On intuitionistic fuzzy implications

I. Bakhadach, S. Melliani* and L. S. Chadli

LMACS, Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal Morocco
e-mail: said.melliani@gmail.com
* Corresponding author

Received: 26 September 2017 Accepted: 15 November 2017

Abstract: In this paper we conduct a systematic algebraic study on the set \( I \) of all intuitionistic fuzzy implications. To this end, we propose a binary operation, denoted by \( * \), which makes a \((I, *)\) a monoid. We determine the largest subgroup \( \mathcal{R} \) of this monoid and using its representation define a group action of \( \mathcal{R} \) that partitions \( I \) into equivalence classes. Also we give novel way of generating newer fuzzy implications from given ones by a bijective transformations.

Keywords: Intuitionistic fuzzy implication, Group action, Bijective transformation.

AMS Classification: 03E72.

1 Introduction

Study in intuitionistic fuzzy subsets and application of intuitionistic fuzzy control have been developed quickly since the definition of intuitionistic fuzzy sets was introduced by Atanassov in 1983. IFSS theory basically defies the claim that from the fact that an element \( x \) ”belongs” to a given degree (say \( \mu \)) to a fuzzy set \( A \), naturally follows that \( x \) should ”not belong” to \( A \) to the extent \( 1 - \mu \), an assertion implicit in the concept of a fuzzy set. On the contrary, IFSS assign to each element of the universe both a degree of membership \( \mu \) and one of non-membership \( \nu \) such that \( \mu + \nu \leq 1 \), thus relaxing the enforced duality \( \nu = 1 - \mu \) from fuzzy set theory. Obviously, when \( \mu + \nu = 1 \) for all elements of the universe, the traditional fuzzy set concept is recovered.

Technology of intuitionistic fuzzy control has been applied to many fields including medical field [7, 8, 9]. But the basic theory of intuitionistic fuzzy control is inferior to its application, especially the theory of intuitionistic fuzzy reasoning. Since Zadeh [10] introduced the compositional rule of inference (CRI), many researchers have take advantage of fuzzy implication operators
to represent the relation between two variables linked together by means of an \( \text{if} - \text{then} \) rule. In intuitionistic fuzzy reasoning theory, intuitionistic fuzzy implication operators play the same important role.

This paper is organized as follows. In Section 2 we propose a binary operation \( * \) on the set of all intuitionistic fuzzy implication \( \mathcal{I} \) that makes \( (\mathcal{I}, *) \) a monoid. This is the first work in which such a rich structure has been obtained on the entire set of intuitionistic fuzzy implications \( \mathcal{I} \). In Section 3 we characterize the largest such subgroup \( \mathcal{K} \) and, based on their representation, propose a group action of \( \mathcal{K} \) on \( \mathcal{I} \). Clearly, this group action partitions \( \mathcal{I} \) into equivalence classes. And in Section 4 we propose a new method for the construction of new intuitionistic fuzzy implications. Finally we draw conclusions and indicate future lines of research.

## 2 Preliminaries

First we give the concept of intuitionistic fuzzy set defined by Atanassov and we recall some elementary definitions that we use in the sequel. Assume that \( X \) is the universe.

**Definition 1** ([1, 2]). The intuitionistic fuzzy subsets (in shorts IFSS) defined on a non-empty set \( X \) as objects having the form

\[
A = \{ (x, \mu(x), \nu(x)) : x \in X \}
\]

where the functions \( \mu : X \to [0, 1] \) and \( \nu : X \to [0, 1] \) denote the degree of membership and the degree of non-membership of each element \( x \in X \) to the set \( A \) respectively, and \( 0 \leq \mu(x) + \nu(x) \leq 1 \) for all \( x \in X \).

For the sake of simplicity, we shall use the symbol \( \langle \mu, \nu \rangle \) for the intuitionistic fuzzy subset \( A = \{ (x, \mu(x), \nu(x)) : x \in X \} \).

