Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 22, 2016, No. 5, 84–97

An introduction to intuitionistic L-fuzzy semi-primary ideals

P. Veerammal[1] and M. Palanivelrajan[2]

[1] Department of Mathematics, Saraswathi Narayanan College,
Madurai 625 022, Tamil Nadu, India.
e-mail: chandruruthresh@gmail.com
[2] Department of Mathematics, Government Arts College,
Paramakudi 623 707, Tamil Nadu, India
e-mail: palanivelrajan1975@gmail.com

Received: 30 April 2016 **Revised:** 12 September 2016 **Accepted:** 20 December 2016

Abstract: Intuitionistic Fuzzy Sets, involving membership,non-membership and hesitancy considerations mathematically present a very general structure.Based on these considerations,we have made an attempt to study some operations on the algebraic nature of an Intuitionistic L-Fuzzy Semi-primary Ideals (ILFSPI).
Keywords: Intuitionistic fuzzy set, Intuitionistic L-fuzzy primary ideals, Intuitionistic L-fuzzy semi-primary ideals.
AMS Classification: 03E72, 06B23, 08A72.

1 Introduction

Ever since the introduction of fuzzy sets by Zadeh [8], the fuzzy concepts have involved almost all branches of Mathematics. The concept of Intuitionistic Fuzzy Sets [1] and Intuitionistic L-Fuzzy Sets [2] were introduced by K. Atanassov and he defined new operations over Intuitionistic Fuzzy sets in [3]. In [4], S. Lehmke discussed some properties of Fuzzy Ideals on a Lattice. Mohammed M. Atallah [5] discussed on the L-Fuzzy Prime Ideal theorem on distributive Lattice. In [6] and [7], M. Palanivelrajan and S. Nandhakumar defined Intuitionistic Fuzzy Semi-Primary Ideals over Rings and discussed some of its properties. The purpose of this paper is to introduce some basic concepts on Operations and Cartesian products over Intuitionistic L-Fuzzy Semi-Primary Ideals and prove some fundamental properties.

2 Preliminaries

Let X be a non-empty set $L = (L, \leq, \vee, \wedge)$ as a complete distributive lattice,which has least and greatest elements,say 0 and 1 respectively and for simplicity of notation we write $\sup$ and $\inf$ for $\vee$ and $\wedge$ respectively. In this section,some well known definitions are recalled for the convenience of reading. These definitions are necessary in order to understand the new concepts introduced in this paper.

Definition 2.1. *Let X be any non-empty set. A mapping $\mu : X \to [0, 1]$ is called a fuzzy subset of X.*

Definition 2.2. *A fuzzy subset μ of a ring R is called a fuzzy ideal of R if,*

- $\mu(x - y) \geq \mu(x) \wedge \mu(y)$
- $\mu(xy) \geq \mu(x) \vee \mu(y)$, *for all $x, y \in R$ and for some $n \in Z_+$.*

Definition 2.3. *A fuzzy ideal μ of a ring R is called a fuzzy primary ideal if for all $x, y \in R$ either $\mu(xy) = \mu(x^n)$ for some $n \in Z_+$ or $\mu(xy) \leq \mu(y^m)$for some $m \in Z_+$.*

Definition 2.4. *A fuzzy ideal μ of a ring R is called a fuzzy semi - primary ideal if for all $x, y \in R$ either $\mu(xy) \leq \mu(x^n)$ for some $n \in Z_+$ or $\mu(xy) \leq \mu(y^m)$for some $m \in Z_+$.*

Definition 2.5. *A fuzzy ideal μ of a ring R is called Intuitionistic fuzzy primary ideal if for all $x, y \in R$ either $\mu(xy) = \mu(x^n)$ and $\nu(xy) = \nu(x^n)$ or $\mu(xy) \leq \mu(y^n)$ and $\nu(xy) \geq \nu(y^n)$ for some $n \in Z_+$.*

Definition 2.6. *A fuzzy ideal μ of a ring R is called Intuitionistic fuzzy semi-primary ideal if for all $x, y \in R$ either $\mu(xy) \leq \mu(x^n)$ and $\nu(xy) \geq \nu(x^n)$ or $\mu(xy) \leq \mu(y^n)$ and $\nu(xy) \geq \nu(y^n)$ for some $n \in Z_+$.*

Definition 2.7. *Let X be a non-empty set. An Intuionistic fuzzy set (IFS) A in X is defined as an object of the form $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\ \rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership function of A respectively and $0 \leq \mu_A(xy) + \nu_A(xy) \leq 1$ for each x inX.*

Definition 2.8. *An L- fuzzy subset A of X is a function $A : X \to L$.*

Definition 2.9. *Let $(R, +, .)$ be a ring.An intutionistic L-fuzzy subset A of Ris said to be an intutionistic L-fuzzy semi-primary ideals(ILFSPI) of R,if*

- $\mu_A(x - y) \geq \mu_A(x^n) \wedge \mu_A(y^n)$
- $\mu_A(xy) \geq \mu_A(x^n) \vee \mu_A(y^n)$
- $\nu_A(x - y) \leq \nu_A(x^n) \vee \nu_A(y^n)$
- $\nu_A(xy) \leq \nu_A(x^n) \wedge \nu_A(y^n)$, *for all $x, y \in R$ and for some $n \in Z_+$.*

Example 2.10. *Consider*

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \in 4Z - \{0\} \\ 0.7 & \text{if } x \in Z - 4Z \end{cases} \qquad \nu_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.4 & \text{if } x \in 4Z - \{0\} \\ 0.2 & \text{if } x \in Z - 4Z \end{cases}$$

Definition 2.11. *Let X be a non-empty set. Let A and B be an intutionistic L-fuzzy semi-primary ideals (ILFSPI) of R such that*

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle \,| x \in X\} \\ B &= \{\langle x, \mu_B(x), \nu_B(x)\rangle \,| x \in X\} \end{aligned}$$

