. We have introduced a notion of intuitionistic fuzzy vector space, intuitionistic fuzzy basis and intuitionistic fuzzy dimension in [10, 11].

In this paper, by synthesizing the definition of fuzzy topologies of Chakraborty and Ahsanullah [8] and of Lowen [17], we extend it to IF setting to introduce a definition of intuitionistic fuzzy topology on intuitionistic fuzzy set. Also we introduce a notion of IF topological vector space associated with an intuitionistic fuzzy vector space [10] and intuitionistic fuzzy topology defined on this intuitionistic fuzzy vector space. Some fundamental properties of IF topological vector spaces have been investigated.

2 Preliminaries

Definition 2.1 ([1]). *Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X is defined as an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.*

Definition 2.2 ([1]). *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a set X. Then*

(1) $A \subseteq B$ *iff* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$.

(2) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$.

(3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$

(4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

(5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

(6) $\Box\, A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$, $\Diamond\, A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3 ([4]). *Let A be an IFS in a set X. Then for $\lambda, \xi \in [0,1]$ with $\lambda + \xi \leq 1$, the set $A^{[\lambda,\xi]} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \xi\}$ is called (λ, ξ)-level subset of A.*

Proposition 2.4 ([4]). *Let A be an IFS in a set X and $(\lambda_1, \xi_1), (\lambda_2, \xi_2) \in Im(A)$. If $\lambda_1 \leq \lambda_2$ and $\xi_1 \geq \xi_2$, then $A^{[\lambda_1,\xi_1]} \supseteq A^{[\lambda_2,\xi_2]}$.*

Definition 2.5 ([5, 18]). *Let X be a vector space over the field K, the field of real and complex numbers, $\alpha \in K$, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of X.Then*

(1) the sum of A and B is defined to be the intuitionistic fuzzy set $A+B = (\mu_A+\mu_B, \nu_A+\nu_B)$ of X given by

$$\mu_{A+B}(x) = \begin{cases} \underset{x=a+b}{sup}\,\{\mu_A(a) \wedge \mu_B(b)\} & if\ x = a+b \\ 0 & otherwise, \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \underset{x=a+b}{inf}\,\{\nu_A(a) \vee \nu_B(b)\} & if\ x = a+b \\ 1 & otherwise. \end{cases}$$

(2) αA is defined to be the IFS $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ of X, where

$$\mu_{\alpha A}(x) = \begin{cases} \mu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \underset{y\in X}{sup}\, \mu_A(y) & if\ \alpha = 0, x = \theta \\ 0 & if\ \alpha = 0, x \neq \theta, \end{cases}$$

$$\nu_{\alpha A}(x) = \begin{cases} \nu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \underset{y\in X}{inf}\, \nu_A(y) & if\ \alpha = 0, x = \theta \\ 1 & if\ \alpha = 0, x \neq \theta. \end{cases}$$

Definition 2.6 ([10]). *An IFS $V = (\mu_V, \nu_V)$ of a vector space X over the field K is said to be intuitionistic fuzzy vector space over X if*

(i) $V + V \subseteq V$

(ii) $\alpha V \subseteq V$, for every scalar α.

We denote the set of all intuitionistic fuzzy vector spaces over a vector space X by $IFVS(X)$.

Lemma 2.7 ([10]). *Let V be an intuitionistic fuzzy set in a vector space X. Then, the following are equivalent:*

(1) V is an intuitionistic fuzzy vector space over X.

(2) For all scalars α, β, we have $\alpha V + \beta V \subseteq V$.

(3) For all scalars α, β and for all $x, y \in X$, we have
$\mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y)\}$ and $\nu_V(\alpha x + \beta y) \leq \nu_V(x) \vee \nu_V(y)$.

Remark 2.8 ([10]). *Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [21] and [9].*

Proposition 2.9 ([10]). *Let $V \in IFVS(X)$. Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x), \forall x \in X$.*

Definition 2.10 ([10]). *For any $(a, b), (c, d) \in [0, 1] \times [0, 1]$ with $a + b \leq 1,\ c + d \leq 1$, we say that:*

(1) $(a, b) \geq (c, d)$ if $a \geq b$ and $c \leq d$.

(2) $(a, b) \leq (c, d)$ if $a \leq b$ and $c \geq d$.

(3) $(a, b) > (c, d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.

(4) $(a, b) < (c, d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.

(5) $(a, b) = (c, d)$ if $a = b$ and $c = d$.

Unless otherwise stated in the rest of the paper the collection of all intuitionistic fuzzy subsets of X is denoted by η^X, where $\eta = \{(k, m) \in [0, 1] \times [0, 1] : k + m \leq 1\}$.

Definition 2.11. *[12] An intuitionistic fuzzy topology on a non-empty set X is a family τ of intuitionistic fuzzy sets in X which satisfies the following conditions:*

(i) $0_\sim, 1_\sim \in \tau$;

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$;

(iii) If $A_i \in \tau$, for each $i \in I$, then $\underset{i \in I}{\cup} \in \tau$.

In this case (X, τ) is called an intuitionistic fuzzy topological space. The members of τ are called the intuitionistic fuzzy open sets and the complement A^C of an intuitionistic fuzzy open set in an intuitionistic fuzzy topological space (X, τ) is called an intuitionistic fuzzy closed set.

Definition 2.12 ([12])**.** *Let (X, τ) and (Y, δ) be two intuitionistic fuzzy topological spaces and let $f : X \to Y$ be a function. Then f is said to be an intuitionistic fuzzy continuous if the pre-image of each intuitionistic fuzzy set of δ is an intuitionistic fuzzy set in τ.*

Definition 2.13 ([22])**.** *Let $(r, s), (p, q) \in \eta$. Define*

(i) $(r, s) \sqcup (p, q) = (r \vee p, s \wedge q)$,

(ii) $(r, s) \sqcap (p, q) = (r \wedge p, s \vee q)$,

where $\vee$ and $\wedge$ are usual maximum and minimum inthe ordered set of real numbers.

Definition 2.14 ([22])**.** *Let X and Y be two non-empty sets and let $A \in \eta^X$ and $B \in \eta^Y$. An intuitionistic fuzzy subset F of $X \times Y$ is said to be an intuitionistic fuzzy proper function from the intuitionistic fuzzy set A to the intuitionistic fuzzy set B if*

(i) $F(x, y) \leq A(x) \sqcap B(y)$, for each $(x, y) \in X \times Y$.

(ii) for each $x \in X$, there exists a unique $y_0 \in Y$ such that $F(x, y_0) = A(x)$ and $F(x, y) = (0, 1)$, if $y \neq y_0$.

Henceforth $F : A \to B$ implies F is an intuitionistic fuzzy proper function from $A \in \eta^X$ into $B \in \eta^Y$.

