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ON MEASURING DISTANCES BETWEEN INTUITIONISTIC FUZZY SETS

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Abstract. Employing a geometrical representation of an intuitionistic fuzzy set we propose some concepts of distances between intuitionistic fuzzy sets. We show that these definitions are consistent with those defined for fuzzy sets.

Keywords: fuzzy set, intuitionistic fuzzy set, distance between intuitionistic fuzzy sets

1. Introduction

We propose some new definitions of distances between intuitionistic fuzzy sets introduced by Atanassov [1-5]. We define the four basic distances between the intuitionistic fuzzy sets: the Hamming distance, the normalized Hamming distance, the Euclidean distance, and the normalized Euclidean distance. While deriving these distances a convenient geometric interpretation of intuitionistic fuzzy sets is employed. It is shown that the definitions proposed are consistent with their counterparts traditionally used for fuzzy sets.

2. Intuitionistic fuzzy sets - a geometrical interpretation, and comparison with fuzzy sets

Let us start with a short review of basic concepts related to intuitionistic fuzzy sets.

Definition. A fuzzy set A' in X={x} is given by [11]:

$$A' = \{< x, \mu_{A'}(x) > / x \in X\} \tag{1}$$

where $\mu_{A'}: X \to [0,1]$ is the membership function of A'; $\mu_{A'}(x) \in [0,1]$ is the membership of x $\in$X in A'.

Definition. An intuitionistic fuzzy set A in X is given by [1]:

$$A = \{< x, \mu_A(x), \nu_A(x) > / x \in X\} \tag{2}$$

where: $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$, and $0 \le \mu_A(x) + \nu_A(x) \le 1, \forall x \in X$. The numbers $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote the degree of membership and non-membership of x to A respectively.

Obviously, every fuzzy set A' may be represented by the following intuitionistic fuzzy set:

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > / x \in X\} \tag{3}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{4}$$

the *intuitionistic index* of x in A. It is hesitancy degree of x to A [1-5]. Obviously, $0 \le \pi_A(x) \le 1, \forall x \in X$.

For each fuzzy set A' in X, we evidently have $\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0$, for each $x \in$ X.

A convenient geometrical interpretations of an intuitionistic fuzzy set [5] is shown in Fig.1 in which a universe E and its subset F in the Euclidean plane with the Cartesian coordinates are considered.

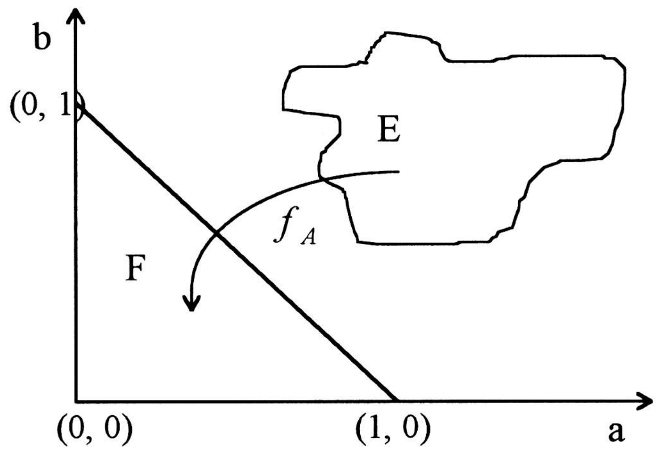

Figure 1

For a fixed intuitionistic fuzzy set A, a function f_A from E to F can be constructed, such that if $x \in E$, then $p = f_A(x) \in F$, and the point $p \in F$ has the coordinates $< a', b' >$ for which

$$0 \le a', b' \le 1$$

where: $a' = \mu_A(x)$, and $b' = \nu_A(x)$.

The above geometrical interpretation can be used e.g. when considering a situation at the beginning of negotiations - cf. Fig. 2 (applications of intuitionistic fuzzy sets for group decision making, negotiations and other real situations are presented in [7-10]).

