

On some methods of study of states on interval valued fuzzy sets

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Abstract: In this paper the state on interval valued fuzzy sets is studied. Two methods are considered: a representation of a state by a Kolmogorov probability and an embedding to an *MV*-algebra. The Butnariu–Klement representation theorem for interval valued fuzzy sets as a relation between probability measure and state is presented.

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1 Introduction

Intuitionistic fuzzy sets were introduced by Prof. Krassimir Atanassov in 1983 [1]. A similar model was introduced by Prof. Lotfi Zadeh in 1975, namely, interval valued fuzzy sets [10]. Although the motivations as well as the applications were different, actually they are isomorphic. Here we will present the isomorphism and show that these two models can be studied from the *MV*-algebra theory point of view.

Let us start with the definition of the basic terms. Consider a measurable space (Ω, \mathcal{S}) with a σ -algebra \mathcal{S} . Let \mathcal{J} be the family of all measurable functions $f : \Omega \rightarrow [0, 1]$. To define the state on \mathcal{J} we need two binary operations. In this paper we will use the Łukasiewicz operations:

^{*} This is one of the last papers of Prof. Beloslav Riečan (11 November 1936 – 13 August 2018).

$$f \hat{+} g = (f + g) \wedge 1_\Omega,$$

$$f \hat{*} g = (f + g - 1_\Omega) \vee 0_\Omega.$$

These binary operations play the same role as the union and the intersection in the set theory. A state on \mathcal{J} is a mapping $m_{\mathcal{J}} : \mathcal{J} \rightarrow [0, 1]$ satisfying the following conditions [5]:

1. $m_{\mathcal{J}}(0_\Omega) = 0, m_{\mathcal{J}}(1_\Omega) = 1.$
2. If $f \hat{*} g = 0_\Omega$, then $m_{\mathcal{J}}(f \hat{+} g) = m_{\mathcal{J}}(f) + m_{\mathcal{J}}(g).$
3. If $f_n \nearrow f$, then $m_{\mathcal{J}}(f_n) \nearrow m_{\mathcal{J}}(f).$

Since \mathcal{J} contains the functions $f : \Omega \rightarrow [0, 1]$, then \mathcal{J} represents the family of fuzzy sets. In the fuzzy set theory, f is called membership function. Now, let us look at the more general structures – intuitionistic fuzzy sets, shortly *IF*-sets, and interval valued fuzzy sets, shortly *IV*-sets.

The *IF*-set is a pair $A = (\mu_A, \nu_A)$ of the functions $\mu_A : \Omega \rightarrow [0, 1], \nu_A : \Omega \rightarrow [0, 1]$ such that $\mu_A + \nu_A \leq 1_\Omega$. Similarly as in the fuzzy set theory, the function μ_A is called the membership function. The second function, ν_A is called the non-membership function. Denote by \mathcal{F} the family of all *IF*-sets such that μ_A, ν_A are \mathcal{S} -measurable. On the set \mathcal{F} we could define the Łukasiewicz operations and the ordering by the following way

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega),$$

$$A \odot B = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega),$$

$$A = (\mu_A, \nu_A) \preceq (\mu_B, \nu_B) = B \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

By this ordering for each $A \in \mathcal{F}$ it holds that $(0_\Omega, 1_\Omega) \preceq A \preceq (1_\Omega, 0_\Omega)$. The family \mathcal{J} can be considered as a subset of \mathcal{F} if for $f \in \mathcal{J}$ we put $A = (f, 1_\Omega - f)$. Then $\mu_A + \nu_A = 1_\Omega$.

By an *IV*-set we shall consider the pair $C = (\pi_C, \rho_C)$ such that $\pi_C : \Omega \rightarrow [0, 1], \rho_C : \Omega \rightarrow [0, 1]$ and $\pi_C \leq \rho_C$. If (Ω, \mathcal{S}) is a measurable space, then by \mathcal{K} we shall denote the family of all *IV*-sets with the measurable functions $\pi_C, \rho_C : (\Omega, \mathcal{S}) \rightarrow [0, 1]$. Then for any $C, D \in \mathcal{K}$ we could define

$$C \boxplus D = ((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega),$$

$$C \boxminus D = ((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega),$$

$$C = (\pi_C, \rho_C) \leq (\pi_D, \rho_D) = D \iff \pi_C \leq \pi_D, \rho_C \leq \rho_D.$$

