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An embedding IF-sets to MV-algebras

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Abstract

MV-algebras and IF-sets were studying in the past. On MV-algebras there is built a good theory of probability. It will be good make use this theory of probability in connection with IF-set. Therefore I essayed about embedding IF-set to MV-algebra. Then we can use known knowledges from MV-algebras. In my contribution I define fundamental notions, like MV-algebra, tribe and IF-set. There I declare and prove, that any IF-set can be embedded to an MV-algebra. This result generalized the previous result of [3], where it was proved only for tribes of all measurable functions.

1 Introduction

In the paper [3] the family $\mathcal{F} = \{\ (f,g);\ f,g \in \mathcal{T}_0\ ,\ f+g \leq 1\}$ was considered where $\mathcal{T}_0$ is the set of all functions $f : \Omega \longrightarrow \langle 0,1 \rangle$ measurable with respect to a σ-algebra of subsets of Ω. It was proved that there exists an MV-algebra $\mathcal{M}$ such that $\mathcal{F}$ can be embedded to $\mathcal{M}$. In the paper we show that instead of $\mathcal{T}_0$ an arbitrary tribe $\mathcal{T}$ can be considered.

Let us considered a non-empty set Ω and let $\mathcal{S}$ be a $\sigma - algebra$ of subset of Ω.

Definition 1.1 *Tribe $\mathcal{T}$ is set of functions $f : \Omega \longrightarrow < 0,1 >$ measurable with respect to $\mathcal{S}$ and we assume that $\mathcal{T}$ satisfies the following conditions:*
$(1) 0_\Omega \in \mathcal{T}, 1_\Omega \in \mathcal{T};$
$(2) f,g \in \mathcal{T} \Longrightarrow f \oplus g = (f+g) \wedge 1 \in \mathcal{T}, f \odot g = (f+g-1) \vee 0 \in \mathcal{T};$
$(3) f_n \in \mathcal{T}, f_n \nearrow f \Longrightarrow f \in \mathcal{T};$
$(4) f \in \tau \Longrightarrow \neg f = 1 - f \in \mathcal{T}$
where $\wedge = min$ and $\vee = max$.

Definition 1.2 *An IF-set is a couple (f,g) of mappings $f,g\ :\ \Omega \longrightarrow \langle 0,1 \rangle$ such that $f+g \leq 1$.*

Definition 1.3 *An MV-algebra is a system* $(M, \oplus, \odot, \neg, 0, 1)$*, where* $\oplus$ *and* $\odot$ *are binary operations and* $\neg$ *is unary a operation.*
$\oplus$ *is commutative and associative operation,*
$a \oplus 0 = a,$
$a \oplus 1 = 1,$
$\neg(\neg a) = a,$
$\neg 0 = 1,$
$a \oplus (\neg a) = 1,$
$\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a,$
$a \odot b = \neg(\neg a \oplus \neg b).$
Every MV-algebra is distributive lattice, where $a \vee b = a \oplus (\neg(a \oplus b))$*, 0 is the least element, and 1 is the greatest element of M.*

2 Embedding

Theorem: Let $\mathcal{T}$ be any tribe, $\mathcal{M} = \{(f,g); f,g \in \mathcal{T}\}$ where for (f,g),(h,k)$\in \mathcal{M}$
$(f,g) \oplus (h,k) = (f \oplus h, g \odot k),$
$(f,g) \odot (h,k) = (f \odot h, g \oplus k),$
$\neg(f,g) = (1-f, 1-g)$
Then $(\mathcal{M}, \oplus, \odot, \neg, (0,1), (1,0))$ is an MV-algebra.

At first we proved that
$\neg(f \wedge 1, g \vee 0) = (\neg f \vee 0, \neg g \wedge 1)$

Indeed
$\neg(f \wedge 1, g \vee 0) = \neg(f,g) = (1-f, 1-g) = ((1-f) \vee 0, (1-g) \wedge 1) = (\neg f \vee 0, \neg g \wedge 1)$

We shall show that this system $\mathcal{M} = \{(f,g); f,g \in \mathcal{T}\}$ satisfies all properties of MV-algebra.