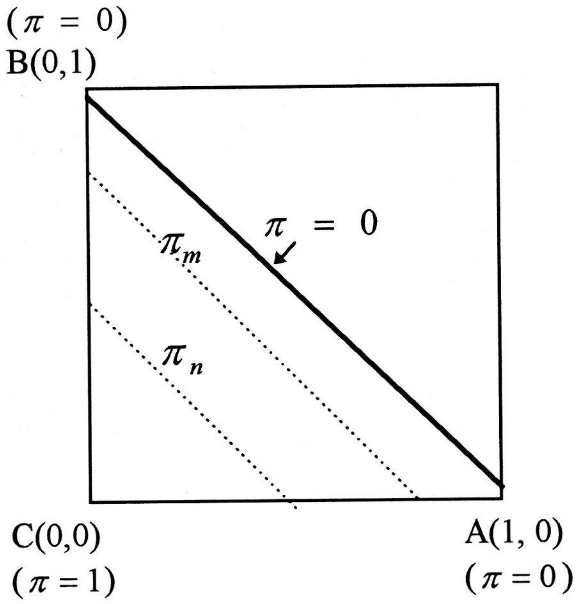

Figure 2

Namely, each expert i is represented as a point having coordinates $<\mu_i, \nu_i, \pi_i>$. Expert A: $<1, 0, 0>$ fully accepts a discussed idea. Expert B: <0, 1, 0> - fully rejects it. The experts placed on the segment AB fixed their points of view (their hesitation margins are equal zero for segment AB, so each expert is convinced to the extent μ_i, is against to the extent ν_i, and $\mu_i + \nu_i = 1$; segment AB represents a fuzzy set). Expert C: <0, 0, 1> is absolutely hesitant i.e. undecided - he or she is the most open to influence of the arguments presented.

A line parallel to AB describes a set of experts with the same level of hesitancy. For example, in Fig. 2, two sets are presented: with intuitionistic indices equal π_m, and π_n, where: $\pi_n > \pi_m$.

In other words, Fig. 2 (the triangle ABC) is an orthogonal projection of the real situation (the triangle ABD) presented in Fig.3.

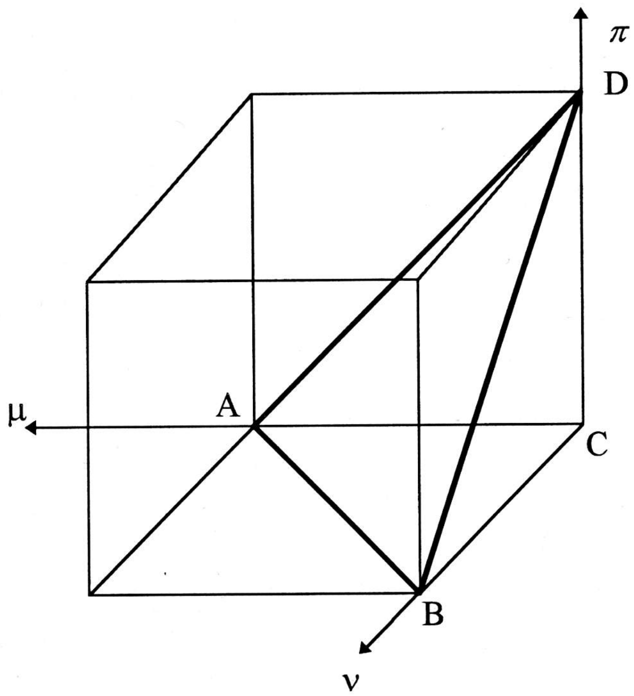

Figure 3

An element of an intuitionistic fuzzy set has three coordinates $<\mu_i, \nu_i, \pi_i>$, such that (4), hence the most natural representation of an intuitionistic fuzzy set is to draw a cube (with unit edge), and because of (4), the triangle ABD (Fig. 3) represents an intuitionistic fuzzy set. As before (Fig. 2), the triangle ABC is the orthogonal projection of ABD.