**Definition 2** ([2]). Let \( A = \langle \mu_A, \nu_A \rangle \) and \( B = \langle \mu_B, \nu_B \rangle \) IFSS of \( X \). Then

\[
\begin{align*}
A & \subset B \iff \mu_A \leq \mu_B \text{ and } \nu_A \geq \nu_B \\
A & = B \iff A \subset B \text{ and } B \subset A \\
A^c & = \langle \nu_A, \mu_A \rangle \\
A \cap B & = \langle \mu_A \land \mu_B, \nu_A \lor \nu_B \rangle \\
A \cup B & = \langle \mu_A \lor \mu_B, \nu_A \land \nu_B \rangle \\
\Box A & = \langle \mu_A, 1 - \mu_A \rangle, \Diamond A = \langle 1 - \nu_A, \nu_A \rangle
\end{align*}
\]

We recall from [5] that \( L^* = \{ \tilde{x} = (x_1, x_2)/x_1 + x_2 \leq 1 \} \) is a complete lattice with the order defined by

\[
\tilde{x} \geq \tilde{y} \quad \text{if and only if} \quad x_1 \geq y_1 \quad \text{and} \quad x_2 \leq y_2
\]

Now we recall the definition of intuitionistic fuzzy implication operator given by Atanassov and Gargov [3].
Definition 3. An intuitionistic fuzzy implication operator (IFIO) is any $I : L^* \rightarrow L^*$ mapping satisfying the border conditions:

$I((0,1), (0,1)) = (1,0)$; $I((0,1), (1,0)) = (1,0)$

$I((1,0), (1,0)) = (1,0)$; $I((1,0), (0,1)) = (0,1)$

and the two following conditions:

1) If $\tilde{x} \leq \tilde{y}$, then $\forall \tilde{z} \in L^* I(\tilde{x}, \tilde{z}) \geq I(\tilde{y}, \tilde{z})$

2) If $\tilde{y} \leq \tilde{z}$, then $\forall \tilde{x} \in L^* I(\tilde{x}, \tilde{y}) \leq I(\tilde{x}, \tilde{z})$

Definition 4 ([6]). If $(X, *)$ is a mathematical system such that $\forall a, b, c \in X, (a*b)*c = a*(b*c)$, then $*$ is called associative and $(X, *)$ is called a semigroup.

3 Monoid structure on the set of all intuitionistic fuzzy implications

Let $\mathbb{I}$ be the set of all intuitionistic fuzzy implications. In this section, we begin by proposing a binary operation $*$ on the set $\mathbb{I}$ of all intuitionistic fuzzy implications and show that $(\mathbb{I}, *)$ forms a monoid and discuss the properties preserved under this operation.

Definition 5. For any two intuitionistic fuzzy implications $I, J$ we define $I * J : L^* \rightarrow L^*$ as $(I * J)(\tilde{x}, \tilde{y}) = I(\tilde{x}, J(\tilde{x}, \tilde{y})), \tilde{x}, \tilde{y} \in L^*$.

The following result shows that $I * J$ is, indeed, an intuitionistic fuzzy implication.

Theorem 1. $I * J$ is an intuitionistic fuzzy implication, i.e., $I * J \in \mathbb{I}$.
Lemma 1. Let $x_1, x_2, y \in L^+$ be such that $x_1 \geq x_2$. Then $J(x_1, y) \leq J(x_2, y)$.

Then $(I * J)(x_1, y) = I(x_1, J(x_1, y)) \leq I(x_2, J(x_2, y)) = (I * J)(x_2, y)$.

Then $I * J$ is decreasing for the first variable. Similarly one can show that $I * J$ is increasing in the second variable.