Proposition 1. *The systems $(\mathcal{F}, \preceq, \oplus, \odot)$ and $(\mathcal{K}, \leq, \boxplus, \boxminus)$ are isomorphism by the mapping $\psi : \mathcal{F} \rightarrow \mathcal{K}$,*

$$\psi((\mu_A, \nu_A)) = (\mu_A, 1_\Omega - \nu_A).$$

Proof. Evidently $A = (\mu_A, \nu_A) \in \mathcal{F}$ if and only if $(\mu_A, 1_\Omega - \nu_A) \in \mathcal{K}$. If $A, B \in \mathcal{F}$, then

$$A = (\mu_A, \nu_A) \preceq (\mu_B, \nu_B) = B \Leftrightarrow \psi(A) = (\mu_A, 1_\Omega - \nu_A) \leq (\mu_B, 1_\Omega - \nu_B) = \psi(B).$$

If $A, B \in \mathcal{F}$, then

$$\begin{aligned} \psi(A \oplus B) &= \psi((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega) = \\ &= ((\mu_A + \mu_B) \wedge 1_\Omega, 1_\Omega - ((\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)) = \\ &= ((\mu_A + \mu_B) \wedge 1_\Omega, (1_\Omega - \nu_A + 1_\Omega - \nu_B) \wedge 1_\Omega). \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi(A) \boxplus \psi(B) &= (\mu_A, 1_\Omega - \nu_A) \boxplus (\mu_B, 1_\Omega - \nu_B) = \\ &= ((\mu_A + \mu_B) \wedge 1_\Omega, (1_\Omega - \nu_A + 1_\Omega - \nu_B) \wedge 1_\Omega) = \psi(A \oplus B). \end{aligned}$$

Similarly the equality

$$\psi(A \odot B) = \psi(A) \boxdot \psi(B)$$

can be obtain. □

Also the states on \mathcal{F} and \mathcal{K} are isomorphism. For the first recall the definitions of these structures.

Definition 2. A state on \mathcal{F} is a mapping $m : \mathcal{F} \rightarrow [0, 1]$ satisfying the following conditions [4]

1. $m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1,$
2. $A \odot B = (0_\Omega, 1_\Omega) \implies m(A \oplus B) = m(A) + m(B),$
3. $A_n \nearrow A$ (i.e $\mu_{A_n} \nearrow \mu_A$ and $\nu_{A_n} \searrow \nu_A$) $\implies m(A_n) \nearrow m(A).$

Definition 3. A state on \mathcal{K} is a mapping $k : \mathcal{K} \rightarrow [0, 1]$ satisfying the following conditions

1. $k((0_\Omega, 0_\Omega)) = 0, k((1_\Omega, 1_\Omega)) = 1,$
2. $C \boxdot D = (0_\Omega, 0_\Omega) \implies k(C \boxplus D) = k(C) + k(D),$
3. $C_n \nearrow C$ (i.e $\pi_{C_n} \nearrow \pi_C$ and $\rho_{C_n} \nearrow \rho_C$) $\implies k(C_n) \nearrow k(C).$

Proposition 4. Let $k : \mathcal{K} \rightarrow [0, 1]$ be a state, $\psi : \mathcal{F} \rightarrow \mathcal{K}$, $\psi((\mu_A, \nu_A)) = (\mu_A, 1_\Omega - \nu_A)$ be a mapping and $m : \mathcal{F} \rightarrow [0, 1]$, $m = k \circ \psi$ be a composition function. Then m is a state.

Proof. Let $k : \mathcal{K} \rightarrow [0, 1]$ and $\psi((\mu_A, \nu_A)) = (\mu_A, 1_\Omega - \nu_A)$. Put $m = k \circ \psi$. Then

$$\begin{aligned} m((0_\Omega, 1_\Omega)) &= k(\psi(0_\Omega, 1_\Omega)) = k(0_\Omega, 0_\Omega) = 0, \\ m((1_\Omega, 0_\Omega)) &= k(\psi(1_\Omega, 0_\Omega)) = k(1_\Omega, 1_\Omega) = 1. \end{aligned}$$

Let $A, B \in \mathcal{F}$,

$$A \odot B = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega) = (0_\Omega, 1_\Omega)$$

if and only if

$$\mu_A + \mu_B \leq 1_\Omega \text{ and } \nu_A + \nu_B \geq 1_\Omega.$$

Then

$$\begin{aligned} \psi(A) \boxdot \psi(B) &= (\mu_A, 1_\Omega - \nu_A) \boxdot (\mu_B, 1_\Omega - \nu_B) = \\ &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (1_\Omega - \nu_A + 1_\Omega - \nu_B - 1_\Omega) \vee 0_\Omega) = (0_\Omega, 0_\Omega). \end{aligned}$$

Moreover

$$\begin{aligned} \psi(A) \boxplus \psi(B) &= (\mu_A, 1_\Omega - \nu_A) \boxplus (\mu_B, 1_\Omega - \nu_B) = \\ &= ((\mu_A + \mu_B) \wedge 1_\Omega, (1_\Omega - \nu_A + 1_\Omega - \nu_B) \wedge 1_\Omega) = \psi(A \oplus B). \end{aligned}$$

