First Int. Workshop on IFSs
Banska Bystrica, 22 Sept. 2005
NIFS 11 (2005), 6, 26-28

Measurable elements on IF-sets

Alžbeta Michalíková

Faculty of Natural Sciences, Matej Bel University
Department of Mathematics
Tajovského 40
974 01 Banská Bystrica, Slovakia
e-mail: *michalik@fpv.umb.sk*

Abstract. In this paper we study an outer measure on IF-sets as a mapping to the set of all compact subintervals of the unit interval. We characterize the properties of the outer measure by the help of the properties of functions given by the edges of the intervals.
Keywords: IF-sets, outer measure, measurable element,

1 Introduction

We shall consider the set $\mathcal{F} = \{(f_A, g_A); f_A, g_A : \Omega \to \langle 0,1\rangle, f_A + g_A \leq 1\}$ as a partially ordered set with the ordering

$$A \subset B \Longleftrightarrow (f_A, g_A) \leq (f_B, g_B) \Longleftrightarrow f_A \leq f_B,\ g_A \geq g_B$$

With respect to this ordering, $\mathcal{F}$ is a lattice with the operations

$$(f_A, g_A) \vee (f_B, g_B) = (f_A \vee f_B, g_A \wedge g_B)$$

$$(f_A, g_A) \wedge (f_B, g_B) = (f_A \wedge f_B, g_A \vee g_B)$$

and the least element (0, 1) and the greatest element (1, 0). In paper [4] there have been introduced the binary operations

$$A \hat{+} B = (f_A, g_A)\ \hat{+}\ (f_B, g_B) = (f_A + f_B,\ g_A + g_B - 1)$$

$$A \hat{-} B = (f_A, g_A)\ \hat{-}\ (f_B, g_B) = (f_A - f_B,\ g_A - g_B + 1)$$

There are also defined Lukasiewicz operations on $\mathcal{F}$

$$A \oplus B = (f_A, g_A)\ \oplus\ (f_B, g_B) = (f_A \oplus f_B,\ g_A \odot g_B)$$

$$A \odot B = (f_A, g_A)\ \odot\ (f_B, g_B) = (f_A \odot f_B,\ g_A \oplus g_B)$$

where

$$f_A \oplus f_B = (f_A + f_B) \wedge 1\ ,\ \ f_A \odot f_B = (f_A + f_B - 1) \vee 0$$

2 Carathéodory outer measure

Remark 2.1 Since outer measure is a function from $\mathcal{F}$ to the set $\mathcal{I}$ of all compact intervals we recall that

$$\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b+d\rangle$$

and

$$\langle a,b\rangle \leq \langle c,d\rangle \Longleftrightarrow a \leq c \ \text{ and } \ b \leq d$$

Definition 2.2 *Let $\mathcal{F}$ be a set of IF-sets. By an outer measure we mean a function $\lambda^* : \mathcal{F} \to \langle 0,1\rangle$ satisfying the following conditions:*
(1) $\lambda^*((0,1)) = 0$;
(2) $\lambda^*((1,0)) = 1$;
(3) $\lambda^*(A \hat{+} B) \leq \lambda^*(A) + \lambda^*(B)$
(4) *if* $A \subset B$ *then* $\lambda^*(A) \leq \lambda^*(B)$

Definition 2.3 *Let $\lambda^* : \mathcal{F} \to \mathcal{I}$ be an outer measure. An element $A \in \mathcal{F}$ is called measurable if*

$$\lambda^*(H) = \lambda^*(H \wedge A) + \lambda^*(H \hat{-} (H \wedge A))$$

for each $H \in \mathcal{F}$.

Remark 2.4 In the following the operations '$\wedge$' and '$\vee$' take precedence over the operations '$+$', '$-$'; thus $(H \hat{-} H \wedge A)$ denotes $(H \hat{-} (H \wedge A))$.