Define the following operations on A and B,

- $A \cap B = \{\langle x, (\mu_A(x) \wedge \mu_B(x)), (\nu_A(x) \vee \nu_B(x))\rangle \,| x \in X\}$.
- $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \,| x \in X\}$.
- $\bar{A} \cap \bar{B} = \{\langle x, \nu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \mu_B(x)\rangle \,| x \in X\}$.
- $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x)\nu_B(x)\rangle \,| x \in X\}$.
- $A.B = \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x))\rangle \,| x \in X\}$.
- $A@B = \{\left\langle x, \dfrac{\mu_A(x) + \mu_B(x)}{2}, \dfrac{\nu_A(x) + \nu_B(x)}{2}\right\rangle |x \in X\}$.
- $A\$B = \{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)}\right\rangle |x \in X\}$.
- $A * B = \{\left\langle x, \dfrac{\mu_A(x) + \mu_B(x)}{2(\mu_A(x)\mu_B(x) + 1)}, \dfrac{\nu_A(x) + \nu_B(x)}{2(\nu_A(x)\nu_B(x) + 1)}\right\rangle |x \in X\}$.
- $A \bowtie B = \{\left\langle x, \dfrac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \dfrac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)}\right\rangle |x \in X\}$.

Definition 2.12. *Define the following operation on Cartesian product of two ILFSPI of A and B,*

$$A \times B = \{\langle\langle x, y\rangle, \mu_A(x) \wedge \mu_B(y), \nu_A(x) \vee \nu_B(y)\rangle\, x, y \in X\}.$$

3 Intuitionistic L-Fuzzy semi-primary ideals

Theorem 3.1. *If A and $B$are any two ILFSPI, then $A \cap B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

$$\begin{aligned} \text{Consider} \quad \mu_{A\cap B}(x-y) &= \mu_A(x-y) \wedge \mu_B(x-y) \\ &\geq (\mu_A(x^n) \wedge \mu_A(y^n)) \wedge (\mu_B(x^n) \wedge \mu_B(y^n)) \\ &= (\mu_A(x^n) \wedge \mu_B(x^n)) \wedge (\mu_A(y^n) \wedge \mu_B(y^n)) \\ &= \mu_{A\cap B}(x^n) \wedge \mu_{A\cap B}(y^n) \\ \text{thus} \quad \mu_{A\cap B}(x-y) &\leq \mu_{A\cap B}(x^n) \wedge \mu_{A\cap B}(y^n) \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Consider} \quad \mu_{A\cap B}(xy) &= \mu_A(xy) \wedge \mu_B(xy) \\ &\geq (\mu_A(x^n) \vee \mu_A(y^n)) \wedge (\mu_B(x^n) \vee \mu_B(y^n)) \\ &= (\mu_A(x^n) \wedge \mu_B(x^n)) \vee (\mu_A(y^n) \wedge \mu_B(y^n)) \\ &= \mu_{A\cap B}(x^n) \vee \mu_{A\cap B}(y^n) \\ \text{thus} \quad \mu_{A\cap B}(xy) &\geq \mu_{A\cap B}(x^n) \vee \mu_{A\cap B}(y^n) \qquad (2) \end{aligned}$$

$$\begin{aligned} \text{Consider} \quad \nu_{A\cap B}(x-y) &= \nu_A(x-y) \vee \nu_B(x-y) \\ &\leq (\nu_A(x^n) \vee \nu_A(y^n)) \vee (\nu_B(x^n) \vee \nu_B(y^n)) \\ &= (\nu_A(x^n) \vee \nu_B(x^n)) \wedge (\nu_A(y^n) \vee \nu_B(y^n)) \\ &= \nu_{A\cap B}(x^n) \wedge \nu_{A\cap B}(y^n) \\ \text{thus} \quad \nu_{A\cap B}(x-y) &\leq \nu_{A\cap B}(x^n) \wedge \nu_{A\cap B}(y^n) \qquad (3) \end{aligned}$$

$$\begin{aligned} \text{Consider} \quad \nu_{A\cap B}(xy) &= \nu_A(xy) \vee \nu_B(xy) \\ &\leq (\nu_A(x^n) \wedge \nu_A(y^n)) \vee (\nu_B(x^n) \wedge \nu_B(y^n)) \\ &= (\nu_A(x^n) \vee \nu_B(x^n)) \wedge (\nu_A(y^n) \vee \nu_B(y^n)) \\ &= \nu_{A\cap B}(x^n) \wedge \nu_{A\cap B}(y^n) \\ \text{thus} \quad \nu_{A\cap B}(xy) &\leq \nu_{A\cap B}(x^n) \wedge \nu_{A\cap B}(y^n) \qquad (4) \end{aligned}$$

Therefore from (1), (2), (3) and (4), we get $A \cap B$ is an ILFSPI. □

Theorem 3.2. *The intersection of a family of all ILFSPI of X is an ILFSPI of X.*

Theorem 3.3. *If A is an ILFSPI of R then $\bar{\bar{A}} = A$.*

Proof. Let A be any ILFSPI.

$$\begin{aligned} \text{Consider} \quad \mu_{\bar{\bar{A}}}(x-y) &\geq \mu_{\bar{\bar{A}}}(x^n) \wedge \mu_{\bar{\bar{A}}}(y^n), \; for\ all\, x, y \in R \\ &= \nu_{\bar{A}}(x^n) \wedge \nu_{\bar{A}}(y^n) \\ &= \mu_A(x^n) \wedge \mu_A(y^n) \\ &= \mu_A(x-y) \\ \text{thus} \quad \mu_{\bar{\bar{A}}}(x-y) &\geq \mu_A(x-y) \qquad (5) \end{aligned}$$

$$\begin{aligned} \text{Consider} \quad \mu_{\bar{\bar{A}}}(xy) &\geq \mu_{\bar{\bar{A}}}(x^n) \vee \mu_{\bar{\bar{A}}}(y^n), \; for\ all\, x, y \in R \\ &= \nu_{\bar{A}}(x^n) \vee \nu_{\bar{A}}(y^n) \\ &= \mu_A(x^n) \vee \mu_A(y^n) \\ &= \mu_A(xy) \\ \text{thus} \quad \mu_{\bar{\bar{A}}}(xy) &\geq \mu_A(xy) \qquad (6) \end{aligned}$$