Definition 2.15 ([22])**.** *Let $F : A \to B$. If $U \subset A$ and $V \subset B$, then $F^{-1}(V) \subset A$ and $F(U) \subset B$ are defined by*

(i) $F^{-1}(V)(x) = \underset{s \in Y}{\sqcup}\{F(x, s) \sqcap V(s)\},\ \forall x \in X.$

(ii) $F(U)(y) = \underset{t \in X}{\sqcup}\{F(t, y) \sqcap U(t)\},\ \forall y \in Y.$

Lemma 2.16 ([22])**.** *Let $F : A \to B$ be an intuitionistic fuzzy proper function. If $V \subset B$, then $F^{-1}(V)(x) = A(x) \sqcap V(y)$, where $y \in Y$ is unique such that $F(x, y) = A(x)$.*

Lemma 2.17 ([22])**.** *Let $F : A \to B$ be an intuitionistic fuzzy proper function and $U \subset A$, $V \subset B$. Then $F(F^{-1}(V)) \subset V$ and $U \subset F^{-1}(F(U))$.*

Definition 2.18 ([22]). *$A \times B : X \times Y \to \eta$ is defined by $A \times B(x,y) = A(x) \sqcap B(y)$, $\forall (x,y) \in X \times Y$.*

Definition 2.19 ([22]). *The intuitionistic fuzzy proper function $p_A : A \times B \to A$ defined by $p_A((x,y),z) = (A \times B)(x,y)$ or $(0,1)$ accordingly as $z = x$ or $z \neq x$ $\forall x, z \in X$ and $\forall y \in Y$ is said to be the intuitionistic fuzzy projection map of $A \times B$ into A. Similarly, the intuitionistic fuzzy projection map $p_B : A \times B \to B$ is defined.*

Lemma 2.20 ([22]). *Let $U \subset A$, $V \subset B$. Then $p_A^{-1}(U) = U \times B$, $p_B^{-1}(V) = A \times V$.*

3 Intuitionistic fuzzy topology on intuitionistic fuzzy set

Definition 3.1. *An intuitionistic fuzzy set A of X is said to be constant which will be denoted by $(k,m)_\sim$ and defined by $\mu_A(x) = k$ and $\nu_A(x) = m$, $(k,m) \in \eta$, $\forall x \in X$.*

Definition 3.2. *Let A be a intuinistic fuzzy subset of X. A collection τ of intuitionistic fuzzy subsets of A satisfying*

(i) $(k,m)_\sim \cap A \in \tau, \forall (k,m) \in \eta$;

(ii) *If $A, B \in \tau$, then $A \cap B \in \tau$;*

(iii) *If $A_i \in \tau$, for each $i \in I$, then $\underset{i \in I}{\cup} \in \tau$.*

is called an intuitionistic fuzzy topology or IF topology on the intuitionistic fuzzy set A. The pair (A, τ) is called an intuitionistic fuzzy topological space. Members of τ will be called intuitionistic fuzzy open sets.

Unless otherwise mentioned by an intuitionistic fuzzy topological space we shall mean it in the sense of Definition 3.2 and (A, τ) will denote an intuitionistic fuzzy topological space.

Proposition 3.3. *If $\mathcal{B}$ be a given collection of intuitionistic fuzzy subsets of an intuitionistic fuzzy set A and the family $\{(k,m)_\sim \cap A \in \tau, (k,m) \in \eta\}$, then the family of all possible unions and finite intersections of the members of $\mathcal{B}$ is an intuitionistic fuzzy topology on A and it will be denoted by $\tau(\mathcal{B})$.*

Definition 3.4. *$\mathcal{B} \subset \tau$ is called an open base of τ if every member of τ can be expressed as a union of some members of $\mathcal{B}$.*

Definition 3.5. *An intuitionistic fuzzy proper function $F : A \to B$ is said to be*

(i) *injective if $F(x_1, y) = A(x_1)(\neq (0,1))$, $F(x_2, y) = A(x_2)(\neq (0,1)) \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in X$, $y \in Y$;*

(ii) *surjective if $\forall y \in Y$ with $B(y) \neq (0,1)$, $\exists\, x \in X$ such that $F(x,y) = A(x) = B(y)$;*

(iii) *bijective if F is both injective and surjective.*

Proposition 3.6. *If $F : A \to B$ is injective, then for all $V \subseteq A$, $F^{-1}(F(V)) = V$.*

Proof. Let $x \in X$ and y be unique such that $F(x, y) = A(x)$.
Then $[F^{-1}(F(V))](x) = \bigsqcup_{s \in Y} \{F(x, s) \sqcap F(V)(s)\}$
$= F(x, y) \sqcap F(V)(y)$
$= A(x) \sqcap \bigsqcup_{t \in X} \{F(t, y) \sqcap V(y)\}$
$= A(x) \sqcap A(x) \sqcap V(x)$ [Since F is injective]
$= V(x)$. □

Proposition 3.7. *If $F : A \to B$ is surjective, then $F(V) = W$ and for all $W \subseteq B$, $F(F^{-1}(W)) = W$.*

Proof. For any $y \in Y$ with $B(y) = (0, 1)$ implies $F(A)(y) = (0, 1)$. Hence for those $y \in Y$, $F(A)(y) = B(y)$.
For any $y \in Y$ with $B(y) \neq (0, 1)$,
$F(A)(y) = \bigsqcup_{t \in X} \{F(t, y) \sqcap A(t)\}$
$= \sqcup\{A(x) : x \in X \ with \ F(x, y) = A(x) = B(y)\}$ [Since F is surjective]
$= B(y)$.
Hence $F(A) = B$.
For any $W \subseteq B$, $F(F^{-1}(W))(y) = \bigsqcup_{x \in X} \{F(x, y) \sqcap F^{-1}(W)(x)\}$
$= \sqcup\{A(x) \sqcap F^{-1}(W)(x) : x \in X, F(x, y) = A(x) = B(y)\}$, [since F is surjective]
$= \sqcup\{F^{-1}(W)(x) : x \in X, F(x, y) = A(x) = B(y)\}$
$\sqcup\{A(x) \sqcap W(y) : x \in X, F(x, y) = A(x) = B(y)\}$, [By Proposition 2.16]
$\sqcup\{B(y) \sqcap W(y) : x \in X, F(x, y) = A(x) = B(y)\}$
$= W(y)$.
Hence $F(F^{-1}(W)) = W$. □

Proposition 3.8. *Let $F : A \to B$ be an intuitionistic fuzzy proper function. If $V, W \subseteq A$, then*

(i) $V \subseteq W \implies F(V) \subseteq F(W)$.

(ii) $F(V \cup W) = F(V) \cup F(W)$.

(iii) $F(V \cap W) \subseteq F(V) \cap F(W)$.

(iv) $F(V \cap W) = F(V) \cap F(W)$, *if F is injective.*

Proof. (i) is obvious.

(ii) For any $y \in Y$,
$F(V \cup W)(y) = \bigsqcup_{x \in X} \{F(x, y) \sqcap (V \cup W)(x)\}$
$= \sqcup\{A(x) \sqcap (V \cup W)(x) : x \in X \ such \ that \ F(x, y) = A(x)\}$
$= \sqcup\{[A(x) \sqcap V(x)] \sqcup [A(x) \sqcap W(x)] : x \in X \ such \ that \ F(x, y) = A(x)\}$
$= [\sqcup\{[A(x) \sqcap V(x)] : x \in X \ such \ that \ F(x, y) = A(x)\}] \sqcup [\sqcup\{[A(x) \sqcap W(x)] : x \in X \ such \ that \ F(x, y) = A(x)\}]$

$= [F(V) \cup F(W)](y)$.
Therefore, $F(V \cup W) = F(V) \cup F(W)$.