Let $(f,g), (h,k), (i,j) \in \mathcal{M}$

1) commutativity

$$(f,g) \oplus (h,k) = (f \oplus h, g \odot k) = (h \oplus f, k \odot g) = (h,k) \oplus (f,g)$$

2) associativity

$$[(f,g) \oplus (h,k)] \oplus (i,j) = [f \oplus h, g \odot k)] \oplus (i,j) = (f \oplus h \oplus i, g \oplus k \oplus j) =$$

$$= (f,g) \oplus (h \oplus i, k \oplus j) = (f,g) \oplus [(h,k) \oplus (i,j)]$$

3) $(f,g) \oplus (0,1) = (f,g)$

$$(f,g) \oplus (0,1) = (f \oplus 0, g \odot 1) =$$

$$= ((f+0) \wedge 1, (g+1-1) \vee 0) = (f \wedge 1, g \vee 0) = (f,g)$$

4) $(f,g) \oplus (1,0) = (1,0)$

$$(f,g) \oplus (1,0) = (f \oplus 1, g \odot 0) = ((f+1) \wedge 1, (g+0-1) \vee 0 = (1,0)$$

5) $\neg(\neg(f,g)) = (f,g)$

$$\neg(\neg(f,g)) = \neg(1-f, 1-g) = (1-1+f, 1-1+g) = (f,g)$$

6) $\neg(0,1) = (1,0)$

$$\neg(0,1) = (1-0, 1-1) = (1,0)$$

7) $(f,g) \oplus \neg(f,g) = (1,0)$

$$(f,g) \oplus \neg(f,g) = (f,g) \oplus (1-f, 1-g) = (f \oplus (1-f), g \odot (1-g)) =$$

$$((f+1-f) \wedge 1, (g+1-g) \vee 0) = (1 \wedge 1, 1 \vee 0) = (1,0)$$

8) $\neg(\neg(f,g) \oplus (h,k)) \oplus (h,k) = \neg((f,g) \oplus \neg(h,k)) \oplus (f,g)$

Left side:

$$\neg(\neg(f,g) \oplus (h,k)) \oplus (h,k) = \neg((1-f, 1-g) \oplus (h,k)) \oplus (h,k) =$$

$$= \neg((1-f) \oplus h, (1-g) \odot k) \oplus (h,k) = \neg((1-f+h) \wedge 1, (1-g+k-1) \vee 0) \oplus (h,k) =$$

$$= \neg((1-f+h)\wedge 1,(k-g)\vee 0)\oplus(h,k) = ((1-1+f-h)\vee 0,(1-k+g)\wedge 1)\oplus(h,k) =$$

$$= ((f-h)\vee 0,(1+g-k)\wedge 1)\oplus(h,k) = (((f-h)\vee 0)\oplus h,((1+g-k)\wedge 1)\odot k) =$$

$$= (((f-h)\vee 0+h)\wedge 1,((1+g-k)\wedge 1+(k-1))\vee 0) = ((f\vee h)\wedge 1,(g\wedge k)\vee 0) = ((f\vee h),(g\wedge k))$$

Right side:

$$\neg((f,g)\oplus\neg(h,k))\oplus(f,g) = \neg((f,g)\oplus(1-h,1-k))\oplus(f,g) =$$

$$= \neg((f+1-h)\wedge 1,(g+1-k-1)\vee 0)\oplus(f,g) = ((1-f-1+h)\vee 0,(1-g+k)\wedge 1)\oplus(f,g) =$$

$$= ((h-f)\vee 0,(1-g+k)\wedge 1)\oplus(f,g) = (((h-f)\vee 0+f)\wedge 1,((1-g+k)\wedge 1+(g-1))\vee 0) =$$

$$= ((h\vee f)\wedge 1,(k\wedge g)\vee 0) = ((f\vee h),(g\wedge k))$$

Left side = Right side

9) $(f,g)\odot(h,k) = \neg(\neg(f,g)\oplus\neg(h,k))$

$$\neg(\neg(f,g)\oplus\neg(h,k)) = \neg((1-f,1-g)\oplus(1-h,1-k)) = \neg((2-f-h)\wedge 1,(1-g-k)\vee 0) =$$

$$= ((1-2+f+h)\vee 0,(1-1+g+k)\wedge 1) = ((f+h-1)\vee 0,(g+k)\wedge 1) = (f\odot h,g\oplus k) = (f,g)\odot(h,k)$$

Such that $(\mathcal{M},\oplus,\odot,\neg,(0,1),(1,0))$ is an MV-algebra.

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