A Measure of Similarity for Intuitionistic Fuzzy Sets

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Abstract

In this article we propose a new measure of similarity for intuitionistic fuzzy sets. The proposed measure takes into account not only a pure distance between elements but measures the whole missing information which may be necessary to say if the considered elements are more similar or more dissimilar. It is shown that even if a distance between objects is small it can happen that the objects are completely dissimilar.

Keywords: intuitionistic fuzzy sets, distances, similarity measure.

1 Introduction

The nature of similarity is broadly explored and discussed by Cross and Sudkamp [2]. They stressed the fundamental role of compatibility and similarity in inference and in applications in approximate reasoning using fuzzy set theory. The analysis of the similarity is as well a fundamental task when employing intuitionistic fuzzy sets (Atanassov, [1]).

In this article we propose a new measure of similarity which takes into account not only a pure distance between compared elements but measures as well the whole missing information which may be necessary to say if the considered elements are more similar or more dissimilar.

Dissimilarity of two elements X and F is specified here by a similarity measure between element X and the complement of element F, i.e. F^C.

We consider here the simplest situation - formulas are given for comparison of any two elements belonging to an intuitionistic fuzzy set. But it is easy to give analogical formulas for more complicated situations (e.g. m experts comparing n options in pairs - see (Szmidt and Kacprzyk, [6]).

The organization of the paper is as follow. First, intuitionistic fuzzy sets (Atanassov, [1]) are presented in a brief way. Next, a concept of distances between intuitionistic fuzzy sets is reminded and a new method of analyzing similarity is introduced. Finally, we give simple example illustrating that a small distance between elements/objects does not guarantee their similarity.

2 Brief introduction to intuitionistic fuzzy sets

As opposed to a fuzzy set in X(Zadeh [10]) , given by

$$A' = \{< x, \mu_{A'}(x) > | x \in X\} \quad (1)$$

where $\mu_{A'} : X \rightarrow [0,1]$ is the membership function of the fuzzy set A', an intuitionistic fuzzy set (Atanassov [1]) A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \quad (2)$$

where: $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad (3)$$

and $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following intuitionistic fuzzy set

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \quad (4)$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

a *hesitation margin* (or an *intuitionistic fuzzy index*) of $x \in A$ and, it expresses lack of knowledge of whether x belongs to A or not (cf. Atanassov [1]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

On the other hand, for each fuzzy set $A^{'}$ in X, we evidently have

$$\begin{aligned} \pi_{A'}(x) &= 1 - \mu_{A'}(x) - [1 - \mu_{A'}(x)] = \\ &= 0 \text{ for each } x \in X. \end{aligned} \tag{6}$$

In our further considerations we will use the notion of the complement elements, which definition is a simple consequence of a complement set A^C

$$A^C = \{< x, \nu_A(x), \mu_A(x) > | x \in X\} \tag{7}$$

The application of intuitionistic fuzzy sets instead of fuzzy sets means the introduction of another degree of freedom into a set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imperfect knowledge what leads to describing many real problems in a more adequate way.

Basically, intuitionistic fuzzy sets based models may be adequate in situations when we face human testimonies, opinions, etc. involving answers of the type:

- yes,
- no,
- does not apply.

Voting can be a good example of such a situation as the human voters may be divided into three groups of those who:

- vote for,
- vote against,
- abstain.

This third "out-of question" area is of a great relevance and interest because it is difficult to deal with it within, say, unceratinty calculi, interval analyses, etc. Intuitionistic fuzzy sets can provide here an effective and efficient tools for representation and processing.

In this paper we propose some similarity measures between intuitionistic fuzzy sets. These measures are relevant in many situations involving testimonies of the above type, i.e. containing "does not apply", "abstain", etc.

2.1 Distances between intuitionistic fuzzy sets

In [3, 4] it is shown why when calculating distances between intuitionistic fuzzy sets it is necessary to take into account all three parameters describing intuitionistic fuzzy sets. One of the reasons is that when taking into account two parameters only, for elements from classical fuzzy sets (which are a special case of intuitionistic fuzzy sets) we obtain distances from a different interval than for elements belonging to intuitionistic fuzzy sets. It practically makes it impossible to consider by the same formula the two types of sets. For more details we refer the interested reader to [3, 4].

In our further considerations we will use the normalized Hamming distance between intuitionistic fuzzy sets A, B in $X = \{x_1, x_2, \ldots, x_n\}$ [3, 4]:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + \\ &+ |\nu_A(x_i) - \nu_B(x_i)| + \\ &+ |\pi_A(x_i) - \pi_B(x_i)|) \end{aligned} \tag{8}$$

For (8) we have:

$$0 \leq l_{IFS}(A,B) \leq 1. \tag{9}$$

3 Similarity measure

We propose here a new similarity measure for intuitionistic fuzzy sets. We use a geometrical interpretation of intuitionistic fuzzy sets which was described in details by Szmidt and Kacprzyk ([4, 5]). Here we remind only that parameters

