Notes on Intuitionistic Fuzzy Sets
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On the embedding of continuous states

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Abstract: There is known the assertion that any state on a family $\mathcal{F}$ of IF-events can be embedded to an MV-algebra generated by $\mathcal{F}$. In the contribution the assertion is extended for non-additive only upper and lower continuous mappings.
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1 Preliminaries

As usually the *IF*-set is a pair of mappings $A = (\mu_A, \nu_A) : \Omega \to [0,1]^2$ such that $\mu_A + \nu_A \leq 1$.

IF A, B are *IF*-sets, $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$, then we write

$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$.

If $A_n = (\mu_{A_n}, \nu_{A_n})$ are *IF*-sets, and $A = (\mu_A, \nu_A)$ is an *IF*-set, then we write

$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$,

$A_n \searrow A \Longleftrightarrow \mu_{A_n} \searrow \mu_A, \nu_{A_n} \nearrow \nu_A$.

Definition 1. A family $\mathcal{F}$ of *IF*-sets on a space Ω will be called continuous if the following two conditions hold:

(i) $A_n \in \mathcal{F}(n = 1, 2, ...), A_n \nearrow A \Longrightarrow A \in \mathcal{F}$,

(ii) $A_n \in \mathcal{F}(n = 1, 2, ...), A_n \searrow A \Longrightarrow A \in \mathcal{F}$.

Definition 2. Let $\mathcal{F}$ be a continuous family of *IF*-sets, and $(0_\Omega, 1_\Omega) \in \mathcal{F}, (1_\Omega, 0_\Omega) \in \mathcal{F}$. A maping $m : \mathcal{F} \to [0,1]$ will be called continuous state on $\mathcal{F}$, if the following conditions holds:

(i) $m((0_\Omega, 1_\Omega)) = 0, m((1_\Omega, 0_\Omega)) = 1$,

(ii) $A_n \in \mathcal{F}, A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$,

(iii) $A_n \in \mathcal{F}, A_n \searrow A \Longrightarrow m(A_n) \searrow m(A)$.

A mapping $m : \mathcal{F} \to [0,1]$ is weakly additive, if

$m((\mu_A, \nu_A)) + m((0_\Omega, 1_\Omega - \nu_A) = m((\mu_A, 0_\Omega))$

for any $A \in \mathcal{F}$.

Example. For $A, B \in \mathcal{F}, A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ define $A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1), A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)$. Weakly additive mapping need not be additive, hence it need not to satisfy the inclusion

$$A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B) = m(A) + m(B).$$

E.g. let $\Omega = [0,1], m((1_\Omega, 0_\Omega)) = 1, m(A) = 0$ for $A \neq (1_\Omega, 0_\Omega)$. Let $A = B = ((\frac{1}{2})_\Omega, (\frac{1}{2})_\Omega)$. Then $A \odot B = (0_\Omega, 1_\Omega)$ but

$$m(A \oplus B) = m((1_\Omega, 0_\Omega)) = 1 \neq 0 + 0 = m(A) + m(B).$$

2 Embedding

Definition 3. Let $\mathcal{F}$ be a continuous family, $\mathcal{M}$ be the family of all pairs $(\mu_A, \nu_A) : \Omega \to [0,1]^2$ such that $(\mu_A, 0_\Omega) \in \mathcal{F}$, and $(0_\Omega, 1_\Omega - \nu_A) \in \mathcal{F}$. We shall call $\mathcal{M}$ the MV-algebra generated by $\mathcal{F}$.

Theorem. Let $\mathcal{F}$ be a continuous family, $\mathcal{M}$ be the MV-algebra generated by $\mathcal{F}$. Let $m : \mathcal{F} \to [0,1]$ be a continuous state on $\mathcal{F}$. Then there exists a maping $\bar{m} : \mathcal{M} \to [0,1]$ with the following properties:

(i) $\bar{m}((0_\Omega, 1_\Omega)) = 0, \bar{m}((1_\Omega, 0_\Omega)) = 1$,

(ii) $A_n \nearrow A \Longrightarrow \bar{m}(A_n) \nearrow \bar{m}(A)$,

(iii) $A_n \searrow A \Longrightarrow \bar{m}(A_n) \searrow \bar{m}(A)$.

If m is weakly additive, then $\bar{m}$ is an extension of m.

Proof. Define for any $A = (\mu_A, \nu_A)$

$$\bar{m}(A) = m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)).$$

Since $\mu_A \geq 0_\Omega$, and $0_\Omega \leq 1 - \nu_A$ we have $m((\mu_A, 0_\Omega)) \geq m((0_\Omega, 1 - \nu_A))$, hence

$$\bar{m}((\mu_A, \nu_A)) = m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)) \geq 0.$$

On the other hand,

$$m((\mu_A, 0_\Omega)) \leq 1 \leq 1 + m((0_\Omega, 1_\Omega - \nu_A))$$

hence

$$\bar{m}((\mu_A, \nu_A)) = m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)) \leq 1.$$

So actually we have obtained a mapping $\bar{m} : \mathcal{M} \to [0,1]$.

Evidently

$$\bar{m}((0_\Omega, 1_\Omega)) = m((0_\Omega, 0_\Omega)) - m((0_\Omega, 0_\Omega)) = 0,$$

$$\bar{m}((1_\Omega, 0_\Omega)) = m((1_\Omega, 0_\Omega)) - m((0_\Omega, 1_\Omega - 0_\Omega)) = 1 - 0 = 1.$$

Let $A_n \in \mathcal{M}$, $A_n = (\mu_{A_n}, \nu_{A_n}) \nearrow (\mu_A, \nu_A) = A$. Then

$$(\mu_{A_n}, 0_\Omega) \nearrow (\mu_A, 0_\Omega), (0_\Omega, 1_\Omega - \nu_{A_n}) \searrow (0_\Omega, 1_\Omega - \nu_A).$$

Therefore,

$$\bar{m}(A_n) = m((\mu_{A_n}, 0_\Omega)) - m(0_\Omega, 1_\Omega - \nu_{A_n}) \nearrow m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)) = \bar{m}(A).$$

Analogously it can be proved the implication

$$A_n \searrow A \Longrightarrow \bar{m}(A_n) \searrow \bar{m}(A).$$

Let m be weakly additive, and let $A = (\mu_A, \nu_A) \in \mathcal{F}$. Since $(\mu_A, \nu_A) \odot (0_\Omega, 1_\Omega - \nu_A) = (0_\Omega, 1_\Omega)$, we have

$$m((\mu_A, 0_\Omega)) = m((\mu_A, \nu_A) \oplus (0_\Omega, 1_\Omega - \nu_A)) = m((\mu_A, \nu_A)) + m((0_\Omega, 1_\Omega - \nu_A)),$$

hence

$$m((\mu_A, \nu_A)) = m((\mu_A, 0_\Omega)) - m((0_\Omega, 1_\Omega - \nu_A)) = \bar{m}((\mu_A, \nu_A)).$$

Actually $\bar{m}$ is an extension of m. □

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