Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy metric space

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Abstract: We propose a method for constructing a Hausdorff intuitionistic fuzzy metric on the set of the nonempty compact subsets of a given intuitionistic fuzzy metric space. We discuss several important properties as completeness, completion and separability.
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1 Introduction

The notion of defining intuitionistic fuzzy sets (IFSs) for fuzzy set generalizations, introduced by Atanassov [1], has proven interesting and useful in various application areas. Since this fuzzy set generalization can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation becomes more meaningful and applicable. Many authors [5, 6, 9] have introduced and discussed several notions of intuitionistic fuzzy metric space from different points of view. Here, we introduce and discuss a suitable notion for the L_p intuitionistic fuzzy metric of a given intuitionistic fuzzy metric space on the set of its non-empty compact subsets. In particular, we explore several properties of the L_p intuitionistic fuzzy metric, as completeness and completion, and separability.

In this way, we provide a new contribution to the development of the theory of intuitionistic fuzzy metrics in a potentially interesting direction due to the undoubted importance of the Hausdorff distance not only in general topology but also in other areas of Mathematics and Computer Science.

The structure of the paper is as follows. In Section 2 we give the background and auxiliary results which will be needed. Sections 3 and 4 are devoted to construct our L_p intuitionistic fuzzy metric and discuss its properties.

2 Intuitionistic fuzzy metric spaces

Let X be an arbitrary non-empty set and IF(X) be the intuitionistic fuzzy subsets of X

$$\text{IF}(X) = \big\{(u,v) \in I^X \times I^X \ : \ 0 \leq u(x) + v(x) \leq 1 \ \ x \in X\big\}$$

A mapping $d : \text{IF}(X) \times \text{IF}(X) \to \overline{\mathbb{R}}$ is said to be an intuitionistic fuzzy metric on IF(X) if it satisfies the following conditions.

1. $d\Big(\langle u_1, v_1\rangle , \langle u_2, v_2\rangle \Big) \geq 0, \forall \langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \text{IF}(X)$

2. $d\Big(\langle u_1, v_1\rangle , \langle u_2, v_2\rangle \Big) = 0$ iff $\langle u_1, v_1\rangle = \langle u_2, v_2\rangle$

3. $d\Big(\langle u_1, v_1\rangle , \langle u_2, v_2\rangle \Big) = d\Big(\langle u_2, v_2\rangle , \langle u_1, v_1\rangle \Big) \ \forall \langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \text{IF}(X)$

4. $d\Big(\langle u_1, v_1\rangle , \langle u_3, v_3\rangle \Big) \leq d\Big(\langle u_1, v_1\rangle , \langle u_2, v_2\rangle \Big) + d\Big(\langle u_2, v_2\rangle , \langle u_3, v_3\rangle \Big)$
   $\forall \langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \langle u_3, v_3\rangle \in \text{IF}(X)$

The pair $(\text{IF}(X), d)$ is called an intuitionistic fuzzy metric space.
Let $X = \mathbb{R}$, we denote by

$$\text{IF}^1 = \text{IF}(\mathbb{R}) = \big\{\langle u,v\rangle \ : \ \mathbb{R} \to [0,1]^2 , \ 0 \leq u(x) + v(x) \leq 1\big\}$$

An element $\langle u, v\rangle$ of IF^1 is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) *supp* $\langle u, v\rangle = \text{cl}\{x \in \mathbb{R} : v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by IF$_1$
For $\alpha \in [0,1]$ and $\langle u, v\rangle \in \text{IF}^1$, the upper and lower α-cuts of $\langle u, v\rangle$ are defined by

$$[\langle u,v\rangle]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u,v\rangle]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$$

Remark 1. *If $\langle u, v\rangle$ if a fuzzy number, so we can see $[\langle u,v\rangle]^\alpha$ as $[u]^\alpha$ and $[\langle u,v\rangle]_\alpha$ as $[1-v]^\alpha$*

Example. A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v :

$$u(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3, \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1^{'}} & \text{if } a_1^{'} \leq x \leq a_2 \\ \dfrac{x-a_2}{a_3^{'}-a_2} & \text{if } a_2 \leq x \leq a_3^{'}, \\ 1 & otherwise. \end{cases}$$

$$\Big[\langle u, v\rangle\Big]_\alpha = \Big[a_1+\alpha(a_2-a_1), a_3-\alpha(a_3-a_2)\Big]$$
$$\Big[\langle u, v\rangle\Big]^\alpha = \Big[a_1'+\alpha(a_2-a_1'), a_3'-\alpha(a_3'-a_2)\Big]$$