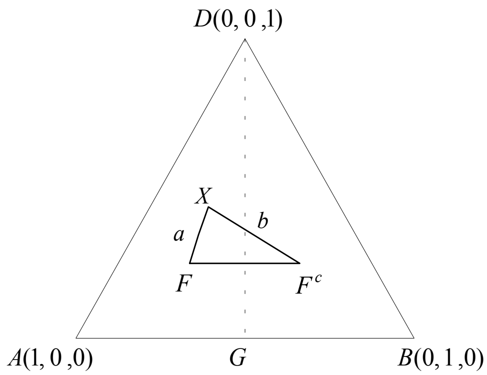

Figure 1: The triangle ABD explaining a ratio-based measure of similarity

(μ, ν, π) of any element belonging to an intuitionistic fuzzy set can be represented as the coordinates of the point (μ, ν, π) belonging to the triangle ABD (Figure 1). Point A represents elements fully belonging to an intuitionistic fuzzy set ($\mu = 1$), point B represents elements fully not belonging to an intuitionistic fuzzy set ($\nu = 1$), point D represents elements with hesitation margin $\pi = 1$ i.e, about which we are not able to say if they belong or not belong to an intuitionistic fuzzy set. Any other combination of the parameters characteristic for elements belonging to an intuitionistic fuzzy set can be represented inside triangle ABD.

In the simplest situations we assess similarity of any two elements X and F belonging to an intuitionistic fuzzy set (or sets). The proposed measure says if X is more similar to F or to F^C, where F^C is a complement of F. In other words, the proposed measure says if X is more similar or more dissimilar to F.

Definition 1

$$Sim(X,F) = \begin{cases} 1 & for \;\; a = 0, \; 0 < b \underline{<} 1 \\ 1 - \frac{a}{b} & for \;\; 0 < a < b, \\ 0 & for \;\; a = b, \; a, b \neq 0 \\ undefined & for \;\; b = 0 \;\; or \;\; b > a \end{cases} \tag{10}$$

where: a (Figure 1) is a distance(X, F) *from* $X(\mu_X, \nu_X, \pi_X)$ *to* $F(\mu_F, \nu_F, \pi_F)$,
b (Figure 1) is the distance(X, F^C) *from* $X(\mu_X, \nu_X, \pi_X)$ *to* $F^C(\nu_F, \mu_F, \pi_F)$,
F^C *is a complement of* F,
the distances $l_{IFS}(X, F)$ *and* $l_{IFS}(X, F^C)$ *are calculated from (8).*

For (10) we have

$$Sim(X, F) = Sim(F, X)$$

Similarity has typically been assumed to be symmetric. Tversky [9] however provided empirical evidence that similarity should not always be treated as a symmetric relation. We stress it to show that the similarity measure (10) has some features which can be useful in some situations but are not welcome in others.

It is worth noticing that when

- $a = l_{IFS}(X, F) = 0$, $b \neq 0$ means identity of X and F,
- $a = b$, $a \neq 0$, and $b \neq 0$ means that X is to the same extent similar to F and F^C, so having in mind that we are interested in more similar than more dissimilar X and F, the similarity measure (10) is assumed to be equal to zero,
- $X = F^C$ (or $X^C = F$), i.e. $b = l_{IFS}(X, F^C)$=$l_{IFS}(X^C, F)$= 0 means complete dissimilarity of X and F (or in other words, identity of X and F^C),
- $X = F = F^C$ ($a = b = 0$) means the highest possible entropy (see [5]) for both elements F and X i.e. the highest "fuzziness" – not too constructive case when looking for similarity.

In other words, the proposed measure (10) was constructed for selecting objects which are more similar than dissimilar (and well-defined in the sense of possessing (or not) attributes we are interested in).

Now we will show that so defined measure of of similarity (10) between $X(\mu_X, \nu_X, \pi_X)$ and $F(\mu_F, \nu_F, \pi_F)$ is more powerful then a simple distance between them.

Example 1 Let X and F be two elements belonging to an intuitionistic fuzzy set (with the coordinates (μ, ν, π),

$$X = (0.5, 0.4, 0.1)$$

$$F = (0.4, 0.5, 0.1)$$

so

$$F^C = (0.5, 0.4, 0.1)$$

and from (10) we have

$$\begin{aligned} l_{IFS}(X,F) &= \frac{1}{2}(|0.5-0.4| + |0.4-0.5| + \\ &+ |0.1-0.1|) = 0.1 \end{aligned} \quad (11)$$

what means that the distance is small - basing on it only we would say that X and F are similar. However

$$\begin{aligned} l_{IFS}(X,F^C) &= \frac{1}{2}(|0.5-0.5| + |0.4-0.4| + \\ &+ |0.1-0.1|) = 0 \end{aligned} \quad (12)$$

what means that X is just the same as F^C . We can not speak at all about similarity of X and F despite that the distance between them is small.
$\square$

Summing up:

- When a distance between two (or more) objects, sets is big it means for sure that the similarity does not exist.
- When a distance is small, we can say nothing sure about similarity basing on a pure distance between two objects only (when not taking into account complements of the objects as in (10)). The distance between objects can be small and the compared objects can be more dissimilar than similar.

4 Concluding remarks

We proposed a new measure of similarity. It was shown that in some situations, pure distance is not a proper measure of similarity.

Intuitionistic fuzzy sets with their possibilities of taking into account non-memberships are the tool which makes it possible to notice the fact that a small distance between the compared objects does not mean that the objects are similar.

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