NIFS 4 (1998) 4, 41-47

The convergence of a sequence of intuitionistic fuzzy sets and intuitionistic (fuzzy) measure

A.I. BAN
Department of Mathematics, University of Oradea,
str. Armatei Române 5, 3700 Oradea, România
e-mails: aiban@hs.uoradea.ro; aban@lego.rdsor.ro

Abstract: In this paper we define the concept of intuitionistic (fuzzy) measure. For this we introduce the limit of a sequence of intuitionistic fuzzy sets and we prove some properties. Finally, we give some example by intuitionistic measures.

Keywords: Triangular norm, sequence of intuitionistic fuzzy sets, intuitionistic measure.

1 Introduction

First, we introduce the limit of a sequence of intuitionistic fuzzy sets, analogous with the classical method, using the upper limit and the lower limit. The properties of the limit of a sequence of intuitionistic fuzzy sets implies the extension of de Morgan laws as countable case (for the finite case, see [2]).

In the last section, we define the concept of intuitionistic f-algebra and intuitionistic f-measure analogous with [5], [6] or [7] for fuzzy sets. It is interesting that the intuitionistic entropy introduced with the help of intuitionistic index (see [3]) is an intuitionistic (fuzzy) measure.

2 Sequences of intuitionistic fuzzy sets

Let $X \neq \emptyset$ be a given set and we will denote with $IFS(X)$ the set of all the intuitionistic fuzzy sets on X.

The following expressions are defined in [1] for all $A, B \in IFS(X)$, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$

$A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

$A = B$ if and only if $A \leq B$ and $B \leq A$

$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$

First, we introduce the upper limit and the lower limit of a sequence of intuitionistic fuzzy sets and we prove some properties.

Theorem 1 *If* $A_n = \{\langle x, \mu_{A_n}(x), \nu_{A_n}(x)\rangle : x \in X\} \in IFS(X)$ *for every* $n \in \mathbf{N}$ *then*

$$(i) \quad \overline{\lim_{n\to\infty}} A_n = \left\{\langle x, \bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty} \mu_{A_n}(x), \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \nu_{A_n}(x)\rangle : x \in X\right\} \in IFS(X)$$

$$(ii) \quad \underline{\lim_{n\to\infty}} A_n = \left\{\langle x, \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \mu_{A_n}(x), \bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty} \nu_{A_n}(x)\rangle : x \in X\right\} \in IFS(X).$$

Proof. (i) Because $\mu_{A_n}(x) + \nu_{A_n}(x) \leq 1$ for all $x \in X, n \in \mathbf{N}$, using the properties of $\vee$ and $\wedge$ we obtain

$$\bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty} \mu_{A_n}(x) + \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \nu_{A_n}(x) \leq \bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty} (1 - \nu_{A_n}(x)) + \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \nu_{A_n}(x) =$$
$$= 1 - \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \nu_{A_n}(x) + \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \nu_{A_n}(x) = 1$$

(ii) Similar with (i) ■

The intuitionistic fuzzy sets $\overline{\lim_{n\to\infty}} A_n$ and $\underline{\lim_{n\to\infty}} A_n$ are called the upper limit, respective the lower limit of the sequence of intuitionistic fuzzy sets $(A_n)_{n\in\mathbf{N}}$.

Theorem 2 *Let* $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$.

$$(i) \quad \underline{\lim_{n\to\infty}} A_n \leq \overline{\lim_{n\to\infty}} A_n$$

$$(ii) \quad \overline{\lim_{n\to\infty}} A_n^c = \left(\underline{\lim_{n\to\infty}} A_n\right)^c$$

$$(iii) \quad \underline{\lim_{n\to\infty}} A_n^c = \left(\overline{\lim_{n\to\infty}} A_n\right)^c.$$

Proof. (i) Obviously.

(ii) If $A_n = \{\langle x, \mu_{A_n}(x), \nu_{A_n}(x)\rangle : x \in X\}$ then

$$\overline{\lim_{n\to\infty}} A_n^c = \left\{\langle x, \bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty} \nu_{A_n}(x), \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty} \mu_{A_n}(x)\rangle : x \in X\right\} = \left(\underline{\lim_{n\to\infty}} A_n\right)^c$$

(iii) Similar with (ii) ■

Now, we define the limit of a sequence of intuitionistic fuzzy sets.