Therefore

$$\begin{aligned} m(A \oplus B) &= k(\psi(A \oplus B)) = k(\psi(A) \boxplus \psi(B)) = \\ &= k(\psi(A)) + k(\psi(B)) = m(A) + m(B). \end{aligned}$$

Finally $A_n \nearrow A$ implies $\mu_{A_n} \nearrow \mu_A$ and $\nu_{A_n} \searrow \nu_A$, hence $\psi(A_n) \nearrow \psi(A)$ and

$$m(A_n) = k(\psi(A_n)) \nearrow k(\psi(A)) = m(A).$$

This completes the proof. □

2 State representation theorem

The isomorphism between \mathcal{F} and \mathcal{K} presented in Proposition 1 and Proposition 4 permits us to translate results holding in \mathcal{F} to \mathcal{K} . As an illustration, we shall present here the state representation theorem. This theorem points on the relation between probability measure and state. In [2] there was proved the Butnariu–Klement theorem:

Theorem 5. *For any state $m_{\mathcal{F}} : \mathcal{F} \rightarrow [0, 1]$ there exists a probability measure $P : \mathcal{S} \rightarrow [0, 1]$ such that*

$$m_{\mathcal{F}}(f) = \int_{\Omega} f dP$$

for any $f \in \mathcal{F}$.

This theorem has been generalized also for Atanassov IF-sets. The generalization of the Butnariu–Klement theorem has the form:

Theorem 6. *If $m : \mathcal{F} \rightarrow [0, 1]$ is a state, then there exists a probability measure $P : \mathcal{S} \rightarrow [0, 1]$ and $\alpha \in [0, 1]$ such that*

$$m(A) = \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} (\mu_A + \nu_A) dP \right)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

This theorem is called Representation theorem for probabilities on IF-states and it was proved in [3]. The Butnariu–Klement theorem can be obtained as a consequence if $\nu_A = 1_\Omega - \mu_A$.

Now we shall use the isomorphism between \mathcal{F} and \mathcal{K} to build representation theorem for IV-states.

Theorem 7. *Let k be a state defined on \mathcal{K} . Then there exists a probability measure $P : \mathcal{S} \rightarrow [0, 1]$ and $\alpha \in [0, 1]$ such that for any $C = (\pi_C, \rho_C) \in \mathcal{K}$ it holds*

$$k(C) = (1 - \alpha) \int_{\Omega} \pi_C dP + \alpha \int_{\Omega} \rho_C dP.$$

Proof. Put $m = k \circ \psi : \mathcal{F} \rightarrow [0, 1]$ and use the results of Theorem 6. Then

$$m(A) = \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} (\mu_A + \nu_A) dP \right)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$. Of course, $k = m \circ \psi^{-1}$, hence for $C = (\pi_C, \rho_C) \in \mathcal{K}$ we obtain

$$\begin{aligned} k(C) &= k(\pi_C, \rho_C) = m(\pi_C, 1 - \rho_C) = \int_{\Omega} \pi_C dP + \alpha \left(1 - \int_{\Omega} (\pi_C + 1 - \rho_C) dP \right) = \\ &= (1 - \alpha) \int_{\Omega} \pi_C dP + \alpha \int_{\Omega} \rho_C dP. \end{aligned}$$

This completes the proof. □

3 States and MV -algebras

Similarly as in the case of \mathcal{F} also \mathcal{K} can be embedded into the suitable MV -algebra ([7]). For the first recall the definition of an MV -algebra.

Definition 8. By an ℓ -group G we consider an algebraic system $(G, +, \leq)$ such that

1. $(G, +)$ is commutative group,
2. (G, \leq) is lattice,
3. $a \leq b \Rightarrow a + c \leq b + c$.

We shall use the form of MV -algebra which is generated by fuzzy sets ([9]).

Definition 9. Let $u \in G, u > 0$. Then, an MV -algebra \mathcal{M} is a set $M = [0, u]$, for which it holds

$$a \oplus b = (a + b) \wedge u,$$

$$a \odot b = (a + b - u) \wedge 0.$$

The element u is supposed to be a strong unit of G .

Then an MV -algebra state [7] is a mapping $\bar{m} : \mathcal{M} \rightarrow [0, 1]$ satisfying the following conditions:

1. $\bar{m}(0) = 0, \bar{m}(u) = 1$.
2. $a \bar{\odot} b = 0 \Rightarrow \bar{m}(a \bar{\oplus} b) = \bar{m}(a) + \bar{m}(b)$.
3. $a_n \nearrow a \Rightarrow \bar{m}(a_n) \nearrow \bar{m}(a)$.