$$\begin{aligned}
\text{Consider } \nu_{\bar{\bar{A}}}(x-y) &\leq \nu_{\bar{\bar{A}}}(x^n) \vee \nu_{\bar{\bar{A}}}(y^n), \ for\ all\ x, y \in R \\
&= \mu_{\bar{A}}(x^n) \vee \mu_{\bar{A}}(y^n) \\
&= \nu_A(x^n) \vee \nu_A(y^n) \\
&= \nu_A(x-y) \\
\nu_{\bar{\bar{A}}}(x-y) &\leq \nu_A(x-y) \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\text{Consider } \nu_{\bar{\bar{A}}}(xy) &\leq \nu_{\bar{\bar{A}}}(x^n) \wedge \nu_{\bar{\bar{A}}}(y^n), \ for\ all\ x, y \in R \\
&= \mu_{\bar{A}}(x^n) \wedge \mu_{\bar{A}}(y^n) \\
&= \nu_A(x^n) \wedge \nu_A(y^n) \\
&= \nu_A(xy) \\
\nu_{\bar{\bar{A}}}(xy) &\leq \nu_A(xy) \qquad (8)
\end{aligned}$$

Therefore, from (5), (6), (7) and (8), we get $\bar{\bar{A}} = A$. □

Theorem 3.4. *If A and B are any two ILFSPI, then $A+B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Consider

$$\begin{aligned}
\mu_{A+B}(x-y) &= \mu_A(x-y)\mu_B(x-y) - \mu_A(x-y)\mu_B(x-y), \ for\ all\ x, y \in R \\
&\geq (\mu_A(x^n) \wedge \mu_A(y^n)) + (\mu_B(x^n) \wedge \mu_B(y^n)) - (\mu_A(x^n) \wedge \mu_A(y^n))(\mu_B(x^n) \wedge \mu_B(y^n)) \\
&= (\mu_A(x^n) + \mu_B(x^n)) \wedge (\mu_A(y^n) + \mu_B(y^n)) - (\mu_A(x^n)\mu_B(x^n) \wedge \mu_A(y^n)\mu_B(y^n)) \\
&= \mu_A(x^n) + \mu_B(x^n) - (\mu_A(x^n).\mu_B(x^n)) \wedge \mu_A(y^n) + \mu_B(y^n) - (\mu_A(y^n).\mu_B(y^n)) \\
&= \mu_{A+B}(x^n) \wedge \mu_{A+B}(y^n)
\end{aligned}$$

thus

$$\mu_{A+B}(x-y) \geq \mu_{A+B}(x^n) \wedge \mu_{A+B}(y^n) \qquad (9)$$

Consider

$$\begin{aligned}
\mu_{A+B}(xy) &= \mu_A(xy)\mu_B(xy) - \mu_A(xy)\mu_B(xy), \ for\ all\ x, y \in R \\
&\geq (\mu_A(x^n) \vee \mu_A(y^n)) + (\mu_B(x^n) \vee \mu_B(y^n)) - (\mu_A(x^n) \vee \mu_A(y^n))(\mu_B(x^n) \vee \mu_B(y^n)) \\
&= (\mu_A(x^n) + \mu_B(x^n)) \vee (\mu_A(y^n) + \mu_B(y^n)) - (\mu_A(x^n)\mu_B(x^n) \vee \mu_A(y^n)\mu_B(y^n)) \\
&= \mu_A(x^n) + \mu_B(x^n) - (\mu_A(x^n).\mu_B(x^n)) \vee \mu_A(y^n) + \mu_B(y^n) - \mu_A(y^n).\mu_B(y^n)) \\
&= \mu_{A+B}(x^n) \vee \mu_{A+B}(y^n)
\end{aligned}$$

thus

$$\mu_{A+B}(xy) \geq \mu_{A+B}(x^n) \vee \mu_{A+B}(y^n) \qquad (10)$$

Consider

$$\begin{aligned}
\nu_{A+B}(x-y) &= \nu_A(x-y)\nu_B(x-y), for\ all\ x,y\ \in\ R \\
&\leq (\nu_A(x^n)\vee\nu_A(y^n))(\nu_B(x^n)\vee\nu_B(y^n)) \\
&= (\nu_A(x^n)\nu_B(x^n))\vee(\nu_A(y^n)\mu_B(y^n)) \\
&= \nu_{A+B}(x^n)\vee\nu_{A+B}(y^n)
\end{aligned}$$

thus

$$\nu_{A+B}(x-y) \leq \nu_{A+B}(x^n)\vee\nu_{A+B}(y^n) \tag{11}$$

Consider

$$\begin{aligned}
\nu_{A+B}(xy) &= \nu_A(xy)\nu_B(xy), for\ all\ x,y\ \in\ R \\
&\leq (\nu_A(x^n)\wedge\nu_A(y^n))(\nu_B(x^n)\wedge\nu_B(y^n)) \\
&= (\nu_A(x^n)\nu_B(x^n))\wedge(\nu_A(y^n)\mu_B(y^n)) \\
&= \nu_{A+B}(x^n)\wedge\nu_{A+B}(y^n)
\end{aligned}$$

thus

$$\nu_{A+B}(xy) \leq \nu_{A+B}(x^n)\wedge\nu_{A+B}(y^n) \tag{12}$$

Therefore, from (9), (10), (11) and (12), we get $A+B$ is an ILFSPI. □

Theorem 3.5. *If A and B are any two ILFSPI, then $A.B$ is an ILFSPI.*

Proof. Let A and B are any two ILFSPI.