We define $0_{(0,1)} \in IF_1$ as

$$0_{(0,1)}(t) = \begin{cases} (1,0) & t=0 \\ (0,1) & t \neq 0 \end{cases}$$

Let $\langle u, v\rangle$,$\langle u', v'\rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by :

$$\Big(\langle u, v\rangle \oplus \langle u', v'\rangle\Big)(z) = \Big(\sup_{z=x+y} \min\left(u(x), u'(y)\right), \inf_{z=x+y} \max\left(v(x), v'(y)\right)\Big)$$

$$\lambda\langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \text{if } \lambda \neq 0 \\ 0_{(0,1)} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u, v\rangle$, $\langle z, w\rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, the addition and scale-multiplication are defined as follows

$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]^\alpha = \Big[\langle u, v\rangle\Big]^\alpha + \Big[\langle z, w\rangle\Big]^\alpha, \quad \Big[\lambda\langle z, w\rangle\Big]^\alpha = \lambda\Big[\langle z, w\rangle\Big]^\alpha$$
$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]_\alpha = \Big[\langle u, v\rangle\Big]_\alpha + \Big[\langle z, w\rangle\Big]_\alpha, \quad \Big[\lambda\langle z, w\rangle\Big]_\alpha = \lambda\Big[\langle z, w\rangle\Big]_\alpha$$

Definition 1. *Let $\langle u, v\rangle$ an element of IF_1 and $\alpha \in [0,1]$, we define the following sets :*

$$\Big[\langle u, v\rangle\Big]_l^+(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\}, \quad \Big[\langle u, v\rangle\Big]_r^+(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$
$$\Big[\langle u, v\rangle\Big]_l^-(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \leq \alpha\}, \quad \Big[\langle u, v\rangle\Big]_r^-(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \leq \alpha\}$$

Remark 2.

$$\Big[\langle u, v\rangle\Big]_\alpha = \Big[\Big[\langle u, v\rangle\Big]_l^+(\alpha), \Big[\langle u, v\rangle\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u, v\rangle\Big]^\alpha = \Big[\Big[\langle u, v\rangle\Big]_l^-(\alpha), \Big[\langle u, v\rangle\Big]_r^-(\alpha)\Big]$$

Proposition 1. *For all* α, $\beta \in [0,1]$ *and* $\langle u, v\rangle \in IF_1$

(i) $\Big[\langle u,v\rangle\Big]_\alpha \subset \Big[\langle u,v\rangle\Big]^\alpha$

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$

(iii) if $\alpha \leq \beta$ *then* $\Big[\langle u,v\rangle\Big]_\beta \subset \Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta \subset \Big[\langle u,v\rangle\Big]^\alpha$

(iv) If $\alpha_n \nearrow \alpha$ *then* $\Big[\langle u,v\rangle\Big]_\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha \in [0,1]$ we denote by

$$M_\alpha = \{x \in \mathbb{R} \ : \ u(x) \geq \alpha\} \quad \text{and} \quad M^\alpha = \{x \in \mathbb{R} \ : \ v(x) \leq 1-\alpha\}$$

Lemma 1. *let* $\Big\{M_\alpha, \ \alpha \in [0,1]\Big\}$ *and* $\Big\{M^\alpha, \ \alpha \in [0,1]\Big\}$ *two families of subsets of* $\mathbb{R}$ *satisfies (i)–(iv) in proposition 1, if* u *and* v *define by*

$$u(x) = \begin{cases} 0 & \text{if } x \notin M_0 \\ \sup\{\alpha \in [0,1] \ : \ x \in M_\alpha\} & \text{if } x \in M_0 \end{cases}$$
$$v(x) = \begin{cases} 1 & \text{if } x \notin M^0 \\ 1 - \sup\{\alpha \in [0,1] \ : \ x \in M^\alpha\} & \text{if } x \in M^0 \end{cases}$$

Then $\langle u, v\rangle \in IF_1$

Proof. From the construction of u and $1-v$, it suffices to show that $0 \leq u(x) + v(x) \leq 1$ $\forall x \in \mathbb{R}$.

