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On the law of large numbers for IF-events

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Abstract: A family $\mathcal{F}$ of IF-events is considered with a state $m : \mathcal{F} \to [0, 1]$ and a version of the law of large numbers is presented for sequences of $\mathcal{F}$-valued observables.
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1 IF-events

In [1], a new fuzzy set theory has been introduced, so-called intuitionistic fuzzy sets theory (IF). Then in [4] a definition of probability on IF events has been introduced and in [5, 7] an axiomatic approach has been studied. Also representation of IF probability by the Kolmogorov probability has been discovered ([2, 3, 6]).

So an IF-event is a pair $A = (\nu_A, \nu_A)$ of mappings $\mu_A, \nu_A : X \to [0, 1]$ defined on a non-empty set X such that

$$\mu_A + \nu_A \leq 1.$$

The mapping μ_A is called the membership function, the mapping ν_A the non-membership function.

We shall use the Lukasiewicz operations with IF-events:

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1)),)$$

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$\neg A = (1 - \mu_A, 1 - \nu_B).$$

The operation $A \odot B$ corresponds to the intersection of sets in the classical case, $A \oplus B$ to the union of sets, $\neg A$ to the complement of a set A. Moreover we define a partial ordering by

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Instead of the probability the therm state of IF-events is considered in the IF-theory. Let $\mathcal{F}$ be a non-empty family of IF-events closed under the operations $\odot, \oplus, \neg$. An IF-state is a mapping $m : \mathcal{F} \to [0, 1]$ satisfying the following conditions:

$$(i) m((1, 0)) = 1, m((0, 1)) = 0,$$

$$(ii) A \odot B = (0, 1) \Longrightarrow m(A \oplus B) = m(A) + m(B),$$

$$(iii) A_n \nearrow A \Longrightarrow\ m(A_n) \nearrow m(A).$$

Recall that $A_n \nearrow A$ means $\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$.

Instead of a random variable the notion an observable is studied. An observable is a mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following properties:

$$(i) x(R) = (1, 0), x(\emptyset) = (o, 1),$$

$$(ii) A \cap B = \emptyset \Longrightarrow x(A) \odot x(B) = (0, 1), x(A \oplus B) = x(A) + x(B),$$

$$(iii) A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A).$$

Recall that here A_n, A are classical sets from $\mathcal{B}(R)$, and $A_n \nearrow A$ means that $A_{n+1} \subset A_n (n = 1, 2, 3, ...)$ and $A = \bigcup_{n=1}^{\infty} A_n$. In the classical case a random variable is a mapping $\xi : \Omega \to R$ such that $\xi^{-1}(A) \in \mathcal{S}$ for any $A \in \mathcal{B}(R)$ where $\mathcal{S}$ is the σ-algebra of events in Ω.

If $x : \mathcal{B}(R) \to \mathcal{F}$ is an observable, and $m : \mathcal{F} \to [0, 1]$ is a IF-state, then $m_x = mox : \mathcal{B}(R \to [0, 1]$ is a probability measure, the probability distribution of the observable x.

Finally we can define the expectation $E(x)$ and the dispersion $D(x)$ (if they exist) by the formulas

$$E(x) = \int_R t dm_x(t), D(x) = \int_R (t - E(x))^2 dm_x(t).$$

2 Independence

Two observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ are independent if there exists a two dimensional observable $\kappa : \mathcal{B}(R^2) \to \mathcal{F}$ such that

$$\kappa(A \times R) = x(A), \kappa(R \times B) = y(B),$$

and

$$m_\kappa(A \times B) = m_x(A) m_y(B)$$

for all $A, B \in \mathcal{B}(R)$. The mapping $\kappa : \mathcal{B}(R^2) \to \mathcal{F}$ is called the joint observable of x, y.

Again it is a generalization of the independence of random variables $\xi, \eta : \Omega \to R$. They are independent if

$$P(\xi^{-1}(A) \cap \eta^{-1}(B)) = P(\xi^{-1}(A)) P(\eta^{-1}(B))$$

for any $A, B \in \mathcal{B}(R)$. Of course,

$$P(\xi^{-1}(A)) = P_\xi(A), P(\eta^{-1}(B)) = P_\eta(B),$$

and if we put $T = (\xi, \eta)) : \Omega \to R^2$, then

$$P(\xi^{-1}(A) \cap \eta^{-1}(B)) = P(T^{-1}(A \times B)) = P_T(A \times B).$$

It is natural to generalize the notion of independence for a finite number of observables $x_1, ..., x_n$. A sequence $(x_n)_n$ of observables is independent, if $x_1, ..., x_n$ are independent for any $n \in N$.

