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A method to obtain trapezoidal approximations of intuitionistic fuzzy numbers from trapezoidal approximations of fuzzy numbers

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Abstract The well-known Karush-Kuhn-Tucker theorem can be used, as in the fuzzy case, to find the trapezoidal approximation of a given intuitionistic fuzzy number. The method is quite technical such that obtaining the trapezoidal approximation of an intuitionistic fuzzy numbers from the trapezoidal approximation of a fuzzy number is proposed in the present paper. Among the advantages of this method is the immediate extension of important properties in fuzzy case to intuitionistic fuzzy case.

Key words: Fuzzy number, Intuitionistic fuzzy number, Trapezoidal fuzzy number

1 Introduction

As a generalization of the concept of fuzzy set, the notion of intuitionistic fuzzy set was introduced (see [1], [2]). In the present paper we refer to intuitionistic fuzzy numbers (see [8]), which are particular intuitionistic fuzzy sets and extensions of fuzzy numbers as well.

We approximate fuzzy numbers by real numbers, real intervals, triangular or trapezoidal fuzzy numbers because in this way it is easy to handle and to have natural interpretations of the results. In [10] the problem of finding nearest trapezoidal approximation of a fuzzy number and in [4] the problem of finding the nearest trapezoidal approximation of a fuzzy number preserving the expected interval, with respect to the same average Euclidean distance, are solved. The Karush-Kuhn-Tucker theorem (proposed in this topic by Grzegorzewski and Mrówka [9]) is used in [5] to find the nearest trapezoidal fuzzy number to an intuitionistic fuzzy number, preserving the expected value. The same method is suitable to find the nearest trapezoidal fuzzy number to an intuitionistic fuzzy number (without condition), with respect to intuitionistic fuzzy version of average Euclidean distance. The procedure is quite technical and complicated such that a method to obtain trapezoidal approximations of intuitionistic fuzzy numbers from trapezoidal approximations of fuzzy numbers is elaborated in this paper.

2 Fuzzy numbers and intuitionistic fuzzy numbers

This section contains some basic on fuzzy numbers and intuitionistic fuzzy numbers.

We consider the following well-known description of a fuzzy number u:

$$u(x) = \begin{cases} 0, & \text{if } x \le a_1, \\ l_u(x), & \text{if } a_1 \le x \le a_2, \\ 1, & \text{if } a_2 \le x \le a_3, \\ r_u(x), & \text{if } a_3 \le x \le a_4, \\ 0, & \text{if } a_4 \le x, \end{cases} \tag{1}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}, l_u : [a_1, a_2] \to [0,1]$ is a nondecreasing continuous function, $l_u(a_1) = 0, l_u(a_2) = 1$, called the left side of the fuzzy number u and $r_u : [a_3, a_4] \to [0,1]$ is a nonincreasing continuous function, $r_u(a_3) = 1, r_u(a_4) = 0$, called the right side of the fuzzy number u. The α-cut, $\alpha \in]0,1]$, of a fuzzy number u is the crisp set defined as

$$u_\alpha = \{x \in \mathbb{R} : u(x) \ge \alpha\}.$$

The support or 0-cut u_0 of a fuzzy number u is defined as the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$, that is

$$u_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}.$$

Every α-cut, $\alpha \in [0,1]$, of a fuzzy number u is a closed interval

$$u_\alpha = [u_L(\alpha), u_R(\alpha)],$$

where

$$u_L(\alpha) = \inf\{x \in \mathbb{R} : u(x) \ge \alpha\},$$
$$u_R(\alpha) = \sup\{x \in \mathbb{R} : u(x) \ge \alpha\}$$

for any $\alpha \in]0,1]$. If the sides of the fuzzy number u are strictly monotone then one can see easily that u_L and u_R are inverse functions of l_u and r_u, respectively. Throughout in this paper we consider fuzzy numbers with strictly monotone sides. We denote by $F(\mathbb{R})$ the set of fuzzy numbers.

