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Taylor's theorem for functions, defined on Atanassov IF-sets

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Abstract: IF-sets were first time defined in the paper [1] and they represent the natural extension of the theory of fuzzy sets. In the paper [6] there was proved that Lagrange theorem holds for functions defined on IF-sets. Therefore it is natural question if it is possible to prove also Taylor's theorem for functions defined on IF-sets. To prove this theorem we defined polynomial function and Taylor's formula for functions defined on IF-sets. Then the Taylor's theorem is proved.
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1 Introduction

By an IF-set we consider a pair $A = (\mu_A, \nu_A)$ of functions $\mu_A, \nu_A : \Omega \to [0, 1]$ such that

$$\mu_A + \nu_A \leq 1.$$

The function μ_A is called a membership function of A, ν_A a nonmembership function of A. If $(\Omega, \mathcal{S})$ is a measurable space and μ_A, ν_A are $\mathcal{S}$-measurable, then A is called an IF-event. Denote by $\mathcal{F}$ the family of all IF-events.
In the paper [3] there was proved that we could construct such ℓ-group G that $\mathcal{F}$ can be embedded into G. Consider the set $A = (\mu_A, \nu_A)$, where $\mu_A, \nu_A : \Omega \to R$. Denote by G the set of all pairs $A = (\mu_A, \nu_A)$ and for any $A, B \in G$ define the following operation

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B - 1)$$

and the relation

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Then the triple $G = (G, +, \leq)$ is the mentioned ℓ-group.
On the set G there are defined some more operations. For our studies we will need following of them

$$\begin{aligned} A - B &= (\mu_A - \mu_B, \nu_A - \nu_B + 1) \\ A.B &= (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B). \end{aligned}$$

Evidently

$$\begin{aligned} c.A &= (c.\mu_A, 1 - c.(1 - \nu_A)) \,,\ c \in R, \\ \sum_{i=1}^{n} A_i &= \left(\sum_{i=1}^{n} \mu_{A_i}, \sum_{i=1}^{n} (\nu_{A_i} - 1) + 1 \right). \end{aligned}$$

The element $(0, 1)$ is a neutral element of the operation $+$ and the element

$$-A = (-\mu_A, 2 - \nu_A)$$

is an inverse element to the element A. If $B > (0, 1)$, whereby $A > B \Leftrightarrow \mu_A > \mu_B, \nu_A < \nu_B$, then

$$\frac{A}{B} = \left(\frac{\mu_A}{\mu_B}, \frac{\nu_A - \nu_B}{1 - \nu_B} \right).$$

In the paper [5] there was given the definition of the function $\bar{f}$ on the family of all IF-sets in the following form:
Let f be a real function. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $A \leq B$ and $[\mu_A, \mu_B] \cup [\nu_B, \nu_A] \subset Domf$. Then the function $\bar{f} : [A, B] \to R^2$ is defined by

$$\bar{f}(X) = (f(\mu_X), 1 - f(1 - \nu_X))$$

where $X = (\mu_X, \nu_X)$ is called a variable.
For example for any natural number n it holds

$$X^n = ((\mu_X)^n, 1 - (1 - \nu_X)^n).$$

In this paper we will use this function in the shorter notation

$$X^n = (\mu_X^n, 1 - (1 - \nu_X)^n).$$

In the paper [4] there was given the definition of the derivation of the function on the family of all IF-sets in the following form

$$\bar{f}'(X) = (f'(\mu_X), 1 - f'(1 - \nu_X)) \,.$$

It is easy to proved that for the n-th derivation it holds

$$\bar{f}^{(n)}(X) = \left(f^{(n)}(\mu_X), 1 - f^{(n)}(1 - \nu_X) \right).$$

2 Polynomial function

To define the Taylor's formula and to prove Taylor's theorem we need first to express the definition of the polynomial function. We will do it in this section.

Definition 2.1 *Let $n \in N$, $X = (\mu_X, \nu_X)$ be the variable and $A_i \in R^2, A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 0, 1, 2, \ldots, n$ be the constants, $A_n \neq (0, 1)$. Then*

$$\bar{p}_n(X) = A_0 + A_1X + A_2X^2 + \ldots + A_nX^n$$

is called the polynomial function.