- If $x \notin M^0$, then $u(x) + v(x) = 1$
- If $x \in M^0 \setminus M_0$, then $u(x) = 0$ and $v(x) = 1 - \sup\{\alpha \in [0,1] \ : \ x \in M^\alpha\}$, we have $0 \leq u(x) + v(x) \leq 1$.
- If $x \in M_0$, then exist $\alpha \in [0,1]$ such that $x \in M_\alpha \subset M^\alpha$. In this case $\{\alpha \in [0,1] \ : \ x \in M_\alpha\} \subset \{\alpha \in [0,1] \ : \ x \in M^\alpha\}$, we deduce $u(x) = \sup\{\alpha \in [0,1] \ : \ x \in M_\alpha\}$ and $v(x) = 1 - \sup\{\alpha \in [0,1] \ : \ x \in M^\alpha\}$, which implies $0 \leq u(x) + v(x) \leq 1$

This completes the proof. □

Lemma 2. *Let* I *a dense subset of* $[0,1]$, *if* $\Big[\langle u,v\rangle\Big]_\alpha = \Big[\langle u',v'\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha = \Big[\langle u',v'\rangle\Big]^\alpha$, *for all* $\alpha \in I$ *then* $\langle u,v\rangle = \langle u',v'\rangle$

Proof. Let $\alpha_0 \notin I$ and $(\alpha_n)_n$ a sequence which converges to α_0, from the proposition , we have

$$\Big[\langle u,v\rangle\Big]_{\alpha_0} = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n} = \bigcap_n \Big[\langle u',v'\rangle\Big]_{\alpha_n} = \Big[\langle u',v'\rangle\Big]_{\alpha_0}$$
$$\Big[\langle u,v\rangle\Big]^{\alpha_0} = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n} = \bigcap_n \Big[\langle u',v'\rangle\Big]^{\alpha_n} = \Big[\langle u',v'\rangle\Big]^{\alpha_0}$$

□

On the space IF_1 we will consider the following L_p-metric,

$$\begin{aligned} d_p\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = \Bigg(&\frac{1}{4}\int_0^1 \Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha) - \Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big|^p d\alpha \\ &+ \frac{1}{4}\int_0^1 \Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big|^p d\alpha \\ &+ \frac{1}{4}\int_0^1 \Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha) - \Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big|^p d\alpha \\ &+ \frac{1}{4}\int_0^1 \Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha) - \Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|^p d\alpha\Bigg)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} d_\infty\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = &\frac{1}{4}\sup_{0<\alpha\leq 1} \Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha) - \Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1} \Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1} \Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha) - \Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1} \Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big| \end{aligned}$$

Proposition 2. *$\Big(IF_1, d_p\Big)$ is a metric space.*

Proof. $d_p\Big(\langle u,v\rangle, \langle z,w\rangle\Big) < \infty$, indeed $supp(u,v)$ and $supp(z,w)$ are bounded.

The functions

$$\alpha \to d_H\bigg(\Big[\langle u,v\rangle\Big]_\alpha, \Big[\langle u',v'\rangle\Big]_\alpha\bigg) \text{ and } \alpha \to d_H\bigg(\Big[\langle u,v\rangle\Big]^\alpha, \Big[\langle u',v'\rangle\Big]^\alpha\bigg) \qquad (1)$$

are measurable where d_H is the Hausdorff metric : if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, with $\alpha_n \to \alpha$, we have

$$\Big[\langle u,v\rangle\Big]_\alpha = \bigcap_{n=1}^{\infty} \Big[\langle u,v\rangle\Big]_{\alpha_n}, \text{ and } \Big[\langle u',v'\rangle\Big]_\alpha = \bigcap_{n=1}^{\infty} \Big[\langle u',v'\rangle\Big]_{\alpha_n}$$
$$\Big[\langle u,v\rangle\Big]^\alpha = \bigcap_{n=1}^{\infty} \Big[\langle u,v\rangle\Big]^{\alpha_n}, \text{ and } \Big[\langle u',v'\rangle\Big]^\alpha = \bigcap_{n=1}^{\infty} \Big[\langle u',v'\rangle\Big]^{\alpha_n}$$

which means that

$$d_H\left(\left[\langle u,v\rangle\right]_{\alpha_n},\left[\langle u,v\rangle\right]_{\alpha}\right)\to 0,\ \text{and}\ d_H\left(\left[\langle u',v'\rangle\right]_{\alpha_n},\left[\langle u',v'\rangle\right]_{\alpha}\right)\to 0$$
$$d_H\left(\left[\langle u,v\rangle\right]^{\alpha_n},\left[\langle u,v\rangle\right]^{\alpha}\right)\to 0,\ \text{and}\ d_H\left(\left[\langle u',v'\rangle\right]^{\alpha_n},\left[\langle u',v'\rangle\right]^{\alpha}\right)\to 0.$$