(iii) For any $y \in Y$,
$[F(V) \cap F(W)](y) = [\sqcup\{[A(x) \sqcap V(x)] : x \in X\ such\ that\ F(x,y) = A(x)\}] \sqcap [\sqcup\{[A(x) \sqcap W(x)] : x \in X\ such\ that\ F(x,y) = A(x)\}]$
$[\sqcup\{V(x) : x \in X\ such\ that\ F(x,y) = A(x)\}] \sqcap [\sqcup\{W(x) : x \in X\ such\ that\ F(x,y) = A(x)\}]$
$\geq \sqcup\{V(x) \sqcap W(x) : x \in X\ such\ that\ F(x,y) = A(x)\}$
$= \sqcup\{A(x) \sqcap [V(x) \sqcap W(x)] : x \in X\ such\ that\ F(x,y) = A(x)\}$
$= F(V \cap W)(y)$.
Therefore, $F(V \cap W) \subseteq F(V) \cap F(W)$.

(iv) If F is injective, for any $y \in Y$,
$[F(V) \cap F(W)](y) = V(x) \sqcap W(x)$, for $x \in X$ unique such that $F(x,y) = A(x)$
$= F(V \cap W)(y)$.
Hence $F(V \cap W) = F(V) \cap F(W)$. □

Proposition 3.9. *Let $F : A \to B$ be an intuitionistic fuzzy proper function. If $V, W \subseteq B$, then*

(i) $V \subseteq W \implies F^{-1}(V) \subseteq F^{-1}(W)$.

(ii) $F^{-1}(V \cup W) = F^{-1}(V) \cup F^{-1}(W)$.

(iii) $F^{-1}(V \cap W) = F^{-1}(V) \cap F^{-1}(W)$.

Proof. (i) is obvious.

(ii) For any $x \in X$,
$F^{-1}(V \cup W)(x) = \underset{y \in Y}{\sqcup}\{F(x,y) \sqcap (V \cup W)(y)\}$
$= A(x) \sqcap (V \cup W)(y)$, for $y \in Y$ unique such that $F(x,y) = A(x)$;
$= [A(x) \sqcap V(y)] \sqcup [A(x) \sqcap W(y)]$, for $y \in Y$ unique such that $F(x,y) = A(x)$;
$= [F^{-1}(V) \cup F^{-1}(W)](x)$.
Therefore, $F^{-1}(V \cup W) = F^{-1}(V) \cup F^{-1}(W)$.

(iii) For any $x \in X$,
$F^{-1}(V \cap W)(x) = \underset{y \in Y}{\sqcup}\{F(x,y) \sqcap (V \cap W)(y)\}$
$= A(x) \sqcap (V \cap W)(y)$, for $y \in Y$ unique such that $F(x,y) = A(x)$;
$= [A(x) \sqcap V(y)] \sqcap [A(x) \sqcap W(y)]$, for $y \in Y$ unique such that $F(x,y) = A(x)$;
$= [F^{-1}(V) \cap F^{-1}(W)](x)$.
Therefore, $F^{-1}(V \cap W) = F^{-1}(V) \cap F^{-1}(W)$. □

Definition 3.10. *The intuitionistic fuzzy proper function $I_A : A \to A$ defined by $I_A(x,y) = A(x)$ or $(0,1)$ according as $y = x$ or $y \neq x$ is said to be the identity proper function on A.*

Definition 3.11. *If $F : A \to B$ and $G : B \to C(C \in \eta^Z)$ are intuitionistic fuzzy proper functions, then the intuitionistic fuzzy proper function $G \circ F : A \to C$ is defined by* $G \circ F(x, z) = \begin{cases} A(x), & if \ \exists \ y \ such \ that \ F(x, y) = A(x), \ G(y, z) = B(y) \\ (0, 1), & otherwise. \end{cases}$

Definition 3.12. *An intuitionistic fuzzy proper function $G : B \to A$ is called an inverse of a bijective proper function $F : A \to B$ if $G \circ F = I_A$ and $F \circ G = I_B$.*
Therefore for a bijective intuitionistic fuzzy proper function $F : A \to B$ defined as in 2.14, its inverse $G : B \to A$ is defined by

(i) $G(y, x) \leq B(y) \sqcap A(x)$;

(ii) for each $y \in Y$, there is unique $x \in X$ such that $G(y, x) = B(y)$ for $F(x, y) = A(x)$ and $G(y, x) = (0, 1)$ otherwise.

Definition 3.13. *An intuitionistic fuzzy proper function $F : (A, \tau) \to (B, \tau_1)$ is said to be*

(i) intuitionistic fuzzy continuous if $F^{-1}(V) \in \tau, \forall V \in \tau_1$,

(ii) intuitionistic fuzzy open if $F(U) \in \tau_1, \forall U \in \tau$,

(iii) intuitionistic fuzzy homeomorphism if F is bijective, intuitionistic fuzzy continuous and inverse of F is also intuitionistic fuzzy continuous.

Proposition 3.14. *Let $A \in \eta^X$, $B \in \eta^Y$ and $C \in \eta^Z$. If $F : (A, \tau) \to (B, \tau_1)$ and $G : (B, \tau_1) \to (C, \tau_2)$ are intuitionistic fuzzy continuous proper functions, then the intuitionistic fuzzy proper function $G \circ F : (A, \tau) \to (C, \tau_2)$ as defined in 3.11 is also intuitionistic fuzzy continuous.*

Proof. Let $C_1 \subseteq C$. Now, $[(G \circ F)^{-1}(C_1)](x) = \bigsqcup_{s \in Z}[(G \circ F)(x, s) \sqcap C_1(s)]$
$= \begin{cases} A(x) \sqcap C_1(s_1), & if \ \exists \ y \in Y, s_1 \in Z \ such \ that \ F(x, y) = A(x) \ and \ G(y, s_1) = B(y) \\ (0, 1), & otherwise. \end{cases}$
Again $[G^{-1}(C_1)](y) = \bigsqcup_{s \in Z}[G(y, s) \sqcap C_1(s)]$
$= B(y) \sqcap C_1(s_y)$, where $s_y \in Y$ unique such that $G(y, s_y) = B(y)$.
Thus $[F^{-1}(G^{-1}(C_1))](x) = \bigsqcup_{t \in Y}[F(x, t) \sqcap G^{-1}(C_1)(t)]$
$= \bigsqcup_{t \in Y}[F(x, t) \sqcap B(t) \sqcap G^{-1}(C_1)(s_t)]$, where $s_t \in Y$ unique such that $G(t, s_t) = B(t)$.
$= \begin{cases} A(x) \sqcap C_1(s_{t'}), & if \ \exists \ t' \in Y, \ s_{t'} \in Z \ such \ that \ F(x, t') = A(x) \ and \ G(t', s_{t'}) = B(t') \\ (0, 1), & otherwise. \end{cases}$
Hence $(G \circ F)^{-1}(C_1) = F^{-1}(G^{-1}(C_1))$. Since G and F are intuitionistic fuzzy continuous, for any $C_1 \in \tau_2, \ G^{-1}(C_1) \in \tau_1$ and $F^{-1}(G^{-1}(C_1)) \in \tau$. Hence $G \circ F$ is intuitionistic fuzzy continuous. □

Definition 3.15. *An element $a \in X$ is called a normal element of A with respect to B if $A(a) \geq B(y), \forall y \in Y$.*

Lemma 3.16. *If (A, τ) and (B, τ_1) are intuitionistic fuzzy topological spaces and $'a'$ be a normal element of B with respect to A, then the intuitionistic fuzzy proper function $F : (A, \tau) \to (B, \tau_1)$ defined by* $F(x, y) = \begin{cases} A(x) & if\ y = a \\ (0, 1) & if\ y \neq a \end{cases}$
is intuitionistic fuzzy continuous.