If $\omega : \mathbb{R} \to [0,1]$ is a fuzzy set such that $1-\omega$ $((1-\omega)(x) = 1-\omega(x), \forall x \in \mathbb{R})$ is a fuzzy number and we denote

$$\omega_\alpha = \{x \in \mathbb{R} : \omega(x) \leq \alpha\}, \alpha \in [0,1[$$
$$\omega_1 = \overline{\{x \in \mathbb{R} : \omega(x) < 1\}},$$

then

$$\omega_\alpha = (1-\omega)_{1-\alpha},$$

for every $\alpha \in [0,1]$. The set ω_α is a closed interval $[\omega_L(\alpha), \omega_R(\alpha)]$, for every $\alpha \in [0,1]$.

Fuzzy numbers with simple membership functions are preferred in practice. The most used such fuzzy numbers are so-called trapezoidal fuzzy numbers given by

$$T(x) = \begin{cases} 0, & \text{if } x \leq t_1, \\ \frac{x-t_1}{t_2-t_1}, & \text{if } t_1 \leq x \leq t_2, \\ 1, & \text{if } t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3}, & \text{if } t_3 \leq x \leq t_4, \\ 0, & \text{if } t_4 \leq x, \end{cases}$$

where $t_1, t_2, t_3, t_4 \in \mathbb{R}$. When $t_2 = t_3$ we obtain so-called triangular fuzzy numbers, when $t_1 = t_2$ and $t_3 = t_4$ we obtain closed intervals and in the case $t_1 = t_2 = t_3 = t_4$ we obtain crisp numbers (by convention, we ignore the situations where the denominator is equal to zero). We denote $T = (t_1, t_2, t_3, t_4)$ a trapezoidal fuzzy number as above and $F^T(\mathbb{R})$ the set of all trapezoidal fuzzy numbers. It is easily to see that $T_L(\alpha) = t_1 + (t_2 - t_1)\alpha$ and $T_R(\alpha) = t_4 - (t_4 - t_3)\alpha$, for every $\alpha \in [0,1]$.

Let $u, v \in F(\mathbb{R}), u_\alpha = [u_L(\alpha), u_R(\alpha)], v_\alpha = [v_L(\alpha), v_R(\alpha)], \alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We consider the sum $u+v$ and the scalar multiplication $\lambda \cdot u$ by

$$(u+v)_\alpha = u_\alpha + v_\alpha = [u_L(\alpha) + v_L(\alpha), u_R(\alpha) + v_R(\alpha)] \tag{2}$$

and

$$(\lambda \cdot u)_\alpha = \lambda u_\alpha = \begin{cases} [\lambda u_L(\alpha), \lambda u_R(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda u_R(\alpha), \lambda u_L(\alpha)], & \text{if } \lambda < 0, \end{cases} \tag{3}$$

respectively, for every $\alpha \in [0,1]$. In the case of the trapezoidal fuzzy numbers $T = (t_1, t_2, t_3, t_4)$ and $S = (s_1, s_2, s_3, s_4)$ we obtain

$$T + S = (t_1 + s_1, t_2 + s_2, t_3 + s_3, t_4 + s_4)$$

and

$$\lambda \cdot T = \begin{cases} (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4), & \text{if } \lambda \geq 0, \\ (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1), & \text{if } \lambda < 0. \end{cases}$$

Definition 1 *([1], [2]) Let $X \neq \emptyset$ be a given set. An intuitionistic fuzzy set in X is an object A given by*

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle ; x \in X\},$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ satisfy the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

for every $x \in X$.

Definition 2 *An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle ; x \in \mathbb{R}\}$ such that μ_A and $1 - \nu_A$ are fuzzy numbers, where*

$$(1 - \nu_A)(x) = 1 - \nu_A(x), \forall x \in \mathbb{R},$$

is called an intuitionistic fuzzy number.

We denote by $A = \langle \mu_A, \nu_A \rangle$ an intuitionistic fuzzy number and by $IF(\mathbb{R})$ the set of all intuitionistic fuzzy numbers. It is obvious that any fuzzy number u can be represented as an intuitionistic fuzzy number by $\langle u, 1 - u \rangle$.

With respect to the α-cuts of the fuzzy number $1 - \nu_A$ the following equalities are immediate:

$$(1 - \nu_A)_L(\alpha) = \nu_{A_L}(1 - \alpha) \tag{4}$$

and

$$(1 - \nu_A)_R(\alpha) = \nu_{A_R}(1 - \alpha), \tag{5}$$

for every $\alpha \in [0,1]$.