Consider

$$\begin{aligned}
\mu_{A.B}(x-y) &= \mu_A(x-y)\mu_B(x-y) \\
&\geq (\mu_A(x^n)\wedge\mu_A(y^n))(\mu_B(x^n)\wedge\mu_B(y^n)) \\
&= (\mu_A(x^n)\mu_B(x^n))\wedge(\mu_A(y^n)\mu_B(y^n)) \\
&= \mu_{A.B}(x^n)\wedge\mu_{A.B}(y^n)
\end{aligned}$$

thus

$$\mu_{A.B}(x-y) \geq \mu_{A.B}(x^n)\wedge\mu_{A.B}(y^n) \tag{13}$$

Consider

$$\begin{aligned}
\mu_{A.B}(xy) &= \mu_A(xy)\mu_B(xy) \\
&\geq (\mu_A(x^n)\vee\mu_A(y^n))(\mu_B(x^n)\vee\mu_B(y^n)) \\
&= (\mu_A(x^n)\mu_B(x^n))\vee(\mu_A(y^n)\mu_B(y^n)) \\
&= \mu_{A.B}(x^n)\vee\mu_{A.B}(y^n)
\end{aligned}$$

thus

$$\mu_{A.B}(xy) \geq \mu_{A.B}(x^n)\vee\mu_{A.B}(y^n) \tag{14}$$

Consider

$$\begin{aligned}
\nu_{A.B}(x-y) &= \nu_A(x-y)+\nu_B(x-y)-\nu_A(x-y)\nu B(x-y), for\ all\ x,y\ \in\ R\\
&\leq (\nu_A(x^n)\vee\nu_A(y^n))+(\nu_B(x^n)\vee\nu_B(y^n))-(\nu_A(x^n)\vee\nu_A(y^n))(\nu_B(x^n)\vee\nu_B(y^n))\\
&= (\nu_A(x^n)+\nu_B(x^n))\vee(\nu_A(y^n)+\nu_B(y^n))-((\nu_A(x^n)\nu_B(x^n))\vee(\nu_B(y^n)\nu_B(y^n)))\\
&= \nu_A(x^n)+\nu_B(x^n)-(\nu_A(x^n).\nu_B(x^n))\vee\nu_A(y^n)+\nu_B(y^n)-(\nu_A(y^n)\nu_B(y^n))\\
&= \nu_{A.B}(x^n)\vee\nu_{A.B}(y^n)
\end{aligned}$$

thus

$$\nu_{A.B}(x-y)\leq\nu_{A.B}(x^n)\vee\nu_{A.B}(y^n) \tag{15}$$

Consider

$$\begin{aligned}
\nu_{A.B}(xy) &= \nu_A(xy)+\nu_B(xy)-\nu_A(xy)\nu B(xy), for\ all\ x,y\ \in\ R\\
&\leq (\nu_A(x^n)\wedge\nu_A(y^n))+(\nu_B(x^n)\wedge\nu_B(y^n))-(\nu_A(x^n)\wedge\nu_A(y^n))(\nu_B(x^n)\wedge\nu_B(y^n))\\
&= (\nu_A(x^n)+\nu_B(x^n))\wedge(\nu_A(y^n)+\nu_B(y^n))-((\nu_A(x^n)\nu_B(x^n))\wedge(\nu_B(y^n)\nu_B(y^n)))\\
&= \nu_A(x^n)+\nu_B(x^n)-(\nu_A(x^n)\nu_B(x^n))\wedge\nu_A(y^n)+\nu_B(y^n)-(\nu_A(y^n)\nu_B(y^n))\\
&= \nu_{A.B}(x^n)\wedge\nu_{A.B}(y^n)
\end{aligned}$$

thus

$$\nu_{A.B}(xy)\ \leq\ \nu_{A.B}(x^n)\wedge\nu_{A.B}(y^n) \tag{16}$$

Therefore, from (13), (14), (15) and (16), we get $A.B$ is an ILFSPI.

□

Theorem 3.6. *If A and B are any two ILFSPI, then $A@B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Consider

$$\begin{aligned}
\mu_{A@B}(x-y) &= \frac{\mu_A(x-y)+\mu_B(x-y)}{2}\\
&\geq \frac{(\mu_A(x^n)\wedge\mu_A(y^n))+(\mu_B(x^n)\wedge\mu_B(y^n))}{2}\\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\wedge(\mu_A(y^n)+\mu_B(y^n))}{2}\\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))}{2}\wedge\frac{(\mu_A(y^n)+\mu_B(y^n))}{2}\\
&= \mu_{A@B}(x^n)\wedge\mu_{A@B}(y^n)
\end{aligned}$$

thus

$$\mu_{A@B}(x-y)\ \geq\ \mu_{A@B}(x^n)\wedge\mu_{A@B}(y^n) \tag{17}$$

Consider

$$
\begin{aligned}
\mu_{A@B}(xy) &= \frac{\mu_A(xy)+\mu_B(xy)}{2}\\
&\geq \frac{(\mu_A(x^n)\vee\mu_A(y^n))+(\mu_B(x^n)\vee\mu_B(y^n))}{2}\\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\vee(\mu_A(y^n)+\mu_B(y^n))}{2}\\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))}{2}\vee\frac{(\mu_A(y^n)+\mu_B(y^n))}{2}\\
&= \mu_{A@B}(x^n)\vee\mu_{A@B}(y^n)
\end{aligned}
$$

thus

$$\mu_{A@B}(xy) \geq \mu_{A@B}(x^n)\vee\mu_{A@B}(y^n) \tag{18}$$

Consider

$$
\begin{aligned}
\nu_{A@B}(x-y) &= \frac{\nu_A(x-y)+\nu_B(x-y)}{2}\\
&\leq \frac{(\nu_A(x^n)\vee\nu_A(y^n))+(\nu_B(x^n)\vee\nu_B(y^n))}{2}\\
&= \frac{(\nu_A(x^n)+\nu_B(x^n))\vee(\nu_A(y^n)+\nu_B(y^n))}{2}\\
&= \frac{\nu_A(x^n)+\nu_B(x^n)}{2}\vee\frac{\nu_A(y^n)+\nu_B(y^n))}{2}\\
&= \nu_{A@B}(x^n)\vee\nu_{A@B}(y^n)
\end{aligned}
$$