3. Distances in fuzzy sets and intuitionistic fuzzy sets

We will first reconsider some better known distances for the fuzzy sets in an intuitionistic setting, and then extend those distances to the intuitionistic fuzzy sets.

3.1. Distances for fuzzy sets

The most widely used distances for fuzzy sets *A, B* in $X = \{x_1, x_2, \ldots, x_n\}$ are [6]:

- the Hamming distance *d(A, B)*

$$d(A,B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \qquad (5)$$

- the normalized Hamming distance *l(A, B)*:

$$l(A,B) = \frac{1}{n}\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \qquad (6)$$

- the Euclidean distance *e(A, B)*:

$$e(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \qquad (7)$$

- the normalized Euclidean distance *q(A, B)*:

$$q(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \qquad (8)$$

As we may remember from Section 2, we can represent a fuzzy set *A'* in *X* in an equivalent intuitionistic-type representation (3) given as

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > / x \in X\}$$

and we will employ such a representation while rewriting the distances (5) - (8).

So, first, taking into account an intuitionistic-type representation of a fuzzy set, we can express the very essence of the Hamming distance by

$$d'(A,B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) =$$

$$= \sum_{i=1}^{n}\left(|\mu_A(x_i) - \mu_B(x_i)| + |1 - \mu_A(x_i) - 1 + \mu_B(x_i)|\right) =$$

$$= 2\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)| = 2d(A,b) \tag{9}$$

i.e. it is twice as large as the Hamming distance of a fuzzy set (5).

And similarly, the normalized Hamming distance $l'(A,B))$ taking into account an intuitionistic-type representation of a fuzzy set is in turn equal to

$$l'(A,B) = \frac{1}{n}\cdot d'(A,B) = \frac{2}{n}\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)| \tag{10}$$

i.e. the result of (10) is two times multiplied as to compared to (6)

And, similarly, the Euclidean distance, is equal to

$$e'(A,B) = \sqrt{\sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2} =$$

$$= \sqrt{\sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(1 - \mu_A(x_i) - 1 + \mu_B(x_i)\right)^2} =$$

$$= \sqrt{2\sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)^2} \tag{11}$$

i.e. it is multiplied by $\sqrt{2}$ as compared to the Euclidean distance for the usual representation of fuzzy sets given by (7).

Analogously, the normalized Euclidean distance $q'(A,B)$ is

$$q'(A,B) = \sqrt{\frac{1}{n}}\cdot e'(A,B) = \sqrt{\frac{2}{n}\sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)^2} \tag{12}$$

so again the result of (11) is multiplied by $\sqrt{2}$ as compared to (8).

Example 1. For simplicity let us consider "degenerated" fuzzy sets A, B, L, M, N in $X = \{1\}$. A full description of each of them is is $A = (\mu_A, \nu_A)/1$ exemplified by:

$$A = (1,0)/1,\ B = (0,1)/1,\ L = \left(\tfrac{1}{3},\tfrac{2}{3}\right)/1,\ N = \left(\tfrac{2}{3},\tfrac{1}{3}\right)/1,\ M = \left(\tfrac{1}{2},\tfrac{1}{2}\right)/1$$

The geometrical interpretation (in the sense of Fig. 2) of these one-element fuzzy sets is Fig. 4.