Remark 1. For associativity of $*$, let $I, J, K \in \mathbb{I}$ and $\tilde{x}, \tilde{y} \in L^*$. Then

$$(I * (J * K))(\tilde{x}, \tilde{y}) = I(\tilde{x}, (J * K)(\tilde{x}, \tilde{y}))$$

$$= I(\tilde{x}, J(K(\tilde{x}, \tilde{y})))$$

$$= (I * J)(\tilde{x}, K(\tilde{x}, \tilde{y}))$$

$$= ((I * J) * K)(\tilde{x}, \tilde{y})$$

Further,

$$(I * I_D)(\tilde{x}, \tilde{y}) = I(\tilde{x}, I_D(\tilde{x}, \tilde{y}))$$

$$= \begin{cases} 
\tilde{1} & \text{if } \tilde{x} = \tilde{0} \\
I(\tilde{x}, \tilde{y}) & \text{if } \tilde{x} \neq \tilde{0}
\end{cases}$$

Similarly $I_D * I = I$ then $I_D$ becomes the identity element in $\mathbb{I}$. 

Remark 1. $(\mathbb{I}, *)$ is not a group. Indeed, take $a$

$$I_1(\tilde{x}, \tilde{y}) = \begin{cases} 
\tilde{y} & \text{if } \tilde{x} = \tilde{1} \\
\tilde{1} & \text{otherwise}
\end{cases}$$

and we have $I * I_1 = I_1$ for all $I \in \mathbb{I}$. Thus there does not exist any $J \in \mathbb{I}$ such that $J * I_1 = I_D$.

Lemma 1. Let $I \in \mathbb{I}$; then $I$ is invertible w.r.t $*$ if and only if there exists a unique $J \in \mathbb{I}$ such that for any $\tilde{x}, \tilde{y} \in L^+$ with $\tilde{x} \neq \tilde{0}$, $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$

Proof. Let $I$ be an invertible element w.r.t $*$, i.e., there exists a unique $J \in \mathbb{I}$ such that $I * J = I_D = J * I$. In other words,

$I(\tilde{x}, J(\tilde{x}, \tilde{y})) = I_D(\tilde{x}, \tilde{y}) = J(\tilde{x}, I(\tilde{x}, \tilde{y})), \tilde{x}, \tilde{y} \in L^*$.

But for $\tilde{x} \neq \tilde{0}$ we have $I_D(\tilde{x}, \tilde{y}) = \tilde{y}$ thus for $\tilde{x} \neq \tilde{0}$, $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$.
Conversely, assume that there exists a unique \( J \in \mathbb{I} \) such that for \( \hat{x} \neq \hat{0} \) \( I(\hat{x}, J(\hat{x}, \hat{y})) = I_D(\hat{x}, \hat{y}) = J(\hat{x}, I(\hat{x}, \hat{y})). \)

Since \( I, J \in \mathbb{I} \) and \( I \ast J, J \ast I \in \mathbb{I} \) we have \( I(\hat{x}, J(\hat{x}, \hat{y})) = I_D(\hat{x}, \hat{y}) = J(\hat{x}, I(\hat{x}, \hat{y})). \) Then \( I \) is invertible w.r.t. \( \ast. \)

Theorem 3 ([4]). A function \( \varphi : L^* \rightarrow L^* \) is a continuous increasing bijection if, and only if, there exists a continuous increasing bijection \( \lambda : [0, 1] \rightarrow [0, 1] \) such that \( \varphi(x) = (\lambda(x_1), 1 − \lambda(1 − x_2)). \)

Theorem 4. The solutions of \( I(\hat{x}, J(\hat{x}, \hat{y})) = \hat{y} = J(\hat{x}, I(\hat{x}, \hat{y})) \) are of the forms \( I(\hat{x},) = \varphi(\hat{y}) \) and \( J(\hat{x}, \hat{y}) = \varphi^{-1}(\hat{y}) \) for some continuous increasing bijection \( \varphi \)