thus

$$\nu_{A@B}(x-y) \leq \nu_{A@B}(x^n)\vee\nu_{A@B}(y^n) \tag{19}$$

Consider

$$
\begin{aligned}
\nu_{A@B}(xy) &= \frac{\nu_A(xy)+\nu_B(xy)}{2}\\
&\leq \frac{(\nu_A(x^n)\wedge\nu_A(y^n))+(\nu_B(x^n)\wedge\nu_B(y^n))}{2}\\
&= \frac{(\nu_A(x^n)+\nu_B(x^n))\wedge(\nu_A(y^n)+\nu_B(y^n))}{2}\\
&= \frac{\nu_A(x^n)+\nu_B(x^n)}{2}\wedge\frac{\nu_A(y^n)+\nu_B(y^n)}{2}\\
&= \nu_{A@B}(x^n)\wedge\nu_{A@B}(y^n)
\end{aligned}
$$

thus

$$\nu_{A@B}(xy) \leq \nu_{A@B}(x^n)\wedge\nu_{A@B}(y^n) \tag{20}$$

Therefore, from (17), (18), (19) and (20), we get $A@B$ is an ILFSPI. □

Theorem 3.7. *If A and B are any two ILFSPI, then $A\$B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Consider

$$\begin{aligned}\mu_{A\$B}(x-y) &= \sqrt{\mu_A(x-y)\mu_B(x-y)} \\ &\geq \sqrt{(\mu_A(x^n)\wedge\mu_A(y^n))(\mu_B(x^n)\wedge\mu_B(y^n))} \\ &= \sqrt{\mu_A(x^n)\mu_B(x^n)\wedge\mu_A(y^n)\mu_B(y^n)} \\ &= \sqrt{\mu_A(x^n)\mu_B(x^n)}\wedge\sqrt{\mu_A(y^n)\mu_B(y^n)} \\ &= \mu_{A\$B}(x^n)\wedge\mu_{A\$B}(y^n)\end{aligned}$$

thus

$$\mu_{A\$B}(x-y) \geq \mu_{A\$B}(x^n)\wedge\mu_{A\$B}(y^n) \tag{21}$$

Consider

$$\begin{aligned}\mu_{A\$B}(xy) &= \sqrt{\mu_A(xy)\mu_B(xy)} \\ &\geq \sqrt{(\mu_A(x^n)\vee\mu_A(y^n))(\mu_B(x^n)\vee\mu_B(y^n))} \\ &= \sqrt{\mu_A(x^n)\mu_B(x^n)\vee\mu_A(y^n)\mu_B(y^n)} \\ &= \sqrt{\mu_A(x^n)\mu_B(x^n)}\vee\sqrt{\mu_A(y^n)\mu_B(y^n)} \\ &= \mu_{A\$B}(x^n)\vee\mu_{A\$B}(y^n)\end{aligned}$$

thus

$$\mu_{A\$B}(xy) \geq \mu_{A\$B}(x^n)\vee\mu_{A\$B}(y^n) \tag{22}$$

Consider

$$\begin{aligned}\nu_{A\$B}(x-y) &= \sqrt{\nu_A(x-y)\nu_B(x-y)} \\ &\leq \sqrt{(\nu_A(x^n)\vee\nu_A(y^n))(\nu_B(x^n)\vee\nu_B(y^n))} \\ &= \sqrt{\nu_A(x^n)\nu_B(x^n)\vee\nu_A(y^n)\nu_B(y^n)} \\ &= \sqrt{\nu_A(x^n)\nu_B(x^n)}\vee\sqrt{(\nu_A(y^n)\nu_B(y^n))} \\ &= \nu_{A\$B}(x^n)\vee\nu_{A\$B}(y^n)\end{aligned}$$

thus

$$\nu_{A\$B}(x-y) \leq \nu_{A\$B}(x^n)\vee\nu_{A\$B}(y^n) \tag{23}$$

Consider

$$\begin{aligned}\nu_{A\$B}(xy) &= \sqrt{\nu_A(xy)\nu_B(xy)} \\ &\leq \sqrt{(\nu_A(x^n)\wedge\nu_A(y^n))(\nu_B(x^n)\wedge\nu_B(y^n))} \\ &= \sqrt{\nu_A(x^n)\nu_B(x^n)\wedge\nu_A(y^n)\nu_B(y^n)} \\ &= \sqrt{\nu_A(x^n)\nu_B(x^n)}\wedge\sqrt{\nu_A(y^n)\nu_B(y^n)} \\ &= \nu_{A\$B}(x^n)\wedge\nu_{A\$B}(y^n)\end{aligned}$$

thus

$$\nu_{A\$B}(xy) \leq \nu_{A\$B}(x^n)\wedge\nu_{A\$B}(y^n) \tag{24}$$

Therefore, from (21), (22), (23) and (24), we get $A\$B$ is an ILFSPI. □

Theorem 3.8. *If A and B are any two ILFSPI, then $A * B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Consider

$$\begin{aligned}
\mu_{A*B}(x-y) &= \frac{\mu_A(x-y)+\mu_B(x-y)}{2(\mu_A(x-y)\mu_B(x-y)+1)} \\
&\geq \frac{(\mu_A(x^n)\wedge\mu_A(y^n))+(\mu_B(x^n)\wedge\mu_B(y^n))}{2((\mu_A(x^n)\wedge\mu_A(y^n))(\mu_B(x^n)\wedge\mu_B(y^n))+1)} \\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\wedge(\mu_A(y^n)+\mu_B(y^n))}{2((\mu_A(x^n)\mu_B(x^n)\wedge\mu_A(y^n)\mu_B(y^n))+1)} \\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\wedge(\mu_A(y^n)+\mu_B(y^n)}{2((\mu_A(x^n)\mu_B(x^n)+1)\wedge(\mu_A(y^n)\mu_B(y^n)+1))} \\
&= \frac{\mu_A(x^n)+\mu_B(x^n)}{2(\mu_A(x^n)\mu_B(x^n)+1)}\wedge\frac{\mu_A(y^n)+\mu_B(y^n)}{2(\mu_A(y^n)\mu_B(y^n)+1)} \\
&= \mu_{A*B}(x^n)\wedge\mu_{A*B}(y^n)
\end{aligned}$$