We define the addition $A + B \in IF(\mathbb{R})$ of $A = \langle \mu_A, \nu_A \rangle, B = \langle \mu_B, \nu_B \rangle \in IF(\mathbb{R})$ by

$$A + B = \langle \mu_{A+B}, \nu_{A+B} \rangle,$$

where $\mu_{A+B} = \mu_A + \mu_B$ and ν_{A+B} is given by

$$1 - \nu_{A+B} = (1 - \nu_A) + (1 - \nu_B).$$

We define the scalar multiplication $\lambda \cdot A \in IF(\mathbb{R})$ of $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R}), \lambda \in \mathbb{R}$ by

$$\lambda \cdot A = \langle \mu_{\lambda \cdot A}, \nu_{\lambda \cdot A} \rangle,$$

where $\mu_{\lambda \cdot A} = \lambda \cdot \mu_A$ and $\nu_{\lambda \cdot A}$ is given by

$$1 - \nu_{\lambda \cdot A} = \lambda \cdot (1 - \nu_A).$$

A distance on the set of fuzzy numbers, called average Euclidean distance, is defined by (see, e. g., [4])

$$d^2(u,v) = \int_0^1 (u_L(\alpha) - v_L(\alpha))^2 \, d\alpha + \int_0^1 (u_R(\alpha) - v_R(\alpha))^2 \, d\alpha,$$

where u and v are arbitrary fuzzy numbers with α-cuts $u_\alpha = [u_L(\alpha), u_R(\alpha)]$ and $v_\alpha = [v_L(\alpha), v_R(\alpha)], \alpha \in [0,1]$.

In the intuitionistic fuzzy case the distance d becomes (see [3])

$$\begin{aligned} \widetilde{d}^2(A,B) &= \frac{1}{2}\int_0^1 (\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha))^2 \, d\alpha + \frac{1}{2}\int_0^1 (\mu_{A_R}(\alpha) - \mu_{B_R}(\alpha))^2 \, d\alpha \\ &+ \frac{1}{2}\int_0^1 (\nu_{A_L}(\alpha) - \nu_{B_L}(\alpha))^2 \, d\alpha + \frac{1}{2}\int_0^1 (\nu_{A_R}(\alpha) - \nu_{B_R}(\alpha))^2 \, d\alpha, \end{aligned}$$

where $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R})$ and $B = \langle \mu_B, \nu_B \rangle \in IF(\mathbb{R})$.

3 Trapezoidal approximation of fuzzy numbers

The nearest trapezoidal fuzzy number to a given fuzzy number u, or the trapezoidal approximation of u, denoted by $T_d(u)$, is the trapezoidal fuzzy number which minimizes $d(u,T)$, where $T \in F^T(\mathbb{R})$. The following result (with other notations: $l_e = \int_0^1 u_L(\alpha)\,d\alpha$, $u_e = \int_0^1 u_R(\alpha)\,d\alpha$, $x_e = 12\int_0^1 \left(\alpha - \frac{1}{2}\right) u_L(\alpha)\,d\alpha$, $y_e = -12\int_0^1 \left(\alpha - \frac{1}{2}\right) u_R(\alpha)\,d\alpha$) was already proved in paper [10], Theorem 4.4 (see also [6], Corollary 8).

Theorem 3 *Let $u, u_\alpha = [u_L(\alpha), u_R(\alpha)], \alpha \in [0,1]$, be a fuzzy number and*

$$T_d(u) = (t_1(u), t_2(u), t_3(u), t_4(u)),$$

the nearest (with respect to the metric d) trapezoidal fuzzy number to fuzzy number u.