This implies that (1) are left continuous and therefore measurable. It is easy to verify the triangle inequality and symmetry of d_p.

It remains to show that $d_p\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)=0$ implies $\langle u,v\rangle=\langle u',v'\rangle$.

So, if $d_p\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)=0$, implies

$$\left[\langle u,v\rangle\right]_{\alpha}=\left[\langle u',v'\rangle\right]_{\alpha}\ \text{and}\ \left[\langle u,v\rangle\right]^{\alpha}=\left[\langle u',v'\rangle\right]^{\alpha}$$

almost everywhere, therefore, by Lemma 2 $\langle u,v\rangle=\langle u',v'\rangle$.

It is the same for d_∞. □

3 Topology induced by an intuitionistic fuzzy metric

Let $\Big(\mathrm{IF}_1,d_p\Big)$ be an intuitionistic fuzzy metric space and $r>0$. For an element $\langle u,v\rangle$ of IF_1 and $r>0$

Definition 2.

(i) The set $B\Big(\langle u,v\rangle,r\Big)=\Big\{\langle u',v'\rangle\in IF_1, d_p\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)<r\Big\}$ *is called fuzzy open ball with center* $\langle u,v\rangle$ *and radius* r.

(ii) The set $B\Big[\langle u,v\rangle,r\Big]=\Big\{\langle u',v'\rangle\in IF_1, d_p\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)\le r\Big\}$ *is called fuzzy open ball with center* $\langle u,v\rangle$ *and radius* r.

Definition 3. *A subset G of IF_1 is called an open set in IF_1 if for all $\langle u,v\rangle\in G$, there exist a number $r>0$ such that $B\Big(\langle u,v\rangle,r\Big)\subset G$.*
Thus G is an open set, iff each point of G is the center of some fuzzy open ball contained in G.

Definition 4. *A subset N of IF_1 is called a neighborhood of $\langle u,v\rangle$, if there exists an open set G such that $\forall\langle u,v\rangle\subset G\subset N$. Also N is called a neighborhood of subset A of IF_1 if there exists an open set G such that $A\subset G\subset N$. The collection of all neighborhoods of $\forall\langle u,v\rangle$ is termed neighborhood system of $\langle u,v\rangle$.*

Definition 5. *Let (IF_1,d_p) be a fuzzy metric space. A subset F of IF_1 is called a closed set in IF_1 if and only if its compliments an open set.*

Remark 3. *The collection of all open subsets of any intuitionistic fuzzy metric space (IF_1,d_p) satisfies the conditions of topology.*

Definition 6. *Let A subset IF_1 is called bounded if there exist $\langle u, v\rangle \in IF_1$ and $r > 0$ such that $A \subset B\Big(\langle u, v\rangle, r\Big)$.*

Definition 7. *A sequence $\Big(\langle u_n, v_n\rangle \Big)_n$ in IF_1 converges to $\langle u, v\rangle$, if and only if*

$$\lim_{n\to\infty} \Big(\langle u_n, v_n\rangle \Big) = \langle u, v\rangle$$

in the sense of the topology induced by the metric d_p.