Definition 3 *The sequence $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$ is called convergent if and only if the upper limit is equal with the lower limit. The intuitinistic fuzzy set $A = \overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n$ is called the limit of $(A_n)_{n\in\mathbf{N}}$ and we denote this by $A_n \to A$ or $\lim_{n\to\infty} A_n = A$.*

Example 4 *If $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$ is increasing,that is $A_n \le A_{n+1}$ for every $n \in \mathbf{N}$ then $(A_n)_{n\in\mathbf{N}}$ is convergent and*

$$\lim_{n\to\infty} A_n = \left\{\langle x, \lim_{n\to\infty} \mu_{A_n}(x), \lim_{n\to\infty} \nu_{A_n}(x)\rangle : x \in X\right\}$$

because $\mu_{A_n}(x) \le \mu_{A_{n+1}}(x)$ and $\nu_{A_n}(x) \ge \nu_{A_{n+1}}(x)\, \forall x \in X$, $\forall n \in \mathbf{N}$. Similar if $(A_n)_{n\in\mathbf{N}}$ is decreasing.

Theorem 5 *If $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$ is convergent then $(A_n^c)_{n\in\mathbf{N}}$ is convergent and*

$$\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} A_n^c$$

Proof. Using Theorem 2, (ii), (iii) and Definition 3 ■

We can extend at countable case the operations on $IFS(X)$ introduced with the help of a triangular norm. Moreover, we obtain the corresponding de Morgan laws (for fuzzy sets see, for example, [7]).

We recall (for details see [2]) that for every triangular norm or triangular conorm and $A, B \in IFS(X)$, $A = \{\langle x, \mu_A(x), \nu_A(x) \mid x \in X\rangle\}$, $B = \{\langle x, \mu_B(x), \nu_B(x) \mid x \in X\rangle\}$ we can define

$$A\tilde{f}B = \{\langle x, f(\mu_A(x), \mu_B(x)), f^c(\nu_A(x), \nu_B(x))\rangle : x \in X\}$$

where f^c is the dual of f, that is $f^c(x, y) = 1 - f(1 - x, 1 - y)\ \forall x, y \in X$.

For any f, $A^c\widetilde{f^c}B^c = \left(A\tilde{f}B\right)^c$ hence the de Morgan laws are satisfied (see [2]).

Let $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$. Since each triangular norm (or triangular conorm) is associative it make sense to consider $\widetilde{f}(A_1, ..., A_n)$ defined recursively by

$$\widetilde{f}(A_1, ..., A_n, A_{n+1}) = \widetilde{f}\left(\widetilde{f}(A_1, ..., A_n), A_{n+1}\right)$$

and afterwards

$$\underset{n\in\mathbf{N}}{\widetilde{f}} A_n = \lim_{n\to\infty} \widetilde{f}(A_1, ..., A_n)$$

because the sequence $\left(\widetilde{f}(A_1, ..., A_n)\right)_{n\in\mathbf{N}} \subseteq IFS(X)$ is decreasing if f is a triangular norm and increasing if f is a triangular conorm.

Theorem 6 *Let $(A_n)_{n\in\mathbf{N}} \subseteq IFS(X)$. Then*

$$\underset{n\in\mathbf{N}}{\widetilde{f^c}} A_n^c = \left(\underset{n\in\mathbf{N}}{\widetilde{f}} A_n\right)^c$$

for every triangular norm or triangular conorm f.

Proof.

$$\begin{aligned}\underset{n\in\mathbf{N}}{\widetilde{f^c}} A_n^c &= \lim_{n\to\infty} \widetilde{f^c}(A_1^c, ..., A_n^c) = \lim_{n\to\infty} \left(\widetilde{f}(A_1, ..., A_n)\right)^c = \\ &= \left(\lim_{n\to\infty} \widetilde{f}(A_1, ..., A_n)\right)^c = \left(\underset{n\in\mathbf{N}}{\widetilde{f}} A_n\right)^c \blacksquare\end{aligned}$$

3 Intuitionistic fuzzy measure and examples

Similar with the introduction of a fuzzy measure (for details see [5],[6],[7]) we define the intuitionistic fuzzy measure.

Definition 7 *Let f be a triangular norm or a triangular conorm. A subfamily $\mathcal{I} \subseteq IFS(X)$ which satisfies:*

(i) $\emptyset = \{\langle x, 0, 1\rangle \,|x \in X\} \in \mathcal{I}$
(ii) $A \in \mathcal{I}$ *implies* $A^c \in \mathcal{I}$
(iii) $(A_n)_{n\in\mathbf{N}} \subseteq \mathcal{I}$ *implies* $\underset{n\in\mathbf{N}}{\widetilde{f}} A_n \in \mathcal{I}$

will be called an intuitionistic (fuzzy) f-algebra on X.