Proof. Let \( I \) and \( J \in \mathbb{I} \) such that \( I(\hat{x}, J(\hat{x}, \hat{y})) = \hat{y} = J(\hat{x}, I(\hat{x}, \hat{y})) \) for all \( \hat{x} \neq \hat{0} \) and \( \hat{y} \in L^* \). Let \( \hat{x} \neq \hat{0} \) be fixed arbitrary and define two functions \( \varphi_{\hat{x}0}, \psi_{\hat{x}0}, \varphi_{\hat{x}0}(\hat{y}) = I(\hat{x}_0, \hat{y}) \) and \( \psi_{\hat{x}0}(\hat{y}) = J(\hat{x}_0, \hat{y}). \) Clearly, both \( \varphi_{\hat{x}0}, \psi_{\hat{x}0} \) are increasing function on \( L^* \). Then \( I(x_0, J(x_0, \hat{y})) = \varphi_{\hat{x}0}(\varphi_{\hat{x}0}(\hat{y})) = (\varphi_{\hat{x}0} \circ \psi_{\hat{x}0})(\hat{y}) = \hat{y} \) for all \( \hat{y} \in L^* \). Similarly, \( J(x_0, I(x_0, \hat{y})) = (\psi_{\hat{x}0} \circ \varphi_{\hat{x}0})(\hat{y}) = \hat{y} \) for every \( \hat{y} \in L^* \). Thus \( \psi_{\hat{x}0} = \varphi_{\hat{x}0}^{-1} \) and \( \psi_{\hat{x}0} \) is a bijection. Hence \( \psi_{\hat{x}0} \) increasing bijection on \( L^* \) for every \( \hat{x}_0 \neq \hat{0} \). Since \( \hat{x}_0 \) is chosen arbitrarily, \( \psi_{\hat{x}} = \varphi_{\hat{x}}^{-1} \) for all \( \hat{x} \neq \hat{0} \) Then for \( \hat{x} \neq \hat{0} \) \( I(\hat{x}, \hat{y}) = \psi_{\hat{x}}(\hat{y}) \) and \( J(\hat{x}, \hat{y}) = \psi_{\hat{x}}^{-1}(\hat{y}). \)

Let \( \hat{x}_1, \hat{x}_2 \) not null such that \( \hat{x}_1 \leq \hat{x}_2 \). Then \( I(\hat{x}_1, \hat{y}) \leq I(\hat{x}_2, \hat{y}) \) implies that \( \psi_{\hat{x}_1}(\hat{y}) \leq \psi_{\hat{x}_2}(\hat{y}) \) and \( \psi_{\hat{x}_1}^{-1}(\hat{y}) \leq \psi_{\hat{x}_2}^{-1}(\hat{y}) \). And we have

\[
\psi_{\hat{x}_1}^{-1} \leq \psi_{\hat{x}_2}^{-1} \implies \psi_{\hat{x}_1} \circ \psi_{\hat{x}_1}^{-1} \leq \psi_{\hat{x}_1} \circ \psi_{\hat{x}_2}^{-1} \\
\implies \text{id} \leq \psi_{\hat{x}_1} \circ \psi_{\hat{x}_2}^{-1} \\
\implies \text{id} \leq \psi_{\hat{x}_1} \circ \psi_{\hat{x}_2}^{-1} \leq \psi_{\hat{x}_2} \circ \psi_{\hat{x}_2}^{-1} \\
\implies \text{id} \leq \psi_{\hat{x}_1} \circ \psi_{\hat{x}_2}^{-1} \leq \text{id}
\]

Hence \( \psi_{\hat{x}_1} \circ \psi_{\hat{x}_2}^{-1} \equiv \text{id} \) i.e \( \psi_{\hat{x}_1}(\hat{y}) = \psi_{\hat{x}_2}(\hat{y}) \) for all \( \hat{y} \in L^* \). Since \( \hat{x}_1 \) and \( \hat{x}_2 \) are arbitrarily chosen \( \psi_{\hat{x}_1} \equiv \psi_{\hat{x}_2} \). Thus \( I(\hat{x}, \hat{y}) = \psi(\hat{y}) \) and \( J(\hat{x}, \hat{y}) = \psi^{-1}(\hat{y}) \) for some increasing bijection on \( L^* \). \( \square \)