Definition 8. *A sequence $(\langle u_n, v_n\rangle)_n$ is called a Cauchy sequence if*

$$\lim_{n,m\to\infty} d_p\Big(\langle u_n, v_n\rangle, \langle u_m, v_m\rangle \Big) = 0 \qquad \forall n, m \in \mathbb{N}$$

Definition 9. *An intuitionistic fuzzy space (IF_1, d) is called complete if and only if every Cauchy sequence in (IF_1, d) converges.*

4 The completeness of intuitionistic fuzzy metric spaces

Theorem 1. $\Big(IF_1, d_p\Big)$ *is a complete space, for $p \in [0, +\infty]$.*

Proof. Let $\Big(\langle u_n, v_n\rangle \Big)_n$ a Cauchy sequence, for $\varepsilon > 0$ there exist an integer n_0 such that for n, $m \geq n_0$ we have

(i) Case $p = \infty$.

$$\sup_{\alpha\in(0,1]} \left| \Big[\langle u_n, v_n\rangle \Big]_l^+(\alpha) - \Big[\langle u_m, v_m\rangle \Big]_l^+(\alpha) \right| \leq \varepsilon$$
$$\sup_{\alpha\in(0,1]} \left| \Big[\langle u_n, v_n\rangle \Big]_r^+(\alpha) - \Big[\langle u_m, v_m\rangle \Big]_r^+(\alpha) \right| \leq \varepsilon$$

According to the Cauchy criterion, we obtain the uniform convergence

$$\Big[\langle u_n, v_n\rangle \Big]_l^+(\alpha) \to \phi_l(\alpha) \text{ and } \Big[\langle u_n, v_n\rangle \Big]_r^+(\alpha) \to \phi_r(\alpha)$$

By Lemma 1, the family and appendix 7 in [7] $\Big([\phi_l(\alpha), \phi_r(\alpha)]\Big)_{\alpha\in(0,1]}$ defined a fuzzy number u. In the same manner,

$$\Big[\langle u_n, v_n\rangle \Big]_l^-(\alpha) \to \psi_l(\alpha) \quad \text{and} \quad \Big[\langle u_n, v_n\rangle \Big]_r^-(\alpha) \to \psi_r(\alpha)$$
$$\Big[\psi_l(\alpha), \psi_r(\alpha)\Big] = \Big[\langle u, v\rangle \Big]^\alpha$$

Thus $d_\infty\Big(\langle u_n, v_n\rangle, \langle u, v\rangle \Big) \to 0$, as $n \to \infty$.

By Lemma 1 and appendix 7 in [7], $\Big([\psi_l(\alpha), \psi_r(\alpha)]\Big)^{\alpha\in(0,1]}$ defined a fuzzy number $1 - v$. Further the families checking $(i) - (iii)$ of proposition (3.1). But $\Big[\langle u_n, v_n\rangle \Big]_\alpha \subset \Big[\langle u_n, v_n\rangle \Big]^\alpha$,

$\forall\alpha \in [0,1]$, by passing to the limit we have (vi) of the proposition (3.1), we deduce $\langle u, v\rangle \in \mathrm{IF}_1$.

(ii) Case $p \in [1, \infty)$.

$$\int_0^1 \left| \Big[\langle u_n, v_n\rangle \Big]_l^+(\alpha) - \Big[\langle u_m, v_m\rangle \Big]_l^+(\alpha) \right|^p d\alpha \le \varepsilon$$
$$\int_0^1 \left| \Big[\langle u_n, v_n\rangle \Big]_r^+(\alpha) - \Big[\langle u_m, v_m\rangle \Big]_r^+(\alpha) \right|^p d\alpha \le \varepsilon$$

By theorem of Freschet-Riesz, see [4],

$$\Big[\langle u_n, v_n\rangle \Big]_r^+(\alpha) \to \phi_r(\alpha) \text{ and } \Big[\langle u_n, v_n\rangle \Big]_l^+(\alpha) \to \phi_l(\alpha) \quad \text{in } L^p\Big((0,1]\Big)$$

so there exists a sub-sequence $\Big(\langle u_{n_k}, v_{n_k}\rangle \Big)_k$ such that

$$\Big[\langle u_{n_k}, v_{n_k}\rangle \Big]_r^+(\alpha) \to \phi_r(\alpha), \quad \Big[\langle u_{n_k}, v_{n_k}\rangle \Big]_l^+(\alpha) \to \phi_l(\alpha) \text{ a.e.}$$

and

$$\Big[\langle u_{n_k}, v_{n_k}\rangle \Big]_r^-(\alpha) \to \psi_r(\alpha), \quad \Big[\langle u_{n_k}, v_{n_k}\rangle \Big]_l^-(\alpha) \to \psi_r(\alpha) \text{ a.e.}$$

almost everywhere.