The pair $(X, \mathcal{I})$ will be called an intuitionistic f-measurable space.

Remark 1 *Thanks to Theorem 6 the condition (iii) can be replaced by*

(iii)' $(A_n)_{n\in\mathbf{N}} \subseteq \mathcal{I}$ *implies* $\widetilde{f^c}_{n\in\mathbf{N}} A_n \in \mathcal{I}$.

Remark 2 *It is obvious that every intuitionistic f-algebra on X is an intuitionistic f^c-algebra on X.*

Example 8 *Let $(X, \mathcal{A})$ be a measurable space. The family*

$$\mathcal{I}_{\mathcal{A}}(X) = \left\{ A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X) \,\middle|\, \begin{array}{c}\mu_A \text{ and } \nu_A \text{ are} \\ \mathcal{A}\text{-measurable}\end{array} \right\}$$

is an intuitionistic f-algebra on X with respect to any continuous triangular norm f.

Definition 9 *Let $(X, \mathcal{I})$ be an intuitionistic f-measurable space. A function $\widetilde{m} : \mathcal{I} \to \overline{\mathbf{R}}$, which assume at most one of the values $-\infty$ and $+\infty$, will be called an intuitionistic f-measure on $\mathcal{I}$ if it satisfies the following conditions*

(i) $\widetilde{m}(\emptyset) = 0$

(ii) $A, B \in \mathcal{I}$ *implies* $\widetilde{m}\left(A\tilde{f}B\right) + \widetilde{m}\left(A\widetilde{f^c}B\right) = \widetilde{m}(A) + \widetilde{m}(B)$

(iii) $(A_n)_{n\in\mathbf{N}} \subseteq \mathcal{I}$, $A_n \leq A_{n+1}$ *for every* $n \in \mathbf{N}$ *and* $\lim\limits_{n\to\infty} A_n \in \mathcal{I}$ *implies*

$$\lim_{n\to\infty} \widetilde{m}(A_n) = \widetilde{m}\left(\lim_{n\to\infty} A_n\right)$$

Example 10 *Let $X = \{x_1, ..., x_N\}$. The intuitionistic entropy defined with help of intuitionistic index (see [3]) $I : IFS(X) \to \mathbf{R}$,*

$$I(A) = \sum_{i=1}^{N} \pi_A(x_i) = \sum_{i=1}^{N} (1 - \mu_A(x_i) - \nu_A(x_i))$$

is an intutionistic f-measure on $IFS(X)$ for every triangular norm f.

Example 11 *Let $X = \{x_1, ..., x_N\}$. We consider the function $\widetilde{m} : IFS(X) \to \mathbf{R}$ defined by*

$$\widetilde{m}(A) = N - E_{IFS}(A)$$

where $E_{IFS}(A) = \sum_{i=1}^{N} (\mu_A^2(x_i) + \nu_A^2(x_i))$ *is the so-called informational intuitionistic energy of the* $A \in IFS(X)$ *(see [4]). It is obvious that* $\widetilde{m}$ *is an intuitionistic* $\wedge-$*measure on* $IFS(X)$ *but not an intuitionistic* t_∞*-measure on* $IFS(X)$ *(the triangular norm* t_∞ *is defined by* $t_\infty(x, y) = \max(x + y - 1, 0)$ $\forall x, y \in [0, 1]$*) since for* $A = B = \left\{\left\langle x_1, \frac{1}{2}, \frac{1}{2}\right\rangle, \langle x_2, 1, 0\rangle, ..., \langle x_N, 1, 0\rangle\right\}$,

$$\frac{3}{2} = \widetilde{m}(A) + \widetilde{m}(B) \neq \widetilde{m}(A\widetilde{t_\infty}B) + \widetilde{m}(A\widetilde{s_\infty}B) = 0.$$

Example 12 *Let* $(X, \mathcal{A}, m)$ *be a measure space and* $\varpi : [0, 1] \to [0, 1]$ *a continuous and additive function so that* $\varpi(0) + \varpi(1) = 0$*. If the triangular norm* f *is a continuous function and verifies* $f(x, y) + f^c(x, y) = x + y$, $\forall x, y \in [0, 1]$ *then the function* $\widetilde{m} : \mathcal{I}_{\mathcal{A}}(X) \to \overline{\mathbf{R}}_+$ *defined by*

$$\widetilde{m}(A) = \int_X (\varpi(\mu_A(x)) + \varpi(\nu_A(x)))\, dm$$

is an intuitionistic f*-measure because*

$$\widetilde{m}(\emptyset) = \widetilde{m}(\{\langle x, 0, 1\rangle : x \in X\}) = \int_X (\varpi(0) + \varpi(1))\, dm = 0,$$