Proof. Let $V \in \tau_1$. Then $\forall x \in X$, $F^{-1}(V)(x) = \bigsqcup_{y \in Y} \{F(x, y) \sqcap V(y)\} = A(x) \sqcap V(a)$, [By definition].
Therefore $F^{-1}(V) = A \cap (k, m)_\sim \in \tau$, where $(k, m) = V(a)$. Hence proved. □

Lemma 3.17. *$U \subset A \in \eta^X$, $V \subset B \in \eta^Y$. Then $p_A(U \times V) = U \cap (k, m)_\sim$, where $(k, m) = sup\{V(y) : y \in Y\}$ and $p_B(U \times V) = V \cap (k_1, m_1)_\sim$, where $(k_1, m_1) = sup\{U(x) : x \in X\}$.*

Proof. For any $z \in X$,
$p_A(U \times V)(z)$
$= \bigsqcup_{(x,y) \in X \times Y} \{p_A((x, y), z) \sqcap (U \times V)(x, y)\}$
$= \bigsqcup_{y \in Y} \{(A \times B)(z, y) \sqcap (U \times V)(z, y)\}$
$= \bigsqcup_{y \in Y} \{(U \times V)(z, y)\}$
$= U(z) \sqcap \bigsqcup_{y \in Y} \{V(y)\}$.
Hence $p_A(U \times V) = U \cap (k, m)_\sim$, where $(k, m) = sup\{V(y) : y \in Y\}$.
Similarly it can be proved that $p_B(U \times V) = V \cap (k_1, m_1)_\sim$, where $(k_1, m_1) = sup\{U(x) : x \in X\}$. □

Remark 3.18. *$p_A(A \times B)$ (or $p_B(A \times B)$) may not be equal to A (or B). However, if there exists a normal element of B (or A) with respect to A (or B), then $p_A(A \times B) = A$ (or $p_B(A \times B) = B$).*

Proposition 3.19. *The collection $\mathcal{B} = \{U \times V : U \in \tau, V \in \tau_1\}$ forms an open base of an intuitionistic fuzzy topology on $A \times B$.*

Proof. For $(k, l) \in \eta$, $(k, l)_\sim \cap (A \times B) = ((k, l)_\sim \cap A) \times ((k, l)_\sim \cap B) \in \tau \times \tau_1$. Hence $(k, l)_\sim \cap (A \times B) \in \mathcal{B}$, for all $(k, l) \in \eta$.
Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$. Then $U_1, U_2 \in \tau$, $V_1, V_2 \in \tau_1$.
Now $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$.
Therefore $\mathcal{B}$ forms an open base for an intutionistic fuzzy topology on $A \times B$. □

Definition 3.20. *The intutionistic fuzzy topology in $A \times B$ induced by $\mathcal{B} = \{U \times V : U \in \tau, V \in \tau_1\}$ is called the product intuitionistic fuzzy topology of τ and τ_1 and is denoted by $\tau \times \tau_1$. The intuitionistic fuzzy topological space $(A \times B, \tau \times \tau_1)$ is called the product of the intuitionistic fuzzy topological spaces (A, τ) and (B, τ_1).*

Theorem 3.21. *$p_A : (A \times B, \tau \times \tau_1) \to (A, \tau)$ and $p_B : (A \times B, \tau \times \tau_1) \to (B, \tau_1)$ are intuitionistic fuzzy continuous and intuitionistic fuzzy open. $\tau \times \tau_1$ is the smallest intuitionistic fuzzy topology in $A \times B$ with respect to which p_A and p_B are intuitionistic fuzzy continuous.*

Proof. The p_A and p_B are intuitionistic fuzzy continuous and open follows from Lemma 2.20 and 3.17. That $\tau \times \tau_1$ is the smallest intuitionistic fuzzy topology in $A \times B$ with respect to which p_A and p_B are intuitionistic fuzzy continuous follows from the fact that if $U \in \tau$, $V \in \tau_1$, then $U \times V = p_A^{-1}(U) \cap p_B^{-1}(V)$. □

Lemma 3.22. *If 'a' be a normal element of B with respect to A, then the intuitionistic fuzzy proper function $F_a : (A, \tau) \to (A \times B, \tau \times \tau_1)$ defined by*

$$F_a(x, (x_1, y_1)) = \begin{cases} A(x) & if\ (x_1, y_1) = (x, a) \\ (0, 1) & if\ (x_1, y_1) \neq (x, a) \end{cases}$$

is intuitionistic fuzzy continuous.

Proof. Let $V \times V' \in \tau \times \tau_1$ and $x \in X$. Then
$F_a^{-1}(V \times V')(x) = \bigsqcup_{(x_1,y_1) \in X \times Y} \{F_a(x, (x_1, y_1)) \sqcap (V \times V')(x_1, y_1)\}$
$= A(x) \sqcap (V \times V')(x, a)$
$= A(x) \sqcap (k, m) \sqcap V(x)$, [where $V'(a) = (k, m)$].
Therefore $F_a^{-1}(V \times V') = A \cap (k, m)_{\sim} \cap V \in \tau$. Hence F_a is intuitionistic fuzzy continuous. □

Similarly we have,

Lemma 3.23. *If 'a' be a normal element of B with respect to A, then the intuitionistic fuzzy proper function $F_a : (A, \tau) \to (B \times A, \tau_1 \times \tau)$ defined by*

$$F_a(x, (y_1, x_1)) = \begin{cases} A(x) & if\ (y_1, x_1) = (a, x) \\ (0, 1) & if\ (y_1, x_1) \neq (a, x) \end{cases}$$

is intuitionistic fuzzy continuous.

Theorem 3.24. *Let (A_i, τ_i) and (B_i, σ_i), $i = 1, 2$ be intuitionistic fuzzy topological spaces and $F_i : (A_i, \tau_i) \to (B_i, \sigma_i)$, $i = 1, 2$ be intuitionistic fuzzy continuous proper functions, where $A_i \in \eta^{X_i}, B_i \in \eta^{Y_i}$. Then for each $i = 1, 2$, $x_i \in X_i$, $\exists$ unique $y_{i_0} \in Y_i$ such that $F_i(x_i, y_{i_0}) = A_i(x_i)$ and $F_i(x_i, y_i) = (0, 1)$ if $y_i \neq y_{i_0}$. Now if we define the proper function $F = F_1 \times F_2 : A_1 \times A_2 \to B_1 \times B_2$ by*

$$F((x_1, x_2), (y_1, y_2)) = \begin{cases} (A_1 \times A_2)(x_1, x_2) & if\ (y_1, y_2) = (y_{1_0}, y_{2_0}) \\ (0, 1) & if\ (y_1, y_2) \neq (y_{1_0}, y_{2_0}) \end{cases}$$

is also intuitionistic fuzzy continuous.