thus

$$\mu_{A*B}(x-y) \geq \mu_{A*B}(x^n)\wedge\mu_{A*B}(y^n) \tag{25}$$

Consider

$$\begin{aligned}
\mu_{A*B}(xy) &= \frac{\mu_A(xy)+\mu_B(xy)}{2(\mu_A(xy)\mu_B(xy)+1)} \\
&\geq \frac{(\mu_A(x^n)\vee\mu_A(y^n))+(\mu_B(x^n)\vee\mu_B(y^n))}{2((\mu_A(x^n)\vee\mu_A(y^n))(\mu_B(x^n)\vee\mu_B(y^n))+1)} \\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\vee(\mu_A(y^n)+\mu_B(y^n))}{2((\mu_A(x^n)\mu_B(x^n)\vee\mu_A(y^n)\mu_B(y^n))+1)} \\
&= \frac{(\mu_A(x^n)+\mu_B(x^n))\vee(\mu_A(y^n)+\mu_B(y^n)}{2((\mu_A(x^n)\mu_B(x^n)+1)\vee(\mu_A(y^n)\mu_B(y^n)+1))} \\
&= \frac{\mu_A(x^n)+\mu_B(x^n)}{2(\mu_A(x^n)\mu_B(x^n)+1)}\vee\frac{\mu_A(y^n)+\mu_B(y^n)}{2(\mu_A(y^n)\mu_B(y^n)+1)} \\
&= \mu_{A*B}(x^n)\vee\mu_{A*B}(y^n)
\end{aligned}$$

thus

$$\mu_{A*B}(xy) \geq \mu_{A*B}(x^n)\vee\mu_{A*B}(y^n) \tag{26}$$

Consider

$$\begin{aligned}
\nu_{A*B}(x-y) &= \frac{\nu_A(x-y)+\nu_B(x-y)}{2(\nu_A(x-y)\nu_B(x-y)+1)}, \text{ for all } x,y\in R \\
&\leq \frac{(\nu_A(x^n)\vee\nu_A(y^n))+(\nu_B(x^n)\vee\nu_B(y^n))}{2((\nu_A(x^n)\vee\nu_A(y^n))(\nu_B(x^n)\vee\nu_B(y^n))+1)} \\
&= \frac{(\nu_A(x^n)+\nu_B(x^n))\vee(\nu_A(y^n)+\nu_B(y^n))}{2((\nu_A(x^n)\nu_B(x^n)\vee\nu_A(y^n)\nu_B(y^n))+1)} \\
&= \frac{(\nu_A(x^n)+\nu_B(x^n))\vee(\nu_A(y^n)+\nu_B(y^n)}{2((\nu_A(x^n)\nu_B(x^n)+1)\vee(\nu_A(y^n)\nu_B(y^n)+1))}
\end{aligned}$$

$$= \frac{\nu_A(x^n) + \nu_B(x^n)}{2(\nu_A(x^n)\nu_B(x^n) + 1)} \vee \frac{\nu_A(y^n) + \nu_B(y^n)}{2(\nu_A(y^n)\nu_B(y^n) + 1)}$$
$$= \nu_{A*B}(x^n) \vee \nu_{A*B}(y^n)$$

thus

$$\nu_{A*B}(x - y) \leq \nu_{A*B}(x^n) \vee \nu_{A*B}(y^n) \qquad (27)$$

Consider

$$\begin{aligned}
\nu_{A*B}(xy) &= \frac{\nu_A(xy) + \nu_B(xy)}{2(\nu_A(xy)\nu_B(xy) + 1)}, \text{ for all } x, y \in R \\
&\leq \frac{(\nu_A(x^n) \wedge \nu_A(y^n)) + (\nu_B(x^n) \wedge \nu_B(y^n))}{2((\nu_A(x^n) \wedge \nu_A(y^n))(\nu_B(x^n) \wedge \nu_B(y^n)) + 1)} \\
&= \frac{(\nu_A(x^n) + \nu_B(x^n)) \wedge (\nu_A(y^n) + \nu_B(y^n))}{2((\nu_A(x^n)\nu_B(x^n) \wedge \nu_A(y^n)\nu_B(y^n)) + 1)} \\
&= \frac{(\nu_A(x^n) + \nu_B(x^n)) \wedge (\nu_A(y^n) + \nu_B(y^n))}{2((\nu_A(x^n)\nu_B(x^n) + 1) \wedge (\nu_A(y^n)\nu_B(y^n) + 1))} \\
&= \frac{\nu_A(x^n) + \nu_B(x^n)}{2(\nu_A(x^n)\nu_B(x^n) + 1)} \wedge \frac{\nu_A(y^n) + \nu_B(y^n)}{2(\nu_A(y^n)\nu_B(y^n) + 1)} \\
&= \nu_{A*B}(x^n) \wedge \nu_{A*B}(y^n)
\end{aligned}$$

thus

$$\nu_{A*B}(xy) \leq \nu_{A*B}(x^n) \wedge \nu_{A*B}(y^n) \qquad (28)$$

Therefore, from (25), (26), (27) and (28), we get $A * B$ is an ILFSPI. □

Theorem 3.9. *If A and B are any two ILFSPI, then $A \bowtie B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Consider