Theorem 2.2 *Let $X = (\mu_X, \nu_X)$ be the variable and $\bar{p}_n$ be a polynomial function. Then it holds*

$$\bar{p}_n((\mu_X, \nu_X)) = (\mu_{A_0} + \mu_{A_1}\mu_X + \mu_{A_2}\mu_X^2 + \ldots + \mu_{A_n}\mu_X^n,$$

$$\nu_{A_0} + (\nu_{A_1} - 1)(1 - \nu_X) + (\nu_{A_2} - 1)(1 - \nu_X)^2 + \ldots + (\nu_{A_n} - 1)(1 - \nu_X)^n)$$

Proof:
From the definitions of the operations it follows

$$X^n = (\mu_X^n, 1 - (1 - \nu_X)^n)$$

$$AX^n = (\mu_A\mu_X^n, \nu_A + (1 - (1 - \nu_X)^n) - \nu_A(1 - (1 - \nu_X)^n)).$$

The second component could be modified by the following way

$$\nu_A + (1 - (1 - \nu_X)^n) - \nu_A(1 - (1 - \nu_X)^n) - 1 + 1 =$$

$$= \nu_A(1 - (1 - (1 - \nu_X)^n)) - 1(1 - (1 - (1 - \nu_X)^n)) + 1 =$$

$$= (\nu_A - 1)(1 - \nu_X)^n + 1.$$

Therefore

$$AX^n = (\mu_A\mu_X^n, (\nu_A - 1)(1 - \nu_X)^n + 1).$$

Then we obtain

$$\begin{aligned}
A_0 &= A_0X^0 = (\mu_{A_0}\mu_X^0, (\nu_{A_0} - 1)(1 - \nu_X)^0 + 1) = \\
&= (\mu_{A_0}, \nu_{A_0}) = B_0 \\
A_1X^1 &= (\mu_{A_1}\mu_X^1, (\nu_{A_1} - 1)(1 - \nu_X)^1 + 1) = B_1 \\
A_2X^2 &= (\mu_{A_2}\mu_X^2, (\nu_{A_2} - 1)(1 - \nu_X)^2 + 1) = B_2 \\
&\vdots \\
A_nX^n &= (\mu_{A_n}\mu_X^n, (\nu_{A_n} - 1)(1 - \nu_X)^n + 1) = B_n
\end{aligned}$$

Therefore

$$\bar{p}_n(X) = \sum_{i=0}^{n} B_i = \Big(\sum_{i=0}^{n} \mu_{A_i}\mu_X^i, \sum_{i=0}^{n} \big(((\nu_{A_i} - 1)(1 - \nu_X)^i + 1) - 1\big) + 1\Big) =$$

$$= \Big(\sum_{i=0}^{n}\mu_{A_i}\mu_X^i, \sum_{i=0}^{n}(((\nu_{A_i}-1)(1-\nu_X)^i)+1)\Big) =$$

$$= \Big(\mu_{A_0}+\mu_{A_1}\mu_X+\mu_{A_2}\mu_X^2+\ldots+\mu_{A_n}\mu_X^n,$$

$$\nu_{A_0}+(\nu_{A_1}-1)(1-\nu_X)+(\nu_{A_2}-1)(1-\nu_X)^2+\ldots+(\nu_{A_n}-1)(1-\nu_X)^n\Big).$$

Remark 2.3 We could use also the approach that polynomial function is a special type of a function. Then

$$\bar{p}_n((\mu_X,\nu_X)) = (p_n(\mu_X), 1-p_n(1-\nu_x))$$

where

$$p_n(\mu_X) = \mu_{A_0}+\mu_{A_1}\mu_X+\mu_{A_2}\mu_X^2+\ldots+\mu_{A_n}\mu_X^n,$$

and

$$1-p_n(1-\nu_X) = 1-[(1-\nu_{A_0})+(1-\nu_{A_1})(1-\nu_X)+$$

$$+(1-\nu_{A_2})(1-\nu_X)^2+\ldots+(1-\nu_{A_n})(1-\nu_X)^n] =$$

$$= \nu_{A_0}+(\nu_{A_1}-1)(1-\nu_X)+(\nu_{A_2}-1)(1-\nu_X)^2+\ldots+(\nu_{A_n}-1)(1-\nu_X)^n$$

3 Taylor's formula and Taylor's theorem

In the paper [6] there was proved that Lagrange mean value theorem could be established also on the family of all IF-sets. Concretely if $\bar{f}$ is continuous on $[A,B]$ and differentiable on (A,B) then there exists $C \in (A,B)$ such that

$$\bar{f}(B)-\bar{f}(A) = (f(\mu_B-\mu_A), f(1-\nu_A)-f(1-\nu_B)+1) =$$

$$= (f'(\mu_C)(\mu_B-\mu_A), f'(1-\nu_C)(\nu_B-\nu_A)+1) = \bar{f}'(C)(B-A).$$

Since this theorem holds we could answer the question if it possible to prove Taylor's theorem on the family of IF-sets. Let us start with the definition of Taylor's formula.