(i) If

$$-\int_0^1 u_L(\alpha)\,d\alpha + \int_0^1 u_R(\alpha)\,d\alpha + 3\int_0^1 \alpha u_L(\alpha)\,d\alpha - 3\int_0^1 \alpha u_R(\alpha)\,d\alpha \leq 0$$

then

$$t_1(u) = 4\int_0^1 u_L(\alpha)\,d\alpha - 6\int_0^1 \alpha u_L(\alpha)\,d\alpha,$$

$$t_2(u) = -2\int_0^1 u_L(\alpha)\,d\alpha + 6\int_0^1 \alpha u_L(\alpha)\,d\alpha,$$
$$t_3(u) = -2\int_0^1 u_R(\alpha)\,d\alpha + 6\int_0^1 \alpha u_R(\alpha)\,d\alpha,$$
$$t_4(u) = 4\int_0^1 u_R(\alpha)\,d\alpha - 6\int_0^1 \alpha u_R(\alpha)\,d\alpha.$$

(ii) If

$$-\int_0^1 u_L(\alpha)\,d\alpha + \int_0^1 u_R(\alpha)\,d\alpha + 3\int_0^1 \alpha u_L(\alpha)\,d\alpha - 3\int_0^1 \alpha u_R(\alpha)\,d\alpha > 0$$
$$-3\int_0^1 u_L(\alpha)\,d\alpha - \int_0^1 u_R(\alpha)\,d\alpha + 5\int_0^1 \alpha u_L(\alpha)\,d\alpha + 3\int_0^1 \alpha u_R(\alpha)\,d\alpha \geq 0$$

and

$$\int_0^1 u_L(\alpha)\,d\alpha + 3\int_0^1 u_R(\alpha)\,d\alpha - 3\int_0^1 \alpha u_L(\alpha)\,d\alpha - 5\int_0^1 \alpha u_R(\alpha)\,d\alpha \geq 0$$

then

$$t_1(u) = \frac{7}{2}\int_0^1 u_L(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 u_R(\alpha)\,d\alpha - \frac{9}{2}\int_0^1 \alpha u_L(\alpha)\,d\alpha - \frac{3}{2}\int_0^1 \alpha u_R(\alpha)\,d\alpha,$$
$$t_2(u) = t_3(u) = -\int_0^1 u_L(\alpha)\,d\alpha - \int_0^1 u_R(\alpha)\,d\alpha$$
$$+ 3\int_0^1 \alpha u_L(\alpha)\,d\alpha + 3\int_0^1 \alpha u_R(\alpha)\,d\alpha,$$
$$t_4(u) = \frac{1}{2}\int_0^1 u_L(\alpha)\,d\alpha + \frac{7}{2}\int_0^1 u_R(\alpha)\,d\alpha - \frac{3}{2}\int_0^1 \alpha u_L(\alpha)\,d\alpha - \frac{9}{2}\int_0^1 \alpha u_R(\alpha)\,d\alpha;$$

(iii) If

$$-\int_0^1 u_L(\alpha)\,d\alpha - 3\int_0^1 u_R(\alpha)\,d\alpha + 3\int_0^1 \alpha u_L(\alpha)\,d\alpha + 5\int_0^1 \alpha u_R(\alpha)\,d\alpha > 0$$

then

$$t_1(u) = \frac{16}{5}\int_0^1 u_L(\alpha)\,d\alpha - \frac{2}{5}\int_0^1 u_R(\alpha)\,d\alpha - \frac{18}{5}\int_0^1 \alpha u_L(\alpha)\,d\alpha,$$
$$t_2(u) = t_3(u) = t_4(u) = -\frac{2}{5}\int_0^1 u_L(\alpha)\,d\alpha + \frac{4}{5}\int_0^1 u_R(\alpha)\,d\alpha + \frac{6}{5}\int_0^1 \alpha u_L(\alpha)\,d\alpha;$$

(iv) If

$$3\int_0^1 u_L(\alpha)\,d\alpha+\int_0^1 u_R(\alpha)\,d\alpha-5\int_0^1 \alpha u_L(\alpha)\,d\alpha-3\int_0^1 \alpha u_R(\alpha)\,d\alpha>0$$

then

$$t_1(u)=t_2(u)=t_3(u)=\frac{4}{5}\int_0^1 u_L(\alpha)\,d\alpha-\frac{2}{5}\int_0^1 u_R(\alpha)\,d\alpha+\frac{6}{5}\int_0^1 \alpha u_R(\alpha)\,d\alpha,$$
$$t_4(u)=-\frac{2}{5}\int_0^1 u_L(\alpha)\,d\alpha+\frac{16}{5}\int_0^1 u_R(\alpha)\,d\alpha-\frac{18}{5}\int_0^1 \alpha u_R(\alpha)\,d\alpha.$$

4 Properties of average Euclidean distance on intuitionistic fuzzy numbers

The following properties are important in the proof of the main result.