$$\begin{aligned}
\mu_{A\bowtie B}(x - y) &= \frac{2\mu_A(x - y)\mu_B(x - y)}{\mu_A(x - y) + \mu_B(x - y)} \\
&\geq \frac{2(\mu_A(x^n) \wedge \mu_A(y^n))(\mu_B(x^n) \wedge \mu_B(y^n))}{(\mu_A(x^n) \wedge \mu_A(y^n)) + (\mu_B(x^n) \wedge \mu_B(y^n))} \\
&= \frac{2(\mu_A(x^n)\mu_B(x^n) \wedge \mu_A(y^n)\mu_B(y^n))}{(\mu_A(x^n) + \mu_B(x^n)) \wedge (\mu_A(y^n) + \mu_B(y^n))} \\
&= 2(\frac{\mu_A(x^n)\mu_B(x^n)}{\mu_A(x^n) + \mu_B(x^n)} \wedge \frac{\mu_A(y^n)\mu_B(y^n)}{\mu_A(y^n) + \mu_B(y^n)}) \\
&= \frac{2\mu_A(x^n)\mu_B(x^n)}{\mu_A(x^n) + \mu_B(x^n)} \wedge \frac{2\mu_A(y^n)\mu_B(y^n)}{\mu_A(y^n) + \mu_B(y^n)} \\
&= \mu_{A\bowtie B}(x^n) \wedge \mu_{A\bowtie B}(y^n)
\end{aligned}$$

thus

$$\mu_{A\bowtie B}(x - y) \geq \mu_{A\bowtie B}(x^n) \wedge \mu_{A\bowtie B}(y^n) \qquad (29)$$

Consider

$$\begin{aligned}
\mu_{A\bowtie B}(xy) &= \frac{2\mu_A(xy)\mu_B(xy)}{\mu_A(xy)+\mu_B(xy)} \\
&\geq \frac{2(\mu_A(x^n)\vee\mu_A(y^n))(\mu_B(x^n)\vee\mu_B(y^n))}{(\mu_A(x^n)\vee\mu_A(y^n))+(\mu_B(x^n)\vee\mu_B(y^n))} \\
&= \frac{2(\mu_A(x^n)\mu_B(x^n)\vee\mu_A(y^n)\mu_B(y^n))}{(\mu_A(x^n)+\mu_B(x^n))\vee(\mu_A(y^n)+\mu_B(y^n))} \\
&= 2(\frac{\mu_A(x^n)\mu_B(x^n)}{\mu_A(x^n)+\mu_B(x^n)}\vee\frac{\mu_A(y^n)\mu_B(y^n)}{\mu_A(y^n)+\mu_B(y^n)}) \\
&= \frac{2\mu_A(x^n)\mu_B(x^n)}{\mu_A(x^n)+\mu_B(x^n)}\vee\frac{2\mu_A(y^n)\mu_B(y^n)}{\mu_A(y^n)+\mu_B(y^n)} \\
&= \mu_{A\bowtie B}(x^n)\vee\mu_{A\bowtie B}(y^n)
\end{aligned}$$

thus

$$\mu_{A\bowtie B}(xy) \geq \mu_{A\bowtie B}(x^n)\vee\mu_{A\bowtie B}(y^n) \tag{30}$$

Consider

$$\begin{aligned}
\nu_{A\bowtie B}(x-y) &= \frac{2\nu_A(x-y)\nu_B(x-y)}{\nu_A(x-y)+\nu_B(x-y)},\ \text{for all } x,y\in R \\
&\leq \frac{2(\nu_A(x^n)\vee\nu_A(y^n))(\nu_B(x^n)\vee\nu_B(y^n))}{(\nu_A(x^n)\vee\nu_A(y^n))+(\nu_B(x^n)\vee\nu_B(y^n))} \\
&= \frac{2(\nu_A(x^n)\nu_B(x^n)\vee\nu_A(y^n)\nu_B(y^n))}{(\nu_A(x^n)+\nu_B(x^n))\wedge(\nu_A(y^n)+\nu_B(y^n))} \\
&= 2(\frac{\nu_A(x^n)\nu_B(x^n)}{\nu_A(x^n)+\nu_B(x^n)}\vee\frac{\nu_A(y^n)\nu_B(y^n)}{\nu_A(y^n)+\nu_B(y^n)}) \\
&= \frac{2\nu_A(x^n)\nu_B(x^n)}{\nu_A(x^n)+\nu_B(x^n)}\vee\frac{2\nu_A(y^n)\nu_B(y^n)}{\nu_A(y^n)+\nu_B(y^n)} \\
&= \nu_{A\bowtie B}(x^n)\vee\nu_{A\bowtie B}(y^n)
\end{aligned}$$

thus

$$\nu_{A\bowtie B}(x-y) \leq \nu_{A\bowtie B}(x^n)\vee\nu_{A\bowtie B}(y^n) \tag{31}$$

Consider

$$\begin{aligned}
\nu_{A\bowtie B}(xy) &= \frac{2\nu_A(xy)\nu_B(xy)}{\nu_A(xy)+\nu_B(xy)},\ \text{for all } x,y\in R \\
&\leq \frac{2(\nu_A(x^n)\wedge\nu_A(y^n))(\nu_B(x^n)\wedge\nu_B(y^n))}{(\nu_A(x^n)\wedge\nu_A(y^n))+(\nu_B(x^n)\wedge\nu_B(y^n))} \\
&= \frac{2(\nu_A(x^n)\nu_B(x^n)\wedge\nu_A(y^n)\nu_B(y^n))}{(\nu_A(x^n)+\nu_B(x^n))\wedge(\nu_A(y^n)+\nu_B(y^n))} \\
&= 2(\frac{\nu_A(x^n)\nu_B(x^n)}{\nu_A(x^n)+\nu_B(x^n)}\wedge\frac{\nu_A(y^n)\nu_B(y^n)}{\nu_A(y^n)+\nu_B(y^n)}) \\
&= \frac{2\nu_A(x^n)\nu_B(x^n)}{\nu_A(x^n)+\nu_B(x^n)}\wedge\frac{2\nu_A(y^n)\nu_B(y^n)}{\nu_A(y^n)+\nu_B(y^n)} \\
&= \nu_{A\bowtie B}(x^n)\wedge\nu_{A\bowtie B}(y^n)
\end{aligned}$$

thus

$$\nu_{A\bowtie B}(xy) \leq \nu_{A\bowtie B}(x^n) \wedge \nu_{A\bowtie B}(y^n) \tag{32}$$

Therefore, from (29), (30), (31) and (32), we get $A \bowtie B$ is an ILFSPI. □

Theorem 3.10. *If A and B are any two ILFSPI, then $A \times B$ is an ILFSPI.*

Proof. Let A and B be any two ILFSPI.