Proof. Let $U \times V \in \sigma_1 \times \sigma_2$. Then $\forall (x_1, x_2) \in X_1 \times X_2$,
$(F_1 \times F_2)^{-1}(U \times V)(x_1, x_2)$
$= \bigsqcup_{(y_1,y_2) \in Y_1 \times Y_2} \{F((x_1, x_2), (y_1, y_2)) \sqcap (U \times V)((y_1, y_2))\}$
$= [(A_1 \times A_2)(x_1, x_2)] \sqcap [(U \times V)(y_{1_0}, y_{2_0})]$
$= (A_1(x_1) \sqcap U(y_{1_0})) \times (A_2(x_2) \sqcap V(y_{2_0}))$.
Now, $(F_1^{-1}(U) \times F_2^{-1}(V))(x_1, x_2)$
$= \bigsqcup_{y_1 \in Y_1} \{F_1(x_1, y_1) \sqcap U(y_1)\} \sqcap \bigsqcup_{y_2 \in Y_2} \{F_2(x_2, y_2) \sqcap V(y_2)\}$
$= (A_1(x_1) \sqcap U(y_{1_0})) \times (A_2(x_2) \sqcap V(y_{2_0}))$.
Therefore $(F_1 \times F_2)^{-1}(U \times V) = F_1^{-1}(U) \times F_2^{-1}(V)$. Since F_1 and F_2 are intuitionistic fuzzy

continuous, $F_1^{-1}(U) \times F_2^{-1}(V)$ is intuitionistic fuzzy open set in $\tau_1 \times \tau_2$. Hence $F = F_1 \times F_2$ is intuitionistic fuzzy continuous. □

Theorem 3.25. *Let (A_i, τ_i) and (B_i, σ_i), $i = 1, 2$ be intuitionistic fuzzy topological spaces and $F_i : (A_i, \tau_i) \to (B_i, \sigma_i)$, $i = 1, 2$ be injective intuitionistic fuzzy open proper functions, where $A_i \in \eta^{X_i}, B_i \in \eta^{Y_i}$. Then for each $i = 1, 2$, $x_i \in X_i$, $\exists$ unique $y_{i_0} \in Y_i$ such that $F_i(x_i, y_{i_0}) = A_i(x_i)$ and $F_i(x_i, y_i) = (0, 1)$ if $y_i \neq y_{i_0}$. Now if we define the proper function $F = F_1 \times F_2 : A_1 \times A_2 \to B_1 \times B_2$ by*

$$F((x_1, x_2), (y_1, y_2)) = \begin{cases} (A_1 \times A_2)(x_1, x_2) & if\ (y_1, y_2) = (y_{1_0}, y_{2_0}) \\ (0, 1) & if\ (y_1, y_2) \neq (y_{1_0}, y_{2_0}) \end{cases}$$

is also intuitionistic fuzzy open.

Proof. Let $U \times V \in \tau_1 \times \tau_2$. Then $\forall (y_1, y_2) \in Y_1 \times Y_2$,
$(F_1 \times F_2)(U \times V)(y_1, y_2)$
$= \bigsqcup_{(x_1,x_2) \in (X_1 \times X_2)} \{F((x_1, x_2), (y_1, y_2)) \sqcap (U \times V)(x_1, x_2)\}$
$= (A_1 \times A_2)(x_1, x_2) \sqcap (U \times V)(x_1, x_2)$, for $(x_1, x_2) \in X_1 \times X_2$ such that $F((x_1, x_2), (y_1, y_2)) = (A_1 \times A_2)(x_1, x_2)$
$= (U \times V)(x_1, x_2)$, for $(x_1, x_2) \in X_1 \times X_2$ such that $F((x_1, x_2), (y_1, y_2)) = (A_1 \times A_2)(x_1, x_2)$.
Again, $[F_1(U) \times F_2(V)](y_1, y_2)$
$= [\bigsqcup_{x_1 \in X_1} \{F_1(x_1, y_1) \sqcap U(x_1)\}] \sqcap [\bigsqcup_{x_2 \in X_2} \{F_2(x_2, y_2) \sqcap V(x_2)\}]$
$= [A_1(x_1) \sqcap U(x_1)] \sqcap [A_2(x_2) \sqcap V(x_2)]$, for $x_1 \in X_1$ such that $F_1(x_1, y_1) = A_1(x_1)$ and $x_2 \in X_2$ such that $F_2(x_2, y_2) = A_2(x_2)$
$= U(x_1) \sqcap V(x_2)$, for $x_1 \in X_1$ such that $F_1(x_1, y_1) = A_1(x_1)$ and $x_2 \in X_2$ such that $F_2(x_2, y_2) = A_2(x_2)$
$= (U \times V)(x_1, x_2)$, for $(x_1, x_2) \in X_1 \times X_2$ such that $F((x_1, x_2), (y_1, y_2)) = (A_1 \times A_2)(x_1, x_2)$.
Therefore $(F_1 \times F_2)(U \times V) = F_1(U) \times F_2(V) \in \sigma_1 \times \sigma_2$.
Hence $F = F_1 \times F_2$ is intuitionistic fuzzy open. □

4 IF topological vector space

Definition 4.1 ([17]). *Given a topological space (X, τ), the collection $\omega(\tau)$, of all fuzzy sets in X which are lower semi-continuous, as functions from X to the unit interval equipped with the usual topology, is a fuzzy topology on X. This fuzzy topology $\omega(\tau)$ is said to be the fuzzy topology generated by the topology τ.*

Definition 4.2 ([16]). *Let $\mathbb{K}$ be the field of real or complex numbers. Then the fuzzy usual topology on $\mathbb{K}$ is the fuzzy topology generated by the usual topology on $\mathbb{K}$.*

Throughout the section we consider V as an intuitionistic fuzzy vector space associated with a vector space X and the ground field $\mathbb{K}$. We consider $\mathbb{K}$ to be equipped with the fuzzy usual topology ν as defined in Definition 4.2.

Definition 4.3. *Let X be a vector space over the field $\mathbb{K}$ with θ as the null vector. Let V be an intuitionistic fuzzy vector space over X, $a \in X$ and $k_0 \in \mathbb{K}$ be fixed. Let us define the intuitionistic fuzzy proper functions*

$F^{\oplus} : V \times V \to V$ *by* $F^{\oplus}((x,y),z) = \begin{cases} (V \times V)(x,y) & if\ x+y=z \\ (0,1) & if\ x+y \neq z, \end{cases}$

$F^{a} : V \to V$ *by* $F^{a}(x,y) = \begin{cases} V(x) & if\ y = a+x \\ (0,1) & if\ y \neq a+x \end{cases};$

$F^{\odot} : (\mathbb{K} \times V) \to V$ *by* $F^{\odot}((k,x),y) = \begin{cases} \mathbb{K} \times V(k,x) & if\ kx=y, k \neq 0 \\ \sup_{x \in X} V(x) & if\ kx = y, k = 0 \\ (0,1) & if\ kx \neq y \end{cases};$

$F^{k_0} : V \to V$ *by* $F^{k_0}(x,y) = \begin{cases} V(x) & if\ y = k_0 x, k_0 \neq 0 \\ \sup_{x \in X} V(x) & if\ k_0 x = y, k_0 = 0 \\ (0,1) & if\ y \neq k_0 x \end{cases};$

$F_V^{L_{(k,m)}} : V \times V \to V$ *by*

$$F_V^{L_{(k,m)}}((x,y),z) = \begin{cases} (V \times V)(x,y) & if\ kx+my=z, k \neq 0, m \neq 0 \\ V(x) & if\ z = kx, k \neq 0, m = 0 \\ V(y) & if\ z = my, k = 0, m \neq 0 \\ \sup_{s \in X} V(s) & if\ kx+my=z, k=0, m=0 \\ (0,1) & if\ kx+my \neq z, \end{cases}$$

for all $x, y, z \in X$, $k, m \in \mathbb{K}$.