Theorem 4 *Let $A=\langle\mu_A,\nu_A\rangle\in IF(\mathbb{R}),B=\langle\mu_B,\nu_B\rangle\in IF(\mathbb{R})$ and $u\in F(\mathbb{R})$. Then*

(i)

$$\widetilde{d}^2(A,B)=\frac{1}{2}d^2(\mu_A,\mu_B)+\frac{1}{2}d^2(1-\nu_A,1-\nu_B);\tag{6}$$

(ii)

$$\widetilde{d}^2(A,u)=d^2\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),u\right)+\frac{1}{4}\int_0^1\left(\mu_{A_L}(\alpha)-(1-\nu_A)_L(\alpha)\right)^2d\alpha$$
$$+\frac{1}{4}\int_0^1\left(\mu_{A_R}(\alpha)-(1-\nu_A)_R(\alpha)\right)^2d\alpha.\tag{7}$$

Proof. (i) The definition of distance d and relations (4) and (5) lead to

$$\begin{aligned}
&d^2(1-\nu_A,1-\nu_B)\\
&=\int_0^1\left((1-\nu_A)_L(\alpha)-(1-\nu_B)_L(\alpha)\right)^2d\alpha\\
&+\int_0^1((1-\nu_A)_R(\alpha)-(1-\nu_B)_R(\alpha))^2d\alpha\\
&=\int_0^1((1-\nu_A)_L(1-\alpha)-(1-\nu_B)_L(1-\alpha))^2d\alpha\\
&+\int_0^1((1-\nu_A)_R(1-\alpha)-(1-\nu_B)_R(1-\alpha))^2d\alpha\\
&=\int_0^1(\nu_{A_L}(\alpha)-\nu_{B_L}(\alpha))^2d\alpha+\int_0^1(\nu_{A_R}(\alpha)-\nu_{B_R}(\alpha))^2d\alpha,
\end{aligned}$$

hence

$$\widetilde{d}^2(A,B)=\frac{1}{2}d^2(\mu_A,\mu_B)+\frac{1}{2}d^2(1-\nu_A,1-\nu_B).$$

(ii) Because $u\in F(\mathbb{R})$ can be represented as an intuitionistic fuzzy number by $\langle u,1-u\rangle$ and taking into account the above result we obtain

$$\begin{aligned}
&\widetilde{d}^2(A,u)\\
&=\frac{1}{2}d^2(\mu_A,u)+\frac{1}{2}d^2(1-\nu_A,u)\\
&=\frac{1}{2}\left(\int_0^1(\mu_{A_L}(\alpha)-u_L(\alpha))^2d\alpha+\int_0^1(\mu_{A_R}(\alpha)-u_R(\alpha))^2d\alpha\right)\\
&+\frac{1}{2}\left(\int_0^1((1-\nu_A)_L(\alpha)-u_L(\alpha))^2d\alpha+\int_0^1((1-\nu_A)_R(\alpha)-u_R(\alpha))^2d\alpha\right)\\
&=\frac{1}{2}\int_0^1\left((\mu_{A_L}(\alpha)-u_L(\alpha))^2+((1-\nu_A)_L(\alpha)-u_L(\alpha))^2\right)d\alpha\\
&+\frac{1}{2}\int_0^1\left((\mu_{A_R}(\alpha)-u_R(\alpha))^2+((1-\nu_A)_R(\alpha)-u_R(\alpha))^2\right)d\alpha\\
&=\int_0^1\left((\frac{1}{2}\mu_{A_L}(\alpha)+\frac{1}{2}(1-\nu_A)_L(\alpha))-u_L(\alpha)\right)^2d\alpha\\
&+\int_0^1\left((\frac{1}{2}\mu_{A_R}(\alpha)+\frac{1}{2}(1-\nu_A)_R(\alpha))-u_R(\alpha)\right)^2d\alpha\\
&+\frac{1}{4}\int_0^1\left(\mu_{A_L}(\alpha)-(1-\nu_A)_L(\alpha)\right)^2d\alpha+\frac{1}{4}\int_0^1\left(\mu_{A_R}(\alpha)-(1-\nu_A)_R(\alpha)\right)^2d\alpha\\
&=d^2\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),u\right)+\frac{1}{4}\int_0^1\left(\mu_{A_L}(\alpha)-(1-\nu_A)_L(\alpha)\right)^2d\alpha\\
&+\frac{1}{4}\int_0^1\left(\mu_{A_R}(\alpha)-(1-\nu_A)_R(\alpha)\right)^2d\alpha.
\end{aligned}$$