Theorem 2.5 *The measurable elements of $\mathcal{F}$ form a lattice.*

Proof.
(i) We show that if A, B are the measurable elements, then $A \wedge B$ is also the measurable element. Because λ^* is subadditive it is sufficient to show an inequality

$$\lambda^*(H) \geq \lambda^*(H \wedge A) + \lambda^*(H \hat{-} H \wedge A)$$

Let A, B be the measurable elements. Then for any $H \in \mathcal{F}$

$$\lambda^*(H) = \lambda^*(H \wedge A) + \lambda^*(H \hat{-} H \wedge A)$$

and $H \wedge A \in \mathcal{F}$ therefore

$$\lambda^*(H \wedge A) = \lambda^*(H \wedge A \wedge B) + \lambda^*(H \wedge A \hat{-} H \wedge A \wedge B)$$

Then

$$\lambda^*(H) = \lambda^*(H \wedge A \wedge B) + \lambda^*(H \wedge A \hat{-} H \wedge A \wedge B) + \lambda^*(H \hat{-} H \wedge A) \geq$$
$$\geq \lambda^*(H \wedge A \wedge B) + \lambda^*(H \wedge A \hat{-} H \wedge A \wedge B \hat{+} H \hat{-} H \wedge A) =$$
$$= \lambda^*(H \wedge A \wedge B) + \lambda^*(H \hat{-} H \wedge A \wedge B)$$

It proves that $A \wedge B$ is the measurable element.
(ii) We show that if A, B are the measurable elements, then $A \vee B$ is also the measurable element. Since $H \hat{-} H \wedge A = H \vee A \hat{-} A$ we have

$$\lambda^*(H \hat{-} H \wedge A) = \lambda^*(H \vee A \hat{-} A)$$

Therefore if A is measurable then for any $H \in \mathcal{F}$

$$\lambda^*(H) - \lambda^*(H \wedge A) = \lambda^*(H \vee A) - \lambda^*(A)$$

or

$$\lambda^*(H) + \lambda^*(A) = \lambda^*(H \wedge A) + \lambda^*(H \vee A)$$

Let $A, B, A \wedge B$ be the measurable elements, then

$$\lambda^*(H \wedge A \wedge B) = \lambda^*((H \wedge A) \wedge (H \wedge B)) =$$

$$= \lambda^*(H \wedge A) + \lambda^*(H \wedge B) - \lambda^*(H \wedge (A \vee B))$$

and also

$$\lambda^*(H \hat{-} H \wedge A \wedge B) = \lambda^*((H \hat{-} H \wedge A) \vee (H \hat{-} H \wedge B)) =$$

$$= \lambda^*(H \hat{-} H \wedge A) + \lambda^*(H \hat{-} H \wedge B) - \lambda^*((H \hat{-} H \wedge A) \wedge (H \hat{-} H \wedge B))$$

$$= \lambda^*(H \hat{-} H \wedge A) + \lambda^*(H \hat{-} H \wedge B) - \lambda^*(H \hat{-} H \wedge (A \vee B))$$

Therefore

$$\lambda^*(H \wedge (A \vee B)) + \lambda^*(H \hat{-} H \wedge (A \vee B)) =$$

$$= \lambda^*(H \wedge A) + \lambda^*(H \hat{-} H \wedge A) + \lambda^*(H \wedge B) + \lambda^*(H \hat{-} H \wedge B) -$$

$$- \lambda^*(H \wedge A \wedge B) - \lambda^*(H \hat{-} H \wedge A \wedge B) =$$

$$= \lambda^*(H) + \lambda^*(H) - \lambda^*(H) = \lambda^*(H)$$

It proves that $A \vee B$ is the measurable element and all measurable elements of $\mathcal{F}$ form a lattice.

Acknowledgements: This paper was supported by Grant VEGA 1/2002/05.

References

[1] Cignoli R., D'Ottaviano I.M.L., Mundici D.(2000), *Foundations of Many - Valued Reasoning, Trends in Logic,* Studia Logica Library, **7**, Kluwer, Dordrecht.

[2] Michalíková, A., *Carathéodory outer measure on MV - algebras*, to appear in Proceeding of IWIFSGN'2005, September 16, Warsaw, Poland.

[3] Riečan, B. (1977), *On the Carathéodory method of the extension of measures and integrals*, Math. Slovaca **27**, No. 4, 365 - 374.

[4] Riečan, B. (2005), *On the entropy on the Lukasiewicz square*, Joint EUSFLAT - LFA 2005, Barcelona, September 7 - 9, 330 - 333

[5] Riečan, B. - Mundici, D. (2002), *Probability on MV-algebras*, Handbook of Measure Theory (E. Pap ed.), Elsevier, Amsterdam, 869 - 909.

[6] Srivastava, P. (2003), *Carathéodory outer measure and measurability*, Indian Journal of Matematics **45**, No. 1, 89 - 104.