Definition 4.4. *An intuitionistic fuzzy topology τ on V is called an IF vector topology if the intuitionistic fuzzy proper functions $F^{\oplus} : (V \times V, \tau \times \tau) \to (V, \tau)$ and $F^{\odot} : (\mathbb{K} \times V, \tau \times \nu) \to (V, \tau)$ are intuitionistic fuzzy continuous. The pair (V, τ) is said to be an IF topological vector space if τ is an IF vector topology on V.*

Remark 4.5. *Here we use the term IF topological vector space as there is a notion of intuitionistic fuzzy topological vector space in [18] where the intuitionistic fuzzy topology is in the sense of Coker and the underlying vector space is crisp vector space.*

Proposition 4.6. *An intuitionistic fuzzy topology τ on V is an IF vector topology if and only if the intuitionistic fuzzy proper function $F_V^{L_{(k,m)}} : (V \times V, \tau \times \tau) \to (V, \tau)$ is intuitionistic fuzzy continuous.*

Proof. Let τ be an IF vector topology on V and $k, m \in \mathbb{K}$. Since $k \in \mathbb{K}$ is normal element of $\mathbb{K}$ with respect to V, by Lemma 3.23, the intuitionistic fuzzy proper function $F_k : (V, \tau) \to (\mathbb{K} \times V, \nu \times \tau)$ defined by $F_k(x, (k_1, x_1)) = \begin{cases} V(x), & if\ (k_1, x_1) = (k, x) \\ (0,1), & if\ (k_1, x_1) \neq (k, x) \end{cases}$
is intuitionistic fuzzy continuous.

Also, by definition of IF vector topology, $F^{\odot} : (\mathbb{K} \times V, \nu \times \tau) \to (V, \tau)$ is intuitionistic fuzzy continuous.
Hence by Proposition 3.14, $F^{\odot} \circ F_k : (V, \tau) \to (V, \tau)$ defined by

$$F^{\odot} \circ F_k(x, y) = \begin{cases} V(x) & if\ y = kx, k \neq 0 \\ \sup\limits_{s \in X} V(s) & if\ y = kx, k = 0 \\ (0, 1) & otherwise \end{cases}$$

is intuitionistic fuzzy continuous.
Similarly, $F^{\odot} \circ F_m : (V, \tau) \to (V, \tau)$ defined by

$$F^{\odot} \circ F_k(z, t) = \begin{cases} V(z) & if\ t = mx, m \neq 0 \\ \sup\limits_{s \in X} V(s) & if\ t = mz, m = 0 \\ (0, 1) & otherwise \end{cases}$$

is intuitionistic fuzzy continuous.
Thus by Theorem 3.24, $(F^{\odot} \circ F_k) \times (F^{\odot} \circ F_m) : (V \times V, \tau \times \tau) \to (V \times V, \tau \times \tau)$ defined by

$$(F^{\odot} \circ F_k) \times (F^{\odot} \circ F_m)((x, z), (y, t)) = \begin{cases} (V \times V)(x, z) & if\ (x, z) = (y, t) \\ (0, 1) & if\ (x, z) \neq (y, t) \end{cases}$$

is intuitionistic fuzzy continuous. Therefore by Proposition 3.14, $F^{\oplus} \circ [(F^{\odot} \circ F_k) \times (F^{\odot} \circ F_m)] = F_V^{L_{(k,m)}}$ is intuitionistic fuzzy continuous.
Conversely, let $F_V^{L_{(k,m)}}$ is intuitionistic fuzzy continuous for all $k, m \in \mathbb{K}$.
We know that the projection mapping $p_V : (\mathbb{K} \times V, \nu \times \tau) \to (V, \tau)$ defined by

$$p_V((k, x), z) = \begin{cases} (\mathbb{K} \times V)(k, x) & if\ z = x \\ (0, 1) & if\ z \neq x \end{cases}$$

and since θ is normal of V with respect to V, by Lemma 3.22,
$F_\theta : (V, \tau) \to (V \times V, \tau \times \tau)$ defined by

$$F_\theta(x, (x_1, y_1)) = \begin{cases} V(x) & if\ (x_1, y_1) = (x, \theta) \\ (0, 1) & if\ (x_1, y_1) \neq (x, \theta) \end{cases}$$

are intuitionistic fuzzy continuous proper functions.
Hence by Proposition 3.14, $F_\theta \circ p_V : (\mathbb{K} \times V, \nu \times \tau) \to (V \times V, \tau \times \tau)$ defined by

$$F_\theta \circ p_V((k, x), (x_1, y_1)) = \begin{cases} (\mathbb{K} \times V)(k, x) & if\ (x_1, y_1) = (x, \theta) \\ (0, 1) & if\ (x_1, y_1) \neq (x, \theta) \end{cases}$$

is intuitionistic fuzzy continuous.
Therefore $F^{\odot} = (F^{L_{(k,0)}} \circ F_\theta \circ p_V) : (\mathbb{K} \times V, \nu \times \tau) \to (V, \tau)$, where

$$(F^{L_{(k,0)}} \circ F_\theta \circ p_V)((k, x), z) = \begin{cases} (\mathbb{K} \times V)(k, x) & if\ z = kx, k \neq 0 \\ \sup\limits_{s \in X} V(s) & if\ z = kx, k = 0 \\ (0, 1) & if\ z \neq kx \end{cases}$$

is intuitionistic fuzzy continuous.
Since $F^{L_{(k,m)}}$ is intuitionistic fuzzy continuous for all $k, m \in \mathbb{K}$, taking $k = 1, m = 1$ we have $F^{\oplus} : (V \times V, \tau \times \tau) \to (V, \tau)$ is intuitionistic fuzzy continuous. Hence proved. □

Proposition 4.7. *If (V, τ) is an IF topological vector space, then F^k is an intuitionistic fuzzy homeomorphism of (V, τ) onto itself, for all $k(\neq 0) \in \mathbb{K}$.*

Proof. Since (V, τ) is an IF topological vector space, $F^{\odot} : (\mathbb{K} \times V, \nu \times \tau) \to (V, \tau)$ is intuitionistic fuzzy continuous.
Also, by Lemma 3.23, the intuitionistic fuzzy proper function $F_k : (V, \tau) \to (\mathbb{K} \times V, \nu \times \tau)$ defined by $F_k(x, (k_1, x_1)) = \begin{cases} V(x), & if\ (k_1, x_1) = (k, x) \\ (0, 1), & if\ (k_1, x_1) \neq (k, x) \end{cases}$
is intuitionistic fuzzy continuous.
Hence for $k \neq 0$, $F^{\odot} \circ F_k = F^k : (V, \tau) \to (V, \tau)$ defined by
$F^k(x, y) = \begin{cases} V(x) & if\ y = kx, \\ (0, 1) & if\ y \neq kx \end{cases}$
is intuitionistic fuzzy continuous.
Similarly, for $k \neq 0$, the intuitionistic fuzzy proper function $(F^k)^{-1} : (V, \tau) \to (V, \tau)$ defined by
$(F^k)^{-1}(x, y) = \begin{cases} V(x) & if\ y = \frac{1}{k}x, \\ (0, 1) & if\ y \neq \frac{1}{k}x \end{cases}$
is intuitionistic fuzzy continuous.
Also $F^k \circ (F^k)^{-1} = I_V = (F^k)^{-1} \circ F^k$. Hence F^k is an intuitionistic fuzzy homeomorphism of (V, τ) onto itself for $k(\neq 0) \in \mathbb{K}$. □

Proposition 4.8. *If (V, τ) is an IF topological vector space and if $'a'$ is a normal element of V with respect to V, then F^a is an intuitionistic fuzzy homeomorphism of (V, τ) onto itself.*

Proof. If $'a'$ is a normal element of V with respect to V, then $F^a = F^{\oplus} \circ F_a$ is intuitionistic fuzzy continuous by continuity of $F^{\oplus}$ and F_a. Also, inverse of F^a is F^{-a} and hence F^{-a} is also intuitionistic fuzzy continuous. Therefore F^a is intuitionistic fuzzy homeomorphism from (V, τ) into itself for any normal $'a'$ of V with respect to V. □

Let V and W be two intuitionistic fuzzy vector spaces in two vector spaces X and Y respectively and θ, θ' be the null vectors of X and Y respectively.