■

5 Trapezoidal approximation of intuitionistic fuzzy numbers

The nearest trapezoidal fuzzy number to a given intuitionistic fuzzy number A, or the trapezoidal approximation of A, denoted by $T_{\widetilde{d}}(A)$, is the trapezoidal fuzzy number which minimizes $\widetilde{d}^2(A,T)$, where $T\in F^T(\mathbb{R})$.

Theorem 5 *If $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number then*

$$T_{\widetilde{d}}(A) = T_d \left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A) \right).$$

Proof. Taking into account (7) we obtain $\widetilde{d}^2(A, T)$, where $T \in F^T(\mathbb{R})$, is minimum if and only if $d^2\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A), T\right)$, where $T \in F^T(\mathbb{R})$, is minimum. But $d^2\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A), T\right)$ is minimum if and only if $T = T_d\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A)\right)$. ■

Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number,

$$(\mu_A)_\alpha = [\mu_{A_L}(\alpha), \mu_{A_R}(\alpha)],$$
$$(\nu_A)_\alpha = [\nu_{A_L}(\alpha), \nu_{A_R}(\alpha)], \alpha \in [0, 1].$$

We denote

$$m_L = \int_0^1 \mu_{A_L}(\alpha)\, d\alpha, m_R = \int_0^1 \mu_{A_R}(\alpha)\, d\alpha,$$
$$n_L = \int_0^1 \nu_{A_L}(\alpha)\, d\alpha, n_R = \int_0^1 \nu_{A_R}(\alpha)\, d\alpha,$$
$$M_L = \int_0^1 \alpha\mu_{A_L}(\alpha)\, d\alpha, M_R = \int_0^1 \alpha\mu_{A_R}(\alpha)\, d\alpha,$$
$$N_L = \int_0^1 \alpha\nu_{A_L}(\alpha)\, d\alpha, N_R = \int_0^1 \alpha\nu_{A_R}(\alpha)\, d\alpha.$$

As an immediate consequence of Theorem 5 and Theorem 3 we obtain the following result:

Theorem 6 *Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number and*

$$T_{\widetilde{d}}(A) = (t_1(A), t_2(A), t_3(A), t_4(A))$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to A.

(i) If

$$m_L - m_R - 3M_L + 3M_R - 2n_L + 2n_R + 3N_L - 3N_R \geq 0,$$

then

$$t_1(A) = 2m_L - 3M_L - n_L + 3N_L,$$
$$t_2(A) = -m_L + 3M_L + 2n_L - 3N_L,$$
$$t_3(A) = -m_R + 3M_R + 2n_R - 3N_R,$$
$$t_4(A) = 2m_R - 3M_R - n_R + 3N_R;$$

(ii) If

$$\begin{aligned}
m_L - m_R - 3M_L + 3M_R - 2n_L + 2n_R + 3N_L - 3N_R &< 0, \\
3m_L + m_R - 5M_L - 3M_R - 2n_L - 2n_R + 5N_L + 3N_R &\le 0, \\
-m_L - 3m_R + 3M_L + 5M_R + 2n_L + 2n_R - 3N_L - 5N_R &\le 0
\end{aligned}$$

then

$$\begin{aligned}
t_1(A) &= \frac{7}{4}m_L + \frac{1}{4}m_R - \frac{9}{4}M_L - \frac{3}{4}M_R - \frac{1}{2}n_L - \frac{1}{2}n_R + \frac{9}{4}N_L + \frac{3}{4}N_R, \\
t_2(A) &= t_3(A) = -\frac{1}{2}m_L - \frac{1}{2}m_R + \frac{3}{2}M_L + \frac{3}{2}M_R + n_L + n_R - \frac{3}{2}N_L - \frac{3}{2}N_R, \\
t_4(A) &= \frac{1}{4}m_L + \frac{7}{4}m_R - \frac{3}{4}M_L - \frac{9}{4}M_R - \frac{1}{2}n_L - \frac{1}{2}n_R + \frac{3}{4}N_L + \frac{9}{4}N_R;
\end{aligned}$$