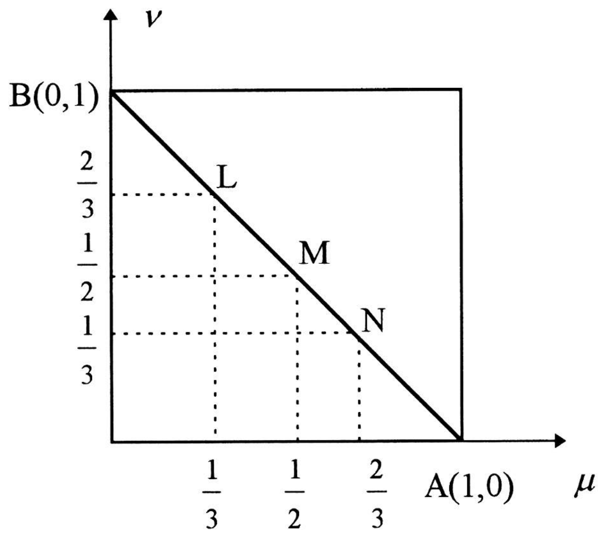

Figure 4

The Euclidean distances between the above fuzzy sets due to (7) are:

$$e(L,N)=\sqrt{\left(\frac{1}{3}-\frac{2}{3}\right)^2}=\frac{1}{3} \quad (13)$$

$$e(L,M)=\sqrt{\left(\frac{1}{3}-\frac{1}{2}\right)^2}=\frac{1}{6} \quad (14)$$

$$e(N,M)=\sqrt{\left(\frac{2}{3}-\frac{1}{2}\right)^2}=\frac{1}{6} \quad (15)$$

$$e(L,A)=\sqrt{\left(\frac{1}{3}-1\right)^2}=\frac{2}{3} \quad (16)$$

$$e(M,A)=\sqrt{\left(1-\frac{1}{2}\right)^2}=\frac{1}{2} \quad (17)$$

$$e(B,M)=\sqrt{0-\left(\frac{1}{2}\right)^2}=\frac{1}{2} \quad (18)$$

$$e(B,A)=\sqrt{1^2}=1 \quad (19)$$

while, using the intuitionistic-type representation (11), they are:

$$e'(L,N)=\sqrt{\left(\frac{1}{3}-\frac{2}{3}\right)^2+\left(\frac{2}{3}-\frac{1}{3}\right)^2}=\frac{\sqrt{2}}{3} \quad (20)$$

$$e'(L,M)=\sqrt{\left(\frac{1}{3}-\frac{1}{2}\right)^2+\left(\frac{2}{3}-\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{6} \quad (21)$$

$$e'(N,M)=\sqrt{\left(\frac{2}{3}-\frac{1}{2}\right)^2+\left(\frac{1}{3}-\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{6} \quad (22)$$

$$e'(L,A) = \sqrt{\left(\frac{1}{3}-1\right)^2 + \left(\frac{2}{3}-0\right)^2} = \frac{2\sqrt{2}}{3} \tag{23}$$

$$e'(M,A) = \sqrt{\left(\frac{1}{2}-1\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} \tag{24}$$

$$e'(B,M) = \sqrt{\left(0-\frac{1}{2}\right)^2 + \left(1-\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} \tag{25}$$

$$e'(B,A) = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{26}$$

i.e. are just multiplied by $\sqrt{2}$. ■

Example 2. Let us consider two fuzzy sets A, B in $X = \{1,2,3,4,5,6,7\}$. The intuitionistic-type representation of them is, e.g.:

$$A = (0.7, 0.3)/1 + (0.2, 0.8)/2 + (0.6, 0.4)/4 + (0.5, 0.5)/5 + (1, 0)/6 \tag{27}$$

$$B = (0.2, 0.8)/1 + (0.6, 0.4)/4 + (0.8, 0.2)/5 + (1, 0)/7 \tag{28}$$

The Hamming distance $d(A,B)$ using (5), is

$$d(A,B) = |0.7-0.2| + |0.2-0| + |0.6-0.6| + |0.5-0.8| + |1-0| + |0-1| = 3 \tag{29}$$

while the normalized distances (6) $l(A,B)$ is

$$l(A,B) = \frac{1}{7} \cdot d(A,B) = \frac{3}{7} = 0.43 \tag{30}$$

When both the membership and non-membership functions are taken into account [cf.(9)], we have