Definition 4.9. *An intuitionistic fuzzy proper function $F : V \to W$ is said to be an intuitionistic fuzzy linear transformation if*

(i) if $F(\theta, \theta') = \underset{(x,y)\in(X\times Y)}{sup} F(x, y)$,

(ii) $F(kx, ky) = \begin{cases} F(x, y) & if\ k \neq 0 \\ \underset{(x,y)\in(X\times Y)}{sup} F(x, y) & if\ k = 0, \end{cases}$

(iii) if $F(kx, ky) = V(kx)$ *and* $F(mz, mw) = V(mz)$ *imply* $F(kx + mz, ky + mw) = V(kx + mz)$,

for all $x, z \in X$, $y, w \in Y$ *and* $k, m \in \mathbb{K}$.

Proposition 4.10. *Let $F : V \to W$ be an intuitionistic fuzzy linear transformation. Then*

(i) *$F^{-1}(W)$ is an intuitionistic fuzzy vector space over X.*

(ii) *$F(V)$ is an intuitionistic fuzzy vector space over Y.*

Proof. (i) For any $x \in X$,
$[F^{-1}(W) + F^{-1}(W)](x) = \underset{x=y+z}{\sqcup}\{[F^{-1}(W)(y)] \sqcap [F^{-1}(W)(z)]\}$
$= \underset{x=y+z}{\sqcup}\{V(y) \sqcap W(t_y) \sqcap V(z) \sqcap W(t_z)\}$, for $t_y, t_z \in Y$ such that $F(y, t_y) = V(y)$ and $F(z, t_z) = V(z)$
$= \underset{x=y+z}{\sqcup}\{(V(y) \sqcap V(z)) \sqcap (W(t_y) \sqcap W(t_z))\}$
$\leq \underset{x=y+z}{\sqcup}\{V(y+z) \sqcap W(t_y + t_z)\}$ [as V, W are intuitionistic fuzzy vector spaces]
$= \underset{x=y+z}{\sqcup}\{F(y+z, t_y+t_z) \sqcap W(t_y + t_z)\}$ [Since F is a linear mapping]
$= \underset{x=y+z}{\sqcup}\{F(x, t_y+t_z) \sqcap W(t_y + t_z)\}$

$$= \begin{cases} V(x) \sqcap W(t_x), & for\ t_x \in Y\ unique\ such\ that\ F(x, t_x) = V(x) \\ (0,1), & otherwise \end{cases}$$

$= F^{-1}(W)(x)$.
Hence $F^{-1}(W) + F^{-1}(W) \subseteq F^{-1}(W)$.
For $k \neq 0$, $[kF^{-1}(W)](x) = V(\frac{x}{k}) \sqcap W(y_{\frac{x}{k}})$, for $y_{\frac{x}{k}} \in Y$ unique such that $F(\frac{x}{k}, y_{\frac{x}{k}}) = V(\frac{x}{k})$
$= V(x, ky_{\frac{x}{k}})$, since F is linear $F(x, ky_{\frac{x}{k}}) = F(\frac{x}{k}, y_{\frac{x}{k}})$
$= [F^{-1}(W)](x)$, for all $x \in X$.
For $k = 0, x \neq \theta$, $[kF^{-1}(W)](x) = (0,1) \leq [F^{-1}(W)](x)$.
Again for $k = 0, x = \theta$, $[kF^{-1}(W)](\theta) = \underset{s\in X}{\sqcup}[F^{-1}(W)(s)]$
$= \underset{s\in X}{\sqcup}\{\underset{t\in Y}{\sqcup}\{F(s,t) \sqcap W(t)\}\}$
$\leq [\underset{(s,t)\in(X\times Y)}{\sqcup} F(s,t)] \sqcap [\underset{t\in Y}{\sqcup} W(t)]$
$= F(\theta, \theta') \sqcap \{\underset{t\in Y}{\sqcup} W(t)\}$ [Since F is linear]
$= F(\theta, \theta') \sqcap W(\theta')$.
Again $[F^{-1}(W)](\theta) = \underset{t\in Y}{\sqcup}\{F(\theta, t) \sqcap W(t)\} = F(\theta, \theta') \sqcap W(\theta')$, [Since F is linear].
Therefore $kF^{-1}(W) \subseteq F^{-1}(W)$, for all $k \in \mathbb{K}$.
Hence (i) is proved.

(ii) For any $z \in Y$,
$[F(V) + F(V)](z) = \underset{z=x+y}{\sqcup}\{[F(V)(x)] \sqcap [F(V)(y)]\}$
$= \underset{z=x+y}{\sqcup}\{[\underset{t\in X}{\sqcup}\{F(t,x) \sqcap V(t)\}] \sqcap [\underset{s\in X}{\sqcup}\{F(s,y) \sqcap V(s)\}]\}$,
$= \underset{z=x+y}{\sqcup}\{\sqcup\{V(t') \sqcap V(s') : t', s' \in X\ such\ that\ F(t', x) = V(t'), F(s', y) = V(s')\}\}$
$\leq \underset{z=x+y}{\sqcup}\{\sqcup\{V(t'+s') : t', s' \in X\ such\ that\ F(t', x) = V(t'), F(s', y) = V(s')\}\}$ [as $V \in IFVS(X)$]
$= \underset{z=x+y}{\sqcup}\{\sqcup\{V(t'+s') : t', s' \in X\ such\ that\ F(t'+s', x+y) = F(t'+s', z) = V(t'+s')\}\}$,
[Since F is a linear].
Now, $F(V)(z)$

$= \underset{t\in X}{\sqcup}\{F(t,z)\sqcap V(t)\}$
$= \sqcup\{V(t_1) : t_1 \in X, F(t_1, z) = V(t_1)\}$.
Hence $F(V) + F(V) \subseteq F(V)$.
For any scalar $k \neq 0, z \in Y$,
$[kF(V)](z) = F(V)(\frac{z}{k})$
$= \underset{t\in X}{\sqcup}\{F(t,\frac{z}{k})\sqcap V(t)\}$
$= \sqcup\{V(t) : t \in X \text{ such that } F(t,\frac{z}{k}) = V(t)\}$
$= \sqcup\{V(kt) : t \in X \text{ such that } F(kt, z) = V(kt)\}$, [since $V \in IFVS(X)$ and F is linear]
$\leq \underset{s\in X}{\sqcup}\{F(s,z)\sqcap V(s)\}$
$= F(V)(z)$
If $k = 0, z \neq \theta'$, then
$[kF(V)](z) = (0, 1) \leq F(V)(z)$.
Again if $k = 0, z = \theta'$, then
$[kF(V)](\theta') = \underset{y\in Y}{\sqcup}\{F(V)(y)\}$
$= \underset{y\in Y}{\sqcup}\{\underset{x\in X}{\sqcup}[F(x,y)\sqcap V(x)]\}$
$\leq [\underset{(x,y)\in X\times Y}{\sqcup}F(x,y)]\sqcap[\underset{x\in X}{\sqcup}V(x)]$
$= F(\theta,\theta')\sqcap V(\theta)$, [since F is linear]
Now, $[F(V)](\theta') = \underset{x\in X}{\sqcup}[F(x,\theta')\sqcap V(x)] = F(\theta,\theta')\sqcap V(\theta)$, [since F is linear].
Therefore $kF(V) \subseteq F(V)$, for all $k \in \mathbb{K}$.
Hence (ii) is proved. □

Proposition 4.11. *Let $F : V \to W$ be an intuitionistic fuzzy linear transformation. If σ be an IF vector topology on W, then $\tau = \{F^{-1}(W_1) : W_1 \in \sigma\}$ is an IF vector topology on V.*

Proof. Obviously τ is an intuitionistic fuzzy topology on V. Let $V_1 \in \tau$. Then there exists $W_1 \in \sigma$ such that $V_1 = F^{-1}(W_1)$.