Definition 3.1 *Let $n \in N$, $X = (\mu_X,\nu_X)$ be the variable and $X_0 = (\mu_{X_0},\nu_{X_0})$, $X_0 \in R^2$ be the fixed point. Let $\bar{f}(X) = (f(\mu_X), 1-f(1-\nu_X))$ be a function defined on the family of all IF-sets. Let the derivations $\bar{f}^{(i)}(X) = (f^{(i)}(\mu_X), 1-f^{(i)}(1-\nu_X))$ exist for $i = 1,2,\ldots,n$. Then the Taylor's formula at the point X_0 has the following form*

$$\bar{T}_n(X) = \bar{f}(X_0)+\frac{\bar{f}^{(1)}(X_0)}{1!}(X-X_0)+\frac{\bar{f}^{(2)}(X_0)}{2!}(X-X_0)^2+\ldots+\frac{\bar{f}^{(n)}(X_0)}{n!}(X-X_0)^n.$$

Theorem 3.2 *Let the assumptions of the previous definition hold. Function $\bar{T}_n$ is the Taylor's formula at the point $X_0 = (\mu_{X_0},\nu_{X_0})$ if and only if for any $X = (\mu_X,\nu_X)$ it holds*

$$\bar{T}_n((\mu_X,\nu_X)) = \Big(f(\mu_{X_0})+\frac{f^{(1)}(\mu_{X_0})}{1!}(\mu_X-\mu_{X_0})+\ldots+\frac{f^{(n)}(\mu_{X_0})}{n!}(\mu_X-\mu_{X_0})^n,$$

$$1-(f(1-\nu_{X_0})+\frac{f^{(1)}(1-\nu_{X_0})}{1!}(\nu_{X_0}-\nu_X)+\ldots+\frac{f^{(n)}(1-\nu_{X_0})}{n!}(\nu_{X_0}-\nu_X)^n)\Big).$$

Proof:
Since

$$\begin{aligned} X^n &= (\mu_X^n, 1-(1-\nu_X)^n) \\ X - X_0 &= (\mu_X - \mu_{X_0}, \nu_X - \nu_{X_0} + 1) \\ \bar{f}^{(n)}(X_0) &= (f^{(n)}(\mu_{X_0}), 1 - f^{(n)}(1-\nu_{X_0})) \\ cX &= (c\mu_X, 1 - c(1-\nu_X)) \end{aligned}$$

where c is any real number. Then

$$(X - X_0)^n = ((\mu_X - \mu_{X_0})^n, 1 - (1 - (\nu_X - \nu_{X_0} + 1))^n) =$$

$$= ((\mu_X - \mu_{X_0})^n, 1 - (\nu_{X_0} - \nu_X)^n)$$

and

$$\bar{f}^{(n)}(X_0)(X - X_0)^n = (f^{(n)}(\mu_{X_0})(\mu_X - \mu_{X_0})^n,$$
$$1 - f^{(n)}(1-\nu_{X_0}) + 1 - (\nu_{X_0} - \nu_X)^n - ((1 - f^{(n)}(1-\nu_{X_0}))(1 - (\nu_{X_0} - \nu_X)^n))).$$

After the modification of the second part we obtain

$$1 - f^{(n)}(1-\nu_{X_0}) + 1 - (\nu_{X_0} - \nu_X)^n - ((1 - f^{(n)}(1-\nu_{X_0}))(1 - (\nu_{X_0} - \nu_X)^n))) =$$

$$= 1 - f^{(n)}(1-\nu_{X_0}) + 1 - (\nu_{X_0} - \nu_X)^n -$$
$$-1 + (\nu_{X_0} - \nu_X)^n + f^{(n)}(1-\nu_{X_0}) - f^{(n)}(1-\nu_{X_0})(\nu_{X_0} - \nu_X)^n =$$
$$= 1 - f^{(n)}(1-\nu_{X_0})(\nu_{X_0} - \nu_X)^n.$$