Then from the obvious theorems we have the following result

Theorem 5. \( I \in \mathbb{I} \) is invertible w.r.t \( \ast \) if and only if

\[
I(\hat{x}, \hat{y}) = \begin{cases} 
\text{id} & \text{if } \hat{x} = \hat{0} \\
\varphi(\hat{y}) & \text{otherwise}
\end{cases}
\]

where the function \( \varphi : L^* \rightarrow L^* \) is an increasing bijection

Let \( \mathcal{K} \) the largest subgroup of the monoid \( \mathbb{I} \)

Now we propose yet another new generating method of intuitionistic fuzzy implications from intuitionistic fuzzy implications and show that this method imposes a semigroup structure on the set \( \mathbb{I} \).
4 Semigroup structure on $\mathbb{I}$

**Definition 6.** Let $I, J \in I$. Define $I \triangleright J : L^2 \rightarrow L^*$ as follows: $(I \triangleright J)(\tilde{x}, \tilde{y}) = I(J(\tilde{1}, \tilde{x}), J(\tilde{x}, \tilde{y})), \tilde{x}, \tilde{y} \in L^*$.

**Theorem 6.** We have $I \triangleright J$ is an intuitionistic fuzzy implication. i.e., $I \triangleright J \in L$.

**Proof.** Let $I, J \in \mathbb{I}$ and $\tilde{x}_1, \tilde{x}_2, \tilde{y} \in L^*$.

Let $\tilde{x}_1 \leq \tilde{x}_2$. Then $J(\tilde{x}_1, \tilde{y}) \geq J(\tilde{x}_2, \tilde{y})$ and $J(1, \tilde{x}_1) \leq J(1, \tilde{x}_2)$

\[
(I \triangleright J)(\tilde{x}_1, \tilde{y}) = I(J(\tilde{1}, \tilde{x}_1), J(\tilde{x}_1, \tilde{y})) \geq I(J(\tilde{1}, \tilde{x}_1), J(\tilde{x}_2, \tilde{y})) \\
\geq I(J(\tilde{1}, \tilde{x}_2), J(\tilde{x}_2, \tilde{y})) = (I \triangleright J)(\tilde{x}_2, \tilde{y})
\]

Thus $\triangleright$ is decreasing in the first variable. Similarly, one can show that $\triangleright$ is increasing in the second variable. Now we have

\[
(I \triangleright J)(0, 0) = I(J(\tilde{1}, 0), J(0, 0)) = I(\tilde{0}, \tilde{1}) = \tilde{1} \\
(I \triangleright J)(\tilde{1}, \tilde{1}) = I(J(\tilde{1}, \tilde{1}), J(\tilde{1}, \tilde{1})) = I(\tilde{1}, \tilde{1}) = \tilde{1} \\
(I \triangleright J)(\tilde{1}, 0) = I(J(\tilde{1}, \tilde{1}), J(\tilde{1}, 0)) = I(\tilde{1}, \tilde{0}) = \tilde{0}
\]

Hence $I \triangleright J$ is an intuitionistic fuzzy implication. $\square$

**Theorem 7.** $(\mathbb{I}, \triangleright)$ is a semigroup.

**Proof.** from the obvious theorem $\triangleright$ is a binary operation on $\mathbb{I}$. Then it is enough to show that $\triangleright$ is associative in $I$. Let $I, J, T \in \mathbb{I}$ and $\tilde{x}, \tilde{y} \in L^*$.