Again by Lemma 1, there exist $\langle u, v\rangle \in \mathrm{IF}_1$ such that

$$\Big[\phi_l(\alpha), \phi_r(\alpha)\Big] = \Big[\langle u, v\rangle \Big]_\alpha \text{ and } \Big[\psi_l(\alpha), \psi_r(\alpha)\Big] = \Big[\langle u, v\rangle \Big]^\alpha$$

This completes the proof. □

Corollary 1. *let $(X_n)_n$ be a sequence of closed subsets in IF_1. Assume that*

$$Int X_n = \emptyset \ \ for \ every \ n \ge 1$$

Then

$$Int \left(\bigcup_{n=1}^{\infty} X_n \right) = \emptyset$$

Proof. (IF_1, d_p) is a complete metric space, it suffices to applying Baire theorem. □

Theorem 2. *$\Big(IF_1, d_p\Big)$ is separable for $p \in [1, \infty)$.*

Proof. The proof follows in several steps. Assume that $\langle u, v\rangle \in \mathrm{IF}_1$.

(a) Construction of $\langle \phi_1, \phi_2\rangle \in \mathrm{IF}_1$.

Since $\mathrm{supp}\Big\{ \langle u, v\rangle \Big\}$ is compact, there exist $S_i = \Big[a_i, b_i\Big]$, $i = 1, \dots, r$ such that $a_i, b_i \in \mathbb{Q}$ with $0 < b_i - a_i = O(\varepsilon)$ and $\mathrm{supp}\Big\{ \langle u, v\rangle \Big\} \subset \bigcup_{i=1}^{r} S_i$.

Consider the corner point $T_i = a_i$ of S_i and define the fuzzy intuitionistic set $\Big(\phi_1, \phi_2\Big) \in \mathrm{IF}_1$

$$\phi_1(x) = \begin{cases} \sup_{x\in\overline{S_i}} u(x) & \text{if } x = T_i, \quad i = 1, \dots, r \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} \inf_{x\in\overline{S_i}} v(x) & \text{if } x = T_i, \quad i = 1, \dots, r \\ 0 & otherwise \end{cases}$$

Obviously we have $\langle\phi_1, \phi_2\rangle \in \mathrm{IF}_1$. Putting $\alpha_i = \phi_1(T_i)$ and $\beta_i = 1 - \phi_2(T_i)$, we reliable $S_1, \dots, S_r$ and $T_1, \dots, T_r$ (if necessary) such that

$$0 = \alpha_0 \le \alpha_1 \le \dots \le \alpha_r = 1 \quad \text{and} \quad 0 = \beta_0 \le \beta_1 \le \dots \le \beta_r = 1$$

We claim that $d_\infty\Big(\langle u, v\rangle, \langle\phi_1, \phi_2\rangle\Big) = O(\varepsilon)$.

Choose $0 < \alpha \le 1$.

Since $\alpha_{i_0-1} \le \alpha \le \alpha_{i_0}$ for some $0 \le i_0 \le 1$ we get $\Big[\langle u, v\rangle\Big]_{\alpha_{i_0}} \subset \Big[\langle u, v\rangle\Big]_\alpha$, $\Big[\langle u, v\rangle\Big]^{\alpha_{i_0}} \subset \Big[\langle u, v\rangle\Big]^\alpha$, $\Big[\langle\phi_1, \phi_2\rangle\Big]_{\alpha_{i_0}} = \Big\{T_{i_0}, \dots, T_r\Big\}$ and $\Big[\langle\phi_1, \phi_2\rangle\Big]^{\alpha_{i_0}} = \Big\{T_{i_0}, \dots, T_r\Big\}$.

As $x \in \Big[\langle u, v\rangle\Big]^\alpha \cap \Big[\langle u, v\rangle\Big]_\alpha$ implies $x \in S_{i_1}$ for some $i_1 \ge i_0$ it is follows that

$$\min_{1\le i\le r} \Big|x - T_i\Big| \le \Big|x - T_{i_1}\Big| = O(\varepsilon)$$

On the other hand, for any $i \ge i_0$,

$$\phi_1(T_i) = \sup_{x\in S_i} u(x) \ge \alpha \text{ and } \phi_2(T_i) = \inf_{x\in S_i} v(x) \le 1 - \alpha$$

since u is upper semi-continuous, it attains its supremum at some point $x_0 \in \overline{S_i} \cap \Big[\langle u, v\rangle\Big]_\alpha$. Also v is lower semi-continuous, it attains its lower bound at some point $y_0 \in \overline{S_i} \cap \Big[\langle u, v\rangle\Big]^\alpha$.