(iii) If

$$-m_L - 3m_R + 3M_L + 5M_R + 2n_L + 2n_R - 3N_L - 5N_R > 0$$

then

$$\begin{aligned}
t_1(A) &= \frac{8}{5}m_L - \frac{1}{5}m_R - \frac{9}{5}M_L - \frac{1}{5}n_L - \frac{1}{5}n_R + \frac{9}{5}N_L, \\
t_2(A) &= t_3(A) = t_4(A) = -\frac{1}{5}m_L + \frac{2}{5}m_R + \frac{3}{5}M_L + \frac{2}{5}n_L + \frac{2}{5}n_R - \frac{3}{5}N_L.
\end{aligned}$$

(iv) If

$$3m_L + m_R - 5M_L - 3M_R - 2n_L - 2n_R + 5N_L + 3N_R > 0$$

then

$$\begin{aligned}
t_1(A) &= t_2(A) = t_3(A) = \frac{2}{5}m_L - \frac{1}{5}m_R + \frac{3}{5}M_R + \frac{2}{5}n_L + \frac{2}{5}n_R - \frac{3}{5}N_R, \\
t_4(A) &= -\frac{1}{5}m_L + \frac{8}{5}m_R - \frac{9}{5}M_R - \frac{1}{5}n_L - \frac{1}{5}n_R + \frac{9}{5}N_R.
\end{aligned}$$

6 Properties of the trapezoidal approximation of intuitionistic fuzzy numbers

The approximation operator of fuzzy numbers by trapezoidal fuzzy numbers $T_d : F(\mathbb{R}) \to F^T(\mathbb{R})$ has the properties of translation invariance (i.e., $T_d(u+z) = T_d(u)+z$, for every $z \in \mathbb{R}$ and $u \in F(\mathbb{R})$), scale invariance (i.e., $T_d(\lambda \cdot u) = \lambda \cdot T_d(u)$, for every $\lambda \in \mathbb{R}$ and $u \in F(\mathbb{R})$) and continuity (i.e., $\forall(\varepsilon > 0), \exists(\delta > 0) : d(A,B) < \delta \Longrightarrow d(T_d(A), T_d(B)) < \varepsilon$). Theorem 5 help us to obtain the same properties for the approximation operator $T_{\widetilde{d}} : IF(\mathbb{R}) \to F^T(\mathbb{R})$.

Theorem 7 *(i)*

$$T_{\widetilde{d}}(A+z) = T_{\widetilde{d}}(A) + z$$

for every $z \in \mathbb{R}$ and $A \in IF(\mathbb{R})$;

(ii)

$$T_{\widetilde{d}}(\lambda \cdot A) = \lambda \cdot T_{\widetilde{d}}(A)$$

for every $\lambda \in \mathbb{R}$ and $A \in IF(\mathbb{R})$;

(iii) $T_{\widetilde{d}}$ is continuous, i.e.,

$$\forall\,(\varepsilon > 0)\,, \exists\,(\delta > 0) : \widetilde{d}\,(A, B) < \delta \Longrightarrow \widetilde{d}\left(T_{\widetilde{d}}\,(A)\,, T_{\widetilde{d}}\,(B)\right) < \varepsilon.$$

Proof. *(i)* Let $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R})$ and $z \in \mathbb{R}$. According to the definition of addition and taking into account Theorem 5 we have

$$\begin{aligned}
&T_{\widetilde{d}}(A+z) \\
&= T_{\widetilde{d}}(\langle \mu_{A+z}, \nu_{A+z} \rangle) \\
&= T_d\left(\frac{1}{2} \cdot (\mu_A + z) + \frac{1}{2} \cdot ((1-\nu_A) + z)\right) \\
&= T_d\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1-\nu_A) + z)\right) \\
&= T_d\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1-\nu_A)\right) + z \\
&= T_{\widetilde{d}}(A) + z.
\end{aligned}$$