If x, y are independent, we can define their sum similarly as in tha classical case. Namely

$$\xi + \eta = goT,$$

where $g : R^2 \to R, g(u, v) = u + v$, hence

$$(\xi + \eta)^{-1}(A) = T^{-1}(g^{-1}(A)).$$

Analogously we can define $x + y : \mathcal{B}(R) \to \mathcal{F}$ by the formula

$$(x + y)(A) = \kappa o g^{-1}(A),$$

$$A \in \mathcal{B}(R).$$

3 Law of large numbers

One of the simplest form of the law of large numbers is the following. Let $(\xi_n)_n$ be a sequence of independent observables with the same expectation $E(\xi_n) = a$ and the same dispersion $D(\xi_n) = \sigma^2$.Then for every $\varepsilon > 0$ there holds

$$\lim_{n\to\infty} P(\{\omega \in \Omega; |\frac{1}{n}\Sigma_{i=1}^n \xi_i(\omega) - a| < \varepsilon\}) = 1.$$

As we have seen before it is not difficult to define the observable

$$\frac{1}{n}\Sigma_{i=1}^n \xi_i - a = \kappa o g_n$$

where $\kappa : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint observable of $x_1, ..., x_n$, and $g_n : R^n \to R$

$$g_n(u_1, ..., u_n) = \frac{1}{n}\Sigma_{i=1}^n - a.$$

Theorem: Let $(x_n)_n$ be a sequence of independent observables with the same expectation $E(x_i) = a$ and the same dispersion $D(x_i) = \sigma^2$. Then for every $\varepsilon > 0$

$$\lim_{n\to\infty} m((\frac{1}{n}\Sigma_{i=1}^n - a)(-\varepsilon, \varepsilon)) = 1.$$

Proof: Put $P_n(A) = m(\kappa_n(A)), A \in \mathcal{B}(R^n)$. Then $P_n : \mathcal{B}(R^n) \to [0, 1]$ is a probability measure. Let $\mathcal{C}$ be the family of all subsets of R^N of the form

$$A = \{\in R^N; u_1 \in A_1, ..., u_n \in A_n\},$$

where $n \in N, A_i \in \mathcal{B}(R)$. Since

$$P_{n+1}(R \times A) = P_n(A)$$

for any $A \in \mathcal{B}(R^n)$, by the Kolmogorov consistency theorem there exists a probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\Pi_n^{-1}(A)) = P_n(A)$$

for every $A \in \mathcal{B}(R^n)$, where $\Pi_n : R^N \to R^n$ is the projection. Then $(R^N, \sigma(\mathcal{C}), P)$ is a probability space. Define $\xi_n : R^N \to R^n$ by the formula

$$\xi_n((u_i)_i) = u_n.$$

Compute

$$P(\xi_1^{-1}(A_1) \cap ... \cap \xi_n^{-1}(A_n) = P(\Pi_n^{-1}(A_1 \times ... \times A_n)) =$$

$$= P_n(A_1 \times ... \times A_n) =$$

$$= m(\kappa_n(A_1 \times ... \times A_n)) = m(x_1^{-1}(A_1)) \times ... \times m(x_n(A_n)) =$$

$$= P_n(A_1 \times R \times ... \times R) \times ... \times P_n(R \times R \times ... \times A_n) =$$

$$= P(\Pi_n^{-1}(A \times R \times ... \times R) \times ... \times P(\Pi_n^{-1}(R \times R \times ... \times A_n) =$$

$$= P(\xi_1^{-1}(A_1)) \times ... \times P(\xi_n^{-1}(A_n)),$$

hence $\xi_1, \xi_2, ...$ are independent. Moreover

$$E(\xi_n) = \int_R t dP_{\xi_n}(t) = \int_R t dm_{x_n}(t) = E(x_n) = a,$$

and simlarly $D(\xi_n) = D(x_i) = \sigma^2$. Put $g_n : R^n \to R$

$$g_n(u)1, ..., u_n) = \frac{1}{n}\Sigma_{n=1}^n u_i - a.$$

Then by the law of large numbers we obtain

$$1 = \lim_{n\to\infty} P(\{u \in R^N; |\frac{1}{n}\Sigma_{i=1}^n \xi_i(u) - a| < \varepsilon\} =$$

$$= \lim_{n\to\infty} P((g_n o \Pi_n)^{-1}((-\varepsilon, \varepsilon))) =$$

$$= \lim_{n\to\infty} P(\Pi_n^{-1}(g_n^{-1}((-\varepsilon, \varepsilon))) =$$

$$= \lim_{n\to\infty} P_n(g_n^{-1}(-\varepsilon, \varepsilon))) =$$

$$= \lim_{n\to\infty} m(\kappa((g_n^{-1}((-\varepsilon, \varepsilon))) =$$

$$= \lim_{n\to\infty} m((\frac{1}{n}\Sigma_{i=1}^n - a)(-\varepsilon, \varepsilon)).$$

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