Therefore

$$\bar{f}^{(n)}(X_0)(X - X_0)^n = (f^{(n)}(\mu_{X_0})(\mu_X - \mu_{X_0})^n, 1 - f^{(n)}(1-\nu_{X_0})(\nu_{X_0} - \nu_X)^n).$$

Finally for any $c \in R$ it holds

$$c\bar{f}^{(n)}(X_0)(X - X_0)^n = (cf^{(n)}(\mu_{X_0})(\mu_X - \mu_{X_0})^n,$$

$$1 - c(1 - (1 - f^{(n)}(1-\nu_{X_0})(\nu_{X_0} - \nu_X)^n)))$$

hence

$$c\bar{f}^{(n)}(X_0)(X - X_0)^n = (cf^{(n)}(\mu_{X_0})(\mu_X - \mu_{X_0})^n, 1 - c(f^{(n)}(1-\nu_{X_0})(\nu_{X_0} - \nu_X)^n)).$$

Put $c_i = \frac{1}{i!}$ for $i = 1, 2, \ldots, n$ then

$$\frac{\bar{f}^{(i)}(X_0)}{i!}(X - X_0)^i = \left(\frac{f^{(i)}(\mu_{X_0})}{i!}(\mu_X - \mu_{X_0})^i, 1 - \frac{f^{(i)}(1-\nu_{X_0})}{i!}(\nu_{X_0} - \nu_X)^i\right)$$

is the i-th member of the Taylor's formula. After the summation of all members of Taylor's formula we get

$$\bar{T}_n((\mu_X, \nu_X)) =$$

$$= \left(\sum_{i=0}^{n} \frac{f^{(i)}(\mu_{X_0})}{i!}(\mu_X - \mu_{X_0})^i, \sum_{i=0}^{n} \left(1 - \frac{f^{(i)}(1-\nu_{X_0})}{i!}(\nu_{X_0} - \nu_X)^i - 1 \right) + 1 \right) =$$

$$= \left(\sum_{i=0}^{n} \frac{f^{(i)}(\mu_{X_0})}{i!}(\mu_X - \mu_{X_0})^i, \sum_{i=0}^{n} \left(- \frac{f^{(i)}(1-\nu_{X_0})}{i!}(\nu_{X_0} - \nu_X)^i \right) + 1 \right) =$$

$$= \Big(f(\mu_{X_0}) + \frac{f^{(1)}(\mu_{X_0})}{1!}(\mu_X - \mu_{X_0}) + \ldots + \frac{f^{(n)}(\mu_{X_0})}{n!}(\mu_X - \mu_{X_0})^n,$$

$$1 - \big(f(1-\nu_{X_0}) + \frac{f^{(1)}(1-\nu_{X_0})}{1!}(\nu_{X_0} - \nu_X) + \ldots + \frac{f^{(n)}(1-\nu_{X_0})}{n!}(\nu_{X_0} - \nu_X)^n \big) \Big).$$

Example 3.3 *Let* $\bar{f}(X) = \overline{\sin}(X) = (\sin(\mu_X), 1 - \sin(1-\nu_X))$ *and* $X_0 = (0, 1)$. *Then* $1 - \nu_{X_0} = 1 - 1 = 0$ *and therefore*

$$\begin{array}{rcl} f(\mu_{X_0}) & = & f(1-\nu_{X_0}) \ = \sin 0 = 0 \\ f^{(1)}(\mu_{X_0}) & = & f^{(1)}(1-\nu_{X_0}) = \cos 0 = 1 \\ f^{(2)}(\mu_{X_0}) & = & f^{(2)}(1-\nu_{X_0}) = -\sin 0 = 0 \\ f^{(3)}(\mu_{X_0}) & = & f^{(3)}(1-\nu_{X_0}) = -\cos 0 = -1 \\ & \vdots & \end{array}$$