Let $(x_1, y_1), (x_2, y_2) \in A \times B$, for every $x_1, x_2 \in A \,\&\, y_1, y_2 \in B$.

Consider

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1)-(x_2,y_2)) &= \mu_{A\times B}(x_1-x_2, y_1-y_2) \\
&= \mu_A(x_1-x_2) \wedge \mu_B(y_1-y_2) \\
&\geq (\mu_A(x_1^n) \wedge \mu_A(x_2^n)) \wedge (\mu_B(y_1^n) \wedge \mu_B(y_2^n)) \\
&= (\mu_A(x_1^n) \wedge \mu_B(y_1^n)) \wedge (\mu_A(x_2^n) \wedge \mu_B(y_2^n)) \\
&= \mu_{A\times B}(x_1^n, y_1^n) \wedge \mu_{A\times B}(x_2^n, y_2^n)
\end{aligned}$$

thus

$$\mu_{A\times B}(x_1,y_1)-(x_2,y_2) \geq \mu_{A\times B}(x_1^n, y_1^n) \wedge \mu_{A\times B}(x_2^n, y_2^n) \tag{33}$$

Consider

$$\begin{aligned}
\mu_{A\times B}((x_1,y_1).(x_2,y_2)) &= \mu_{A\times B}(x_1x_2, y_1y_2) \\
&= \mu_A(x_1x_2) \wedge \mu_B(y_1y_2) \\
&\geq (\mu_A(x_1^n) \vee \mu_A(x_2^n)) \wedge (\mu_B(y_1^n) \vee \mu_B(y_2^n)) \\
&= (\mu_A(x_1^n) \wedge \mu_B(y_1^n)) \vee (\mu_A(x_2^n) \wedge \mu_B(y_2^n)) \\
&= \mu_{A\times B}(x_1^n, y_1^n) \vee \mu_{A\times B}(x_2^n, y_2^n)
\end{aligned}$$

thus

$$\mu_{A\times B}(x_1,y_1)(x_2,y_2) \geq \mu_{A\times B}(x_1^n, y_1^n) \vee \mu_{A\times B}(x_2^n, y_2^n) \tag{34}$$

Consider

$$\begin{aligned}
\nu_{A\times B}((x_1,y_1)-(x_2,y_2)) &= \nu_{A\times B}(x_1-x_2, y_1-y_2) \\
&= \nu_A(x_1-x_2) \vee \nu_B(y_1-y_2) \\
&\leq (\nu_A(x_1^n) \vee \nu_A(x_2^n)) \vee (\nu_B(y_1^n) \vee \nu_B(y_2^n)) \\
&= (\nu_A(x_1^n) \vee \nu_B(y_1^n)) \vee (\nu_A(x_2^n) \vee \nu_B(y_2^n)) \\
&= \nu_{A\times B}(x_1^n, y_1^n) \vee \nu_{A\times B}(x_2^n, y_2^n)
\end{aligned}$$

thus

$$\nu_{A\times B}(x_1,y_1)-(x_2,y_2) \leq \nu_{A\times B}(x_1^n, y_1^n) \vee \nu_{A\times B}(x_2^n, y_2^n) \tag{35}$$

Consider

$$\begin{aligned}
\nu_{A\times B}((x_1,y_1).(x_2,y_2)) &= \nu_{A\times B}(x_1x_2,y_1y_2) \\
&= \nu_A(x_1x_2)\vee\nu_B(y_1y_2) \\
&\leq (\nu_A(x_1^n)\wedge\nu_A(x_2^n))\vee(\nu_B(y_1^n)\wedge\nu_B(y_2^n)) \\
&= (\nu_A(x_1^n)\vee\nu_B(y_1^n))\wedge(\nu_A(x_2^n)\vee\nu_B(y_2^n)) \\
&= \nu_{A\times B}(x_1^n,y_1^n)\wedge\nu_{A\times B}(x_2^n,y_2^n)
\end{aligned}$$

thus

$$\nu_{A\times B}(x_1,y_1)(x_2,y_2) \leq \nu_{A\times B}(x_1^n,y_1^n)\wedge\nu_{A\times B}(x_2^n,y_2^n) \tag{36}$$

Therefore, from (33), (34), (35) and (36), we get $A \times B$ is an ILFSPI. □

References

[1] Atanassov, K. (1986) Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1), 87–96.

[2] Atanassov, K. & Stoeva, S. (1984) Intuitionistic L-fuzzy sets, *Cybernetics and Systems Research*, 2, 539–540.

[3] Atanassov, K. (1994) New operations defined over the Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61, 137–142.

[4] Lehmke, S. (1997) Some properties of fuzzy ideals on a lattice, *Proc. of the* 6^{th} *IEEE Int. Conf. on Fuzzy Systems*, 2, 813–818.

[5] Atallah, M. M. (2001) On the L-fuzzy prime ideal theorem of distributive lattices, *The Journal of Fuzzy Mathematics*, 9(4), 825–831.

[6] Palanivelrajan, M. & Nandhakumar, S. (2012) Some properties of Intuitionistic fuzzy primary and semi primary ideals, *Proc. of 16th Int. Conf. on IFSs, Sofia, 9–10 Sept, Notes on Intuitionistic Fuzzy Sets*, 18(3), 68–74.

[7] Palanivelrajan, M. & Nandhakumar, S. (2012) Some operations of intuitionistic fuzzy primary and semi primary ideals, *Asian Journal of Agebra*, 5(2), 44–49.

[8] Zadeh, L. A. (1965) Fuzzy sets, *Information and Control*, 8, 338–353.