In the paper [6] there was proved that an IF-set could be embedded into the MV -algebra \mathcal{M} by using following construction. Let us take MV -algebra \mathcal{M} such that $G = R^2$. Then the addition of two elements could be defined as

$$C = A \hat{\oplus} B = (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega).$$

This addition represents the addition of two vectors with a fixed point $(0, 1)$. For any point $A = (\mu_A, \nu_A) \in \mathcal{M}$ it holds

$$(\mu_A, \nu_A) \hat{\oplus} (0_\Omega, 1_\Omega - \nu_A) = (\mu_A, 0_\Omega).$$

From this reason, we can define the function $\bar{m} : \mathcal{M} \rightarrow [0, 1]$ by the formula

$$\bar{m}(\mu_A, \nu_A) = m(\mu_A, 0_\Omega) - m(0_\Omega, 1_\Omega - \nu_A).$$

This function represents the state on \mathcal{M} generated by a state defined on \mathcal{F} .

Theorem 10. *The family \mathcal{K} can be embedded into the MV -algebra \mathcal{M} such that to any state $k : \mathcal{K} \rightarrow [0, 1]$ there exist a state $\bar{k} : \mathcal{M} \rightarrow [0, 1]$ such that $\bar{k}|_{\mathcal{K}} = k$.*

Proof. Let $(G, +, \leq)$ be the ℓ -group where G consists of all mappings $C : (\pi_C, \rho_C) \rightarrow R^2$ where π_C, ρ_C are \mathcal{S} -measurable functions. Define

$$C \tilde{\oplus} D = (\pi_C + \pi_D, \rho_C + \rho_D)$$

and

$$C \leq D \Leftrightarrow \pi_C \leq \pi_D, \rho_C \leq \rho_D.$$

Then the element $(0_\Omega, 0_\Omega)$ is the neutral element, i.e.,

$$(\pi_C, \rho_C) \tilde{\oplus} (0_\Omega, 0_\Omega) = (\pi_C, \rho_C),$$

$$C \wedge D = (\pi_C \wedge \pi_D, \rho_C \wedge \rho_D),$$

$$C \vee D = (\pi_C \vee \pi_D, \rho_C \vee \rho_D).$$

Consider the element $(1_\Omega, 1_\Omega)$ and define \mathcal{M} as the structure

$$\mathcal{M} = \{C \in G; (0_\Omega, 0_\Omega) \leq (\pi_C, \rho_C) \leq (1_\Omega, 1_\Omega)\}.$$

Then for any $C, D \in \mathcal{M}$ it holds

$$C \boxplus D = [(\pi_C, \rho_C) + (\pi_D, \rho_D)] \wedge (1_\Omega, 1_\Omega) = ((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega),$$

$$C \boxminus D = [(\pi_C, \rho_C) + (\pi_D, \rho_D) - (1_\Omega, 1_\Omega)] \vee (0_\Omega, 0_\Omega) = ((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega).$$

We have obtained an MV -algebra \mathcal{M} such that $\mathcal{K} \subset \mathcal{M}$. Let $\bar{m} : \mathcal{M} \rightarrow [0, 1]$ be a state on \mathcal{M} . Define $\bar{k} : \mathcal{M} \rightarrow [0, 1]$ by the formula

$$\bar{k}(\pi_C, \rho_C) = \bar{m}(\pi_C, 1_\Omega) - \bar{m}(0_\Omega, 1_\Omega - \rho_C).$$

Then \bar{k} is a state and for $C \in \mathcal{K}$ we have $\pi_C \leq \rho_C$

$$(\pi_C, 1_\Omega) = (\pi_C, \rho_C) \boxplus (0_\Omega, 1_\Omega - \rho_C)$$

hence

$$\bar{m}(\pi_C, 1_\Omega) = \bar{m}(\pi_C, \rho_C) + \bar{m}(0_\Omega, 1_\Omega - \rho_C)$$

and therefore

$$\bar{m}(\pi_C, \rho_C) = \bar{k}(\pi_C, \rho_C).$$

This completes the proof. □

4 Conclusion

Representation theorems as well as assertions about isomorphism of some mathematical structures belong to the most useful mathematical means. In this paper, we used the representation of IF -states by Kolmogorov probability measures and the isomorphism between the Atanassov IF -sets and the Zadeh IV -sets. An important point is the possibility to embed fuzzy families to multivalued algebras. Namely, some useful results of MV -algebra measure theory can be applied to some fuzzy structures. In this way, we have two ways to obtain fuzzy measure constructions. The first is in using set probability theory. The second in using the MV -algebra probability theory.

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