Then

$$\bar{T}_n((\mu_X, \nu_X)) =$$

$$= \Big(\frac{1}{1!}\mu_X - \frac{1}{3!}\mu_X^3 + \ldots + \frac{(-1)^n}{(2n+1)!}\mu_X^{2n+1},$$

$$1 - \Big(\frac{1}{1!}(1-\nu_X) - \frac{1}{3!}(1-\nu_X)^3 + \ldots + \frac{(-1)^n}{(2n+1)!}(1-\nu_X)^{2n+1} \Big) \Big).$$

Theorem 3.4 *Let* $\bar{f}$ *be a function that has continuous derivations* $\bar{f}^{(i)}$, $i = 0, 1, 2, \ldots, n$ *defined on interval* $[X_0, X]$, *let there exists derivation* $\bar{f}^{(n+1)}$ *on interval* (X_0, X) *and* $\bar{T}_n$ *be the Taylor's formula appertaining to* $\bar{f}$ *in the point* X_0. *Then there exist such function* $\bar{R}_n$ *and such* $C = (\mu_C, \nu_C), C \in (X_0, X)$ *that it holds*

$$\bar{f}(X) = \bar{T}_n(X) + \bar{R}_n(X).$$

The function $\bar{R}_n$ *is usually called remainder and it could have following form*

$$\bar{R}_n(X) = \frac{\bar{f}^{(n+1)}(C)}{(n+1)!}(X - X_0)^{n+1}$$

(Lagrange's form).

Proof:
In the first step we will specify the remainder

$$\bar{R}_n(X) = \frac{\bar{f}^{(n+1)}(C)}{(n+1)!}(X - X_0)^{n+1}$$

by using membership and nonmembership functions.

$$\bar{R}_n((\mu_X,\nu_X))=\left(\frac{f^{(n+1)}(\mu_C)}{(n+1)!}(\mu_X-\mu_{X_0})^{n+1},1-\frac{f^{(n+1)}(1-\nu_C)}{(n+1)!}(\nu_{X_0}-\nu_X)^{n+1}\right).$$

It is also important not forget that if $C\in(X_0,X)$ then it holds

$$\mu_{X_0}\le\mu_C\le\mu_X$$

and

$$\nu_{X_0}\ge\nu_C\ge\nu_X.$$

Then from equality

$$\bar{f}(X)=\bar{T}_n(X)+\bar{R}_n(X)$$

for membership function it follows

$$f(\mu_X)=f(\mu_{X_0})+\frac{f^{(1)}(\mu_{X_0})}{1!}(\mu_X-\mu_{X_0})+\ldots+$$

$$+\frac{f^{(n)}(\mu_{X_0})}{n!}(\mu_X-\mu_{X_0})^n+\frac{f^{(n+1)}(\mu_C)}{(n+1)!}(\mu_X-\mu_{X_0})^{n+1}$$

and for nonmembership function it follows

$$f(1-\nu_X)=f(1-\nu_{X_0})+\frac{f^{(1)}(1-\nu_{X_0})}{1!}(\nu_{X_0}-\nu_X)+\ldots+$$

$$+\frac{f^{(n)}(1-\nu_{X_0})}{n!}(\nu_{X_0}-\nu_X)^n+\frac{f^{(n+1)}(1-\nu_C)}{(n+1)!}(\nu_{X_0}-\nu_X)^{n+1}.$$

Since μ_{X_0},μ_X,μ_C are the real numbers $\mu_{X_0}\le\mu_C\le\mu_X$ and in addition Lagrange's theorem holds then for membership function we get the formula which is equal with Taylor's theorem in classical analysis. Therefore also the proof of this part has the same steps as in real analysis.
On the other hand ν_{X_0},ν_X,ν_C are also real numbers and it holds $\nu_{X_0}\ge\nu_C\ge\nu_X$. This inequality is the same as $1-\nu_{X_0}\le1-\nu_C\le1-\nu_X$. Denote $1-\nu_{X_0}=y_0$, $1-\nu_X=y$ and $1-\nu_C=c$. Then

$$\nu_{X_0}-\nu_X=1-\nu_X-(1-\nu_{X_0})=y-y_0.$$

After substitution the values and rewriting the formula for nonmembership function we get

$$f(y)=f(y_0)+\frac{f^{(1)}(y_0)}{1!}(y-y_0)+\ldots+\frac{f^{(n)}(y_0)}{n!}(y-y_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(y-y_0)^{n+1}$$

what is again the formula equal with formula in classical analysis. Therefore Taylor's formula holds also for the family of all IF-sets.

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