(ii) Let $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R})$ and $\lambda \in \mathbb{R}$. The definition of the scalar multiplication, Theorem 5 and the property of scale invariance in the fuzzy case implies

$$\begin{aligned}
&T_{\widetilde{d}}(\lambda \cdot A) \\
&= T_{\widetilde{d}}(\langle \mu_{\lambda \cdot A}, \nu_{\lambda \cdot A} \rangle) \\
&= T_d\left(\frac{1}{2}\lambda \cdot \mu_A + \frac{1}{2}\lambda \cdot (1-\nu_A)\right) \\
&= T_d\left(\lambda \cdot (\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1-\nu_A))\right) \\
&= \lambda \cdot T_d\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1-\nu_A)\right) \\
&= \lambda \cdot T_{\widetilde{d}}(A).
\end{aligned}$$

(iii) Let $A = \langle \mu_A, \nu_A \rangle \in IF(\mathbb{R})$ and $B = \langle \mu_B, \nu_B \rangle \in IF(\mathbb{R})$. According with Theorem 5 we get

$$T_{\widetilde{d}}(A) = T_d\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1-\nu_A)\right)$$

and

$$T_{\widetilde{d}}(B) = T_d\left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right).$$

Because the operator T_d is Lipschitz with constant 1 ([10], Proposition 6.5) we obtain

$$\begin{aligned}
& d\left(T_d\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right), T_d\left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right)\right) \\
& \leq d\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A), \frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& d^2\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A), \frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right) \\
& = \int_0^1 \left(\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right)_L(\alpha) - \left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right)_L(\alpha)\right)^2 d\alpha \\
& + \int_0^1 \left(\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right)_R(\alpha) - \left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right)_R(\alpha)\right)^2 d\alpha \\
& = \int_0^1 \left(\frac{1}{2}\mu_{A_L}(\alpha) + \frac{1}{2}(1-\nu_A)_L(\alpha) - \frac{1}{2}\mu_{B_L}(\alpha) - \frac{1}{2}(1-\nu_B)_L(\alpha)\right)^2 d\alpha \\
& + \int_0^1 \left(\frac{1}{2}\mu_{A_R}(\alpha) + \frac{1}{2}(1-\nu_A)_R(\alpha) - \frac{1}{2}\mu_{B_R}(\alpha) - \frac{1}{2}(1-\nu_B)_R(\alpha)\right)^2 d\alpha \\
& = \frac{1}{4}\int_0^1 \left(\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha) + (1-\nu_A)_L(\alpha) - (1-\nu_B)_L(\alpha)\right)^2 d\alpha \\
& + \frac{1}{4}\int_0^1 \left(\mu_{A_R}(\alpha) - \mu_{B_R}(\alpha) + (1-\nu_A)_R(\alpha) - (1-\nu_B)_R(\alpha)\right)^2 d\alpha \\
& \leq \frac{1}{2}\int_0^1 \left(\mu_{A_L}(\alpha) - \mu_{B_L}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left((1-\nu_A)_L(\alpha) - (1-\nu_B)_L(\alpha)\right)^2 d\alpha \\
& + \frac{1}{2}\int_0^1 \left((\mu_{A_R}(\alpha) - \mu_{B_R}(\alpha)\right)^2 d\alpha + \frac{1}{2}\int_0^1 \left((1-\nu_A)_R(\alpha) - (1-\nu_B)_R(\alpha)\right)^2 d\alpha \\
& = \frac{1}{2}d^2(\mu_A, \mu_B) + \frac{1}{2}d^2(1-\nu_A, 1-\nu_B) \\
& = \widetilde{d}^2(A, B)
\end{aligned}$$

We get

$$\widetilde{d}\left(T_{\widetilde{d}}(A), T_{\widetilde{d}}(A)\right) = d\left(T_{\widetilde{d}}(A), T_{\widetilde{d}}(A)\right) \leq \widetilde{d}(A, B),$$

that is $T_{\widetilde{d}}$ is continuous with respect to distance $\widetilde{d}$. ■

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