$$\begin{aligned} d'(A,B) &= |0.7-0.2| + |0.3-0.8| + |0.2-0| + |0.8-1| + |0.6-0.6| + |0.4-0.4| + |0.5-0.8| + \\ &+ |0.5-0.2| + |1-0| + |0-1| + |0-1| + |1-0| = 6, \end{aligned} \tag{31}$$

i.e. two times the value from (5), while the normalized distance (10) is

$$l'(A,B) = \frac{2}{n} \cdot d'(A,B) = \frac{6}{7} = 0.86 \tag{32}$$

Let us compare the Euclidean distances calculated by (7) and (11). From (7) we have

$$\begin{aligned} e(A,B) &= \sqrt{(0.7-0.2)^2 + (0.2-0)^2 + (0.6-0.6)^2 + (0.5-0.8)^2 + (1-0.2)^2 + (0-1)^2} = \\ &= \sqrt{2.38} = 1.54 \end{aligned} \tag{33}$$

while the normalized Euclidean distance (8) is

$$q(A,B)=\sqrt{\frac{1}{7}}\cdot e(A,B)=\sqrt{\frac{2.38}{7}}=0.58 \quad (34)$$

From (11) we have the Euclidean distance equal to

$$e'(A,B)=\big((0.7-0.2)^2+(0.3-0.8)^2+(0.2-0)^2+(0.8-1)^2+ \\ +(0.6-0.6)^2+(0.4-0.4)^2+(0.5-0.8)^2+(0.5-0.2)^2+(1-0)^2+ \\ +(0-0)^2+(0-1)^2+(1-0)^2\big)^{\frac{1}{2}}=\sqrt{4.76}=2.18 \quad (35)$$

while the normalized Euclidean distance (12) equal to

$$q'(A,B)=\sqrt{\frac{4.76}{7}}=0.83 \quad (36)$$

■

The results obtained in Examples 1 and 2 (see also Fig. 4) show that:

- for the distances calculated between any fuzzy sets A and B, when taking into account the membership functions (5)-(8) only, we have

$$0 \le d(A,B) \le n \quad (37)$$
$$0 \le l(A,B) \le 1 \quad (38)$$
$$0 \le e(A,B) \le \sqrt{n} \quad (39)$$
$$0 \le q(A,B) \le 1 \quad (40)$$

- for the distances calculated for any fuzzy sets A and B, taking into account the intuitionistic-type representation of fuzzy sets (9)-(12), we have

$$0 \le d'(A,B) \le 2n \quad (41)$$
$$0 \le l'(A,B) \le 2 \quad (42)$$
$$0 \le e'(A,B) \le \sqrt{2n} \quad (43)$$
$$0 \le q'(A,B) \le \sqrt{2} \quad (44)$$

3.2. Distances for intuitionistic fuzzy sets

We will now extend the concepts of distances presented in Section 3.1 to the case of intuitionistic fuzzy sets.

Following the line of reasoning presented in Section 3.1, for two intuitionistic fuzzy sets A and B in $X=\{x_1, x_2, \dots, x_n\}$ the Hamming distance is

$$d_{IFS}(A,B)=\sum_{i=1}^{n}\big(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|\big) \quad (45)$$

Taking into account that

$$\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) \qquad \text{and} \qquad \pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i) \tag{46}$$

we have

$$\begin{aligned} &|\pi_A(x_i) - \pi_B(x_i)| = |1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i)| \le \\ &\le |\mu_B(x_i) - \mu_A(x_i)| + |\nu_B(x_i) - \nu_A(x_i)| \end{aligned} \tag{47}$$

Inequality (47) means that the third parameter in (45) cannot be omitted as it was in the case of fuzzy sets, for which taking into account the second parameter would only result in the multiplication by a constant value.