Since $F : (V,\tau) \to (W,\sigma)$ is intuitionistic fuzzy continuous, $F \times F : (V \times V, \tau\times\tau) \to (W \times W, \sigma\times\sigma)$ is also so.
Again since (W,σ) is an IF topological vector space $F_W^{L_{(k,m)}} : (W \times W, \sigma\times\sigma) \to (W,\sigma)$ is intuitionistic fuzzy continuous, hence $(F_W^{L_{(k,m)}})^{-1}(W_1) \in \sigma\times\sigma$ and so, $(F\times F)^{-1}((F_W^{L_{(k,m)}})^{-1}(W_1)) \in \tau\times\tau$.
Now, $(F\times F)^{-1}((F_W^{L_{(k,m)}})^{-1}(W_1))(x_1, x_2)$
$= (V \times V)(x_1, x_2) \sqcap [(F_W^{L_{(k,m)}})^{-1}(W_1)](y_1, y_2)$, where $F(x_i, y_i) = V(x_i)$, for $i = 1, 2$
$= (V \times V)(x_1, x_2) \sqcap (W \times W)(y_1, y_2) \sqcap W_1(ky_1 + my_2)$
$= (V \times V)(x_1, x_2) \sqcap W_1(ky_1 + my_2)$, [since $V(x_i) \leq W(y_i)$, for $i = 1, 2$] .....(I)
Again, $[(F_V^{L_{(k,m)}})^{-1}(V_1)](x_1, x_2)$
$= (V \times V)(x_1, x_2) \sqcap (V_1)(kx_1 + mx_2)$
$= (V \times V)(x_1, x_2) \sqcap [F^{-1}(W_1)](kx_1 + mx_2)$
$= (V \times V)(x_1, x_2) \sqcap [V(kx_1 + mx_2) \sqcap W_1(ky_1 + my_2)]$, [since F is linear]
$= (V \times V)(x_1, x_2) \sqcap W_1(ky_1 + my_2)$, [since $V \in IFVS(X)$].....(II)

From (I) and (II) we have, $(F_V^{L_{(k,m)}})^{-1}(V_1) = (F \times F)^{-1}((F_W^{L_{(k,m)}})^{-1}(W_1)) \in \tau \times \tau$.
Therefore (V, τ) is an IF topological vector space. □

Proposition 4.12. *Let $F : V \to W$ be an injective intuitionistic fuzzy linear transformation. If τ is a IF vector topology on V, then $\sigma = \{W' \subseteq W : F^{-1}(W') \in \tau\}$ is an IF vector topology on $F(V)$. If further F is surjective, then σ is an IF vector topology on W.*

Proof. Since F is injective for all $V_1 \subseteq V$, $F^{-1}(F(V_1)) = V_1$.
It can be easily verified that σ is an intuitionistic fuzzy topology on the vector space $F(V) = W_1$ (say). Since F is injective, $F : (V, \tau) \to (W_1, \sigma)$ is intuitionistic fuzzy open.
Let $W' \in \sigma$. Then $F^{-1}(W') \in \tau$.
Since τ is an IF vector topology on V, $(F_V^{L_{(k,m)}})^{-1}(F^{-1}(W')) \in \tau \times \tau$.
Since $F \times F : (V \times V, \tau \times \tau) \to (W_1 \times W_1, \sigma \times \sigma)$ is intuitionistic fuzzy open, $(F \times F)(F_V^{L_{(k,m)}})^{-1}(F^{-1}(W')) \in \sigma \times \sigma$.
Now $(F_{W_1}^{L_{(k,m)}})^{-1}(W')(y_1, y_2)$
$= \bigsqcup_{y_3 \in Y} \{[F_{W_1}^{L_{(k,m)}}((y_1, y_2), y_3)] \sqcap W'(y_3)\}$
$= (W_1 \times W_1)(y_1, y_2) \sqcap W'(ky_1 + my_2)$
$= [F(V) \times F(V)](y_1, y_2) \sqcap F(V_1)(ky_1 + my_2)$ [Since F is injective, there is $V_1 \subseteq V$ such that $F(V_1) = W'$].
Again, $F(V_1)(ky_1 + my_2) = \bigsqcup_{t \in X} \{F(t, ky_1 + my_2) \sqcap V_1(t)\}$
$= V(t_1) \sqcap V_1(t_1)$, where $t_1 \in X$ with $F(t_1, ky_1 + my_2) = V(t_1)$;
$= V_1(t_1)$, where $t_1 \in X$ with $F(t_1, ky_1 + my_2) = V(t_1)$;
$= V_1(kx_1 + mx_2)$, for $(x_1, x_2) \in X \times X$ such that $F(x_i, y_i) = V(x_i)$, for $i = 1, 2$, as F is linear.
Therefore, $(F_{W_1}^{L_{(k,m)}})^{-1}(W')(y_1, y_2)$
$= (V \times V)(x_1, x_2) \sqcap V_1(kx_1 + mx_2)$, where $F(x_i, y_i) = V(x_i)$, for $i = 1, 2$........(III)
Again $(F_V^{L_{(k,m)}})^{-1}(F^{-1}(W'))(x_1, x_2)$
$= \bigsqcup_{t \in X} \{F_V^{L_{(k,m)}}((x_1, x_2), t) \sqcap F^{-1}(W')(t)\}$
$= (V \times V)(x_1, x_2) \sqcap F^{-1}(W')(kx_1 + mx_2)$
$= (V \times V)(x_1, x_2) \sqcap V_1(kx_1 + mx_2)$, since F is injective, $F^{-1}(W') = F^{-1}(F(V_1)) = V_1$.
Hence $(F \times F)(F_V^{L_{(k,m)}})^{-1}(F^{-1}(W'))(y_1, y_2)$
$= \bigsqcup_{(t,s) \in X \times X} \{(F \times F)((t, s), (y_1, y_2)) \sqcap ((F_V^{L_{(k,m)}})^{-1}(F^{-1}(W'))(t, s)\}$
$= (V \times V)(x_1, x_2) \sqcap V_1(kx_1 + mx_2)$, where $F(x_i, y_i) = V(x_i)$, for $i = 1, 2$.........(IV)
Therefore from (III) and (IV), we have $(F_W^{L_{(k,m)}})^{-1}(W') = (F \times F)(F_V^{L_{(k,m)}})^{-1}(F^{-1}(W')) \in \sigma \times \sigma$ and hence σ is an IF vector topology on $F(V)$.
If further F is surjective, then $F(V) = W$. Hence proved. □

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