We have

$$\inf_{x\in\big[\langle u,v\rangle\big]_\alpha} \Big|T_i - x\Big| \le \Big|T_i - x_0\Big| = O(\varepsilon)$$

and

$$\inf_{x\in\big[\langle u,v\rangle\big]^\alpha} \Big|T_i - x\Big| \le \Big|\Big(1 - T_i\Big) - y_0\Big| = O(\varepsilon)$$

Therefore

$$\begin{aligned} d_\infty\Big(\langle u, v\rangle, \langle\phi_1, \phi_2\rangle\Big) &\le \sup_\alpha d_H\bigg(\Big[\langle u, v\rangle\Big]_\alpha, \Big[\langle\phi_1, \phi_2\rangle\Big]_\alpha\bigg) \\ &\quad + \sup_\alpha d_H\bigg(\Big[\langle u, v\rangle\Big]^\alpha, \Big[\langle\phi_1, \phi_2\rangle\Big]^\alpha\bigg) = O(\varepsilon) \end{aligned}$$

(**b**) Construction of $\langle \psi_1, \psi_2 \rangle \in \mathrm{IF}_1$.

If necessary we reliable $0 \leq \alpha_1 < \cdots < \alpha_s = 1$, with $s \leq r$.

If $\alpha_k \notin \mathbb{Q}$, we choose $\alpha'_k \in \mathbb{Q}$ such that $\max\Big(\alpha_{k-1}, \alpha_k - \epsilon/_M\Big) < \alpha'_k < \alpha_k$, with $M > 2(r-1)\mathrm{diam}\Big(\text{supp}\,\langle u, v\rangle\Big)$, while if $\alpha_k \in \mathbb{Q}$, we set $\alpha'_k = \alpha_k$. Defining $\langle \psi_1, \psi_2 \rangle \in \mathrm{IF}$ by

$$\psi_1 = \begin{cases} \alpha'_k & \text{if } \phi_1(x) = \alpha_k \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_2 = \begin{cases} \beta'_k & \text{if } \phi_2(x) = 1 - \alpha_k \\ 0 & \text{otherwise} \end{cases}$$

yields

$$d_p\Big(\langle \phi_1, \phi_2 \rangle, \langle \psi_1, \psi_2 \rangle \Big) \leq 2\mathrm{diam}\Big\{\text{supp}\,\langle u, v\rangle\Big\}\Big[\Big(\sum_{i=1}^{s}(\alpha_i - \alpha'_i)\Big)^{\frac{1}{p}}\Big].$$

It follows that

$$d_p\Big(\langle \phi_1, \phi_2 \rangle, \langle \psi_1, \psi_2 \rangle \Big) = O\Big(\varepsilon^{\frac{1}{p}}\Big)$$

By the triangle inequality, we have

$$d_p\Big(\langle u, v \rangle, \langle \psi_1, \psi_2 \rangle \Big) \leq d_\infty\Big(\langle u, v \rangle, \langle \phi_1, \phi_2 \rangle \Big) + d_p\Big(\langle \phi_1, \phi_2 \rangle, \langle \psi_1, \psi_2 \rangle \Big) = O(\varepsilon)$$

□

Theorem 3. $\Big(IF_1, d_\infty\Big)$ *is not separable.*

Proof. For $\alpha \in [0,1]$, consider the function

$$\phi_{\alpha,1} = \begin{cases} 1 & \text{if} \quad x = 1 \\ \alpha & \text{if} \quad x \in]0,1[\\ 0 & \text{otherwise} \end{cases},$$

$$\phi_{\alpha,2} = \begin{cases} 0 & \text{if} \quad x = 1 \\ 1-\alpha & \text{if} \quad x \in]0,1[\\ 1 & \text{otherwise} \end{cases}.$$

It is easy to cheek that for $\alpha \neq \beta$, $d_\infty\Big(\langle \phi_{\alpha,1}, \phi_{\alpha,2} \rangle, \langle \phi_{\beta,1}, \phi_{\beta,2} \rangle \Big) = 1$. □

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