A similar situation occurs for the Euclidean distance. Namely, for intuitionistic fuzzy sets A and B in $X = \{x_1, x_2, \ldots, x_n\}$, by following the line of reasoning as in Section 3.1., their Euclidean distance is equal to

$$e_{IFS}(A,B) = \sqrt{\sum_{i=1}^{n} \left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2 + \left(\pi_A(x_i) - \pi_B(x_i)\right)^2} \tag{48}$$

Let us verify the effect of omitting the third parameter (π) in (48). Taking into account (46), we have

$$\begin{aligned} &\left(\pi_A(x_i) - \pi_B(x_i)\right)^2 = \left(1 - \mu_A(x_i) - \nu_A(x_i) - 1 + \mu_B(x_i) + \nu_B(x_i)\right)^2 = \\ &= \left(\mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i)\right)^2 = \\ &= \mu_B(x_i)^2 + \mu_A(x_i)^2 + \nu_B(x_i)^2 + \nu_A(x_i)^2 + 2\big(-\mu_A(x_i)\mu_B(x_i) + \\ &+ \mu_B(x_i)\nu_B(x_i) - \mu_A(x_i)\nu_B(x_i) - \mu_B(x_i)\nu_A(x_i) + \mu_A(x_i)\nu_A(x_i) - \nu_B(x_i)\nu_A(x_i)\big) = \\ &= \left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2 + 2\left(\mu_A(x_i) - \mu_B(x_i)\right)\left(\nu_A(x_i) - \nu_B(x_i)\right) \end{aligned} \tag{49}$$

which means that taking into account the third parameter π when calculating the Euclidean distance for intuitionistic fuzzy sets does have an influence on the final result. It is obvious because a two-dimensional geometrical interpretation (Fig. 2) is an orthogonal projection of a real situation presented in Fig. 3.

Certainly, from (45) and (48) we obtain that:

- the normalized Hamming distance is

$$l_{IFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\right) \tag{50}$$

- the normalized Euclidean distance is

$$q_{IFS}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2 + \left(\pi_A(x_i) - \pi_B(x_i)\right)^2} \tag{51}$$

Example 3. Let us consider for simplicity "degenerated" intuitionistic fuzzy sets $A,B,D,G,F\ in\ X=\{1\}$, exemplified by:

$$A=(1,0,0)/1,\ B=(0,1,0)/1,\ D=(0,0,1)/1,\ G=\left(\frac{1}{2},\frac{1}{2},0\right)/1,\ E=\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)/1 \qquad (52)$$

and their geometrical interpretation is as in Fig 5

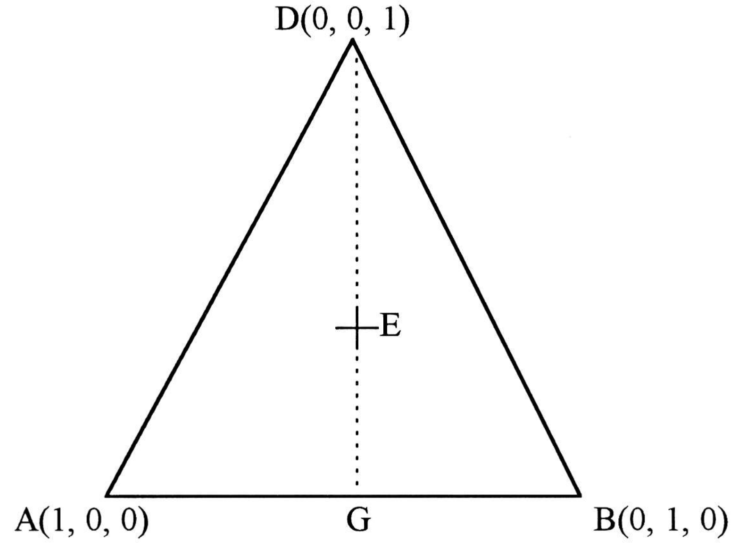

Figure 5

The Euclidean distance between the above intuitionistic fuzzy sets, ommiting the third parameter, is