$$\begin{aligned}
& \widetilde{m}(A\tilde{f}B) + \widetilde{m}(A\tilde{f}B) \\
= & \int_X (\varpi(f(\mu_A(x), \mu_B(x))) + \varpi(f^c(\nu_A(x), \nu_B(x))))\, dm + \\
& + \int_X (\varpi(f^c(\mu_A(x), \mu_B(x))) + \varpi(f(\nu_A(x), \nu_B(x))))\, dm \\
= & \int_X \varpi(f(\mu_A(x), \mu_B(x)) + f^c(\mu_A(x), \mu_B(x)))\, dm \\
& + \int_X \varpi(f(\nu_A(x), \nu_B(x)) + f^c(\nu_A(x), \nu_B(x)))\, dm \\
= & \int_X \varpi(\mu_A(x) + \mu_B(x))\, dm + \int_X \varpi(\nu_A(x) + \nu_B(x))\, dm \\
= & \int_X (\varpi((\mu_A(x)) + \varpi(\mu_B(x))))\, dm + \int_X (\varpi(\nu_A(x)) + \varpi(\nu_B(x)))\, dm \\
= & \widetilde{m}(A) + \widetilde{m}(B), \forall A, B \in \mathcal{I}_{\mathcal{A}}(X)
\end{aligned}$$

and

$$\lim_{n\to\infty} \widetilde{m}(A_n) = \lim_{n\to\infty} \int_X (\varpi(\mu_{A_n}(x)) + \varpi(\nu_{A_n}(x)))\, dm =$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_X \varpi\left(\mu_{A_n}(x)\right)dm + \lim_{n\to\infty}\int_X \varpi\left(\nu_{A_n}(x)\right)dm = \\
&= \int_X \varpi\left(\lim_{n\to\infty}\mu_{A_n}(x)\right)dm + \int_X \varpi\left(\lim_{n\to\infty}\nu_{A_n}(x)\right)dm = \\
&= \widetilde{m}(\lim_{n\to\infty} A_n)
\end{aligned}
$$

if $(A_n)_{n\in\mathbf{N}}$ *is an increasing sequence. with* $A_n \in \mathcal{I}_{\mathcal{A}}(X)$ $\forall n \in \mathbf{N}$.

Theorem 13 *Let* f *be a triangular norm. If* $\mathcal{I}$ *is both an intuitionistic* f*-algebra and an intuitionistic* $\wedge$*-algebra then each intuitionistic* f*-measure is an intuitionistic* $\wedge$*-measure.*

Proof. We assume that $\widetilde{m}$ is an intuitionistic f-measure. Because $\mathcal{I}$ is an intuitionistic $\wedge$-algebra we can write for every $A, B \in \mathcal{I}_{\mathcal{A}}(X)$

$$
\begin{aligned}
\widetilde{m}\left(A\widetilde{\wedge}B\right) + \widetilde{m}\left(A\widetilde{\vee}B\right) &= \widetilde{m}\left(\left(A\widetilde{\wedge}B\right)\widetilde{f}\left(A\widetilde{\vee}B\right)\right) + \widetilde{m}\left(\left(A\widetilde{\wedge}B\right)\widetilde{f^c}\left(A\widetilde{\vee}B\right)\right) = \\
&= \widetilde{m}\left(A\widetilde{f}B\right) + \widetilde{m}\left(A\widetilde{f^c}B\right) = \widetilde{m}(A) + \widetilde{m}(B) \blacksquare
\end{aligned}
$$

References

[1] Atanassov, K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87-96.

[2] Burillo, P., Bustince, H., Two operators on interval-valued intuitionistic fuzzy sets:PART II, Comptes rendus de l 'Académie bulgare de Sciences, Tome 48, No 1, 1995, 17-20.

[3] Burillo, P., Bustince, H., Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy Sets and Systems 78(1996), 305-316.

[4] Bustince, H., Burillo, P., Correlation of interval intuitionistic fuzzy sets, Fuzzy Sets and Systems 74(1995), 237-244.

[5] Butnariu, D., Klement, E. P., Triangular norm based measures and their Markov kernel representation, J. Math. Anal. Appl. 162(1991), 111-143.

[6] Butnariu, D., Klement, E. P., Triangular norm- Based Measures and Games with Fuzzy Coaltions, Kluwer, Dordrecht, 1993.

[7] Klement, E. P., Characterization of fuzzy measures constructed by means of triangular norms, J. Math. Anal. Appl. 86(1982), 345-358.