$$e(A,B)=\sqrt{\sum_{i=1}^{n}\left(\mu_A(x_i)-\mu_B(x_i)\right)^2+\left(\nu_A(x_i)-\nu_B(x_i)\right)^2} \qquad (53)$$

and we obtain:

$$e(A,D)=\sqrt{(1-0)^2+0^2}=1 \qquad (54)$$

$$e(B,D)=\sqrt{0^2+(0-1)^2}=1 \qquad (55)$$

$$e(A,B)=\sqrt{(1-0)^2+(0-1)^2}=\sqrt{2} \qquad (56)$$

$$e(A,G)=\sqrt{\left(1-\frac{1}{2}\right)^2+\left(0-\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}} \qquad (57)$$

$$e(B,G)=\sqrt{\left(0-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}} \qquad (58)$$

$$e(E,G)=\sqrt{\left(\frac{1}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{4} \qquad (59)$$

$$e(D,G)=\sqrt{\left(0-\frac{1}{2}\right)^2+\left(0-\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{2} \qquad (60)$$

The above results are somewhat countintuitive. As it was shown (Fig. 3), the triangle ABD (Fig. 5) has all edges equal to $\sqrt{2}$ (as they are diagonals of squares with sides equal to 1). So we should obtain $e(A,D)=e(B,D)=e(A,B)$. But our results show only that $e(A,D)=e(B,D)$ [cf. (54)-(55)], but $e(A,D)\neq e(A,B)$, and $e(B,D)\neq e(A,B)$. Also $e(E,G)$, which is a half of the height of triangle ABD, is not the value we expect [it is too short, and the same concerns the height of *e(D, G)*].

For the same Euclidean distances using (48), we obtain:

$$e_{IFS}(A,D)=\sqrt{(1-0)^2+0^2+(0-1)^2}=\sqrt{2} \qquad (61)$$

$$e_{IFS}(B,D)=\sqrt{0^2+(1-0)^2+(0-1)^2}=\sqrt{2} \qquad (62)$$

$$e_{IFS}(A,B)=\sqrt{(1-0)^2+(0-1)^2+0^2}=\sqrt{2} \qquad (63)$$

$$e_{IFS}(A,G)=\sqrt{\left(1-\frac{1}{2}\right)^2+\left(0-\frac{1}{2}\right)^2+0^2}=\frac{\sqrt{2}}{2} \qquad (64)$$

$$e_{IFS}(B,G)=\sqrt{\left(0-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2+0^2}=\frac{\sqrt{2}}{2} \qquad (65)$$

$$e_{IFS}(E,G)=\sqrt{\left(\frac{1}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{2}-0^2\right)}=\frac{\sqrt{2}\sqrt{3}}{4} \qquad (66)$$

$$e_{IFS}(D,G)=\sqrt{\left(0-\frac{1}{2}\right)^2+\left(0-\frac{1}{2}\right)^2+(1-0)^2}=\frac{\sqrt{2}\sqrt{3}}{2} \qquad (67)$$

Formula (48) gives the results we may expect, i.e.

$$e(A,D)=e(B,D)=e(A,B)=2e(A,G)=2e(B,G)$$

and $e(E,G)$ is equal to a half of the height of a triangle with all edges equal $\sqrt{2}$, i.e. $\frac{\sqrt{2}\sqrt{3}}{4}$

■

Example 4. For the following intuitionistic fuzzy ts A and B in $X=\{1,2,3,4,5,6,7\}$:

$$A=(0.5,0.3,0.2)/1+(0.2,0.6,0.2)/2+(0.3,0.2,0.5)/4+(0.2,0.2,0.6)/5+(1,0,0)/6 \qquad (68)$$

$$B=(0.2,0.6,0.2)/1+(0.3,0.2,0.5)/4+(0.5,0.2,0.3)/5+(0.9,0,0.1)/7 \qquad (69)$$

the Hamming distance (45), taking into account all three parameters, is

$$\begin{aligned} d_{IFS}(A,B)&=|0.5-0.2|+|0.3-0.6|+|0.2-0.2|+|0.2-0|+|0.6-1|+|0.2-0|+\\ &+|0.3-0.3|+|0.2-0.2|+|0.5-0.5|+|0.2-0.5|+|0.2-0.2|+|0.6-0.3|+\\ &+|1-0|+|0-1|+|0-0|+|0-0.9|+|1-0|+|0-0.1|=6 \end{aligned} \qquad (70)$$

while the normalized Hamming distance (50), taking into account all three parameters, is

$$l_{IFS}(A,B)=\frac{6}{7}=0.86 \tag{71}$$

The Hamming distance taking into account the two parameters only is

$$d(A,B)=|0.5-0.2|+|0.3-0.6|+|0.2-0|+|0.6-1|+|0.3-0.3|+|0.2-0.2|+|0.2-0.5|+|0.2-0.2|+$$
$$+|1-0|+|0-1|+|0-0.9|+|1-0|=5.4 \tag{72}$$

and its normalized counterpart is

$$l(A,B)=\frac{1}{7}\cdot d(A,B)=\frac{5.4}{7}=0.77 \tag{73}$$

The Euclidean distance (48) taking into account all three parameters is

$$e_{IFS}(A,B)=\Big((0.5-0.2)^2+(0.3-0.6)^2+(0.2-0.2)^2+(0.2-0)^2+(0.6-1)^2+(0.2-0)^2+$$
$$+(0.3-0.3)^2+(0.2-0.2)^2+(0.5-0.5)^2+(0.2-0.5)^2+(0.2-0.2)^2+(0.6-0.3)^2+$$
$$+(1-0)^2+(0-1)^2+0^2+(0-0.9)^2+(1-0)^2-(0-0.1)^2\Big)^{0.5}=\sqrt{4.42}=2.1 \tag{74}$$

and the normalized Euclidean distance taking into account all three parameters is

$$q_{IFS}(A,B)=\frac{e(A,B)}{\sqrt{7}}=\sqrt{\frac{4.42}{7}}=0.79 \tag{75}$$

The Euclidean distance (53) taking into account two parameters only is

$$e(A,B)=\Big((0.5-0.2)^2+(0.3-0.6)^2+(0.2-0)^2+(0.6-1)^2+(0.3-0.3)^2+(0.2-0.2)^2+$$
$$+(0.2-0.5)^2+(0.2-0.2)^2+(1-0)^2+(0-1)^2+(0-0.9)^2-(1-0)^2\Big)^{0.5}=\sqrt{4.28}=2.07 \tag{76}$$

and the normalized Euclidean distance taking into account two parameters is

$$q(A,B)=\sqrt{\frac{1}{7}\cdot e(A,B)}=\sqrt{\frac{4.28}{7}}=0.78 \tag{77}$$

■

The results obtained in Examples 3 and 4 confirm that distances in intuitionistic fuzzy sets should be calculated by taking into account all three parameters (membership degree, non-membership degree, and values of hesitancy margin). It is also easy to notice that for the formulas (45), (50), (48), (51) the following is valid

$$0\le d_{IFS}(A,B)\le 2n \tag{78}$$

$$0 \le l_{IFS}(A,B) \le 2 \tag{79}$$

$$0 \le e_{IFS}(A,B) \le \sqrt{2n} \tag{80}$$

$$0 \le q_{IFS}(A,B) \le \sqrt{2} \tag{81}$$

Using two parameters only gives values of distances which are orthogonal projection of the real distances (Fig. 3), and this implies that they are lower.

4. Concluding remarks

In this paper we proposed new definitions of distances between intuitionistic fuzzy sets. It was shown that their definitions should be calculated taking into account three parameters describing an intuitionistic fuzzy set.Taking into account all three parameters describing intuitionistic fuzzy sets when calculating distances ensures that the distances for fuzzy sets and intuitionistic fuzzy sets can be easily compared [cf. (37)-(44) and (78)-(81)].

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