We have

\[
(I \triangleright (J \triangleright T))(\tilde{x}, \tilde{y}) = I((J \triangleright T)(\tilde{1}, \tilde{x}), (J \triangleright T)(\tilde{x}, \tilde{y})) \\
= I(J(T(\tilde{1}, \tilde{x}), T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y}))) \\
= I(J(\tilde{1}, T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y})))
\]

and\(, (I \triangleright J) \triangleright T)(\tilde{x}, \tilde{y}) = (I \triangleright J)(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y})) \\
= I(J(\tilde{1}, T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y}))).
\]

Then $\triangleright$ is associative in $\mathbb{I}$ and $(\mathbb{I}, \triangleright)$ is a semigroup. $\square$

**Theorem 8.** Let $I, J \in \mathcal{K}$. Then $I \triangleright J = I \ast J$.

**Proof.** Let $I, J \in \mathcal{K}$ i.e., for some $\varphi, \psi \in \Theta$,

$I(\tilde{x}, \tilde{y}) = \begin{cases} 
\tilde{1} & \text{if } \tilde{x} = \tilde{0} \\
\varphi(\tilde{y}) & \text{otherwise}
\end{cases}$

and $J(\tilde{x}, \tilde{y}) = \begin{cases} 
\tilde{1} & \text{if } \tilde{x} = \tilde{0} \\
\psi(\tilde{y}) & \text{otherwise}
\end{cases}$

Now we have

\[
(I \triangleright J)(\tilde{x}, \tilde{y}) = I(J(\tilde{1}, \tilde{x}), J(\tilde{x}, \tilde{y})) \\
= I(\psi(\tilde{x}), J(\tilde{x}, \tilde{y})) = \begin{cases} 
\tilde{1} & \text{if } \tilde{x} = \tilde{0} \\
\varphi(\psi(\tilde{y})) & \text{otherwise}
\end{cases}
\]
and \((I \ast J)(\tilde{x}, \tilde{y}) = I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = \tilde{0} \\ \varphi(\psi(\tilde{y})) & \text{otherwise} \end{cases}\)

Hence \(I \triangleright J = I \ast J\)

**Theorem 9.** For all \(I \in \mathbb{I}T \in \mathcal{K}, T \ast (I \triangleright T^{-1}) = (T \ast I) \triangleright T^{-1}\)

*Proof.* Let \(I \in \mathbb{I}\) and \(T \in \mathcal{K}\) we know that \(T(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & \text{otherwise} \end{cases}\) for some \(\varphi \in \Theta\). Also \(T^{-1}\) will be given by

\[ T^{-1}(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = \tilde{0} \\ \varphi^{-1}(\tilde{y}) & \text{otherwise} \end{cases} \]

if \(\tilde{x} = \tilde{0}\). Then \((T \ast (I \triangleright T^{-1}))(\tilde{0}, \tilde{y}) = \tilde{1} = ((T \ast I) \triangleright T^{-1})(\tilde{0}, \tilde{y})\) if \(\tilde{x} \neq \tilde{0}\). Then

\[
(T \ast (I \triangleright T^{-1}))(\tilde{x}, \tilde{y}) = T(\tilde{x}, (I \triangleright T^{-1})(\tilde{x}, \tilde{y}))
= T(\tilde{x}, I(T^{-1}(\tilde{1}, \tilde{x}), T^{-1}(\tilde{x}, \tilde{y})))
= \varphi(I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y})))
\]

and

\[
((T \triangleright I) \ast T^{-1})(\tilde{x}, \tilde{y}) = (T \ast I)(T^{-1}(\tilde{1}, \tilde{x}), T^{-1}(\tilde{x}, \tilde{y}))
= T(T^{-1}(\tilde{1}, \tilde{x}), I(T^{-1}(\tilde{1}, \tilde{x}), T^{-1}(\tilde{x}, \tilde{y})))
= T(\varphi(\tilde{x}), I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y})))
= \varphi(I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y})))
\]

Hence we have proved that \((T \ast (I \triangleright T^{-1}))(\tilde{x}, \tilde{y}) = ((T \ast I) \triangleright T^{-1})(\tilde{x}, \tilde{y})\) for all \(\tilde{x}, \tilde{y} \in L^*\). \(\square\)

## Group action of \(\mathcal{K}\) on \(\mathbb{I}\)

In this section we define the group action of \(\mathcal{K}\) on \(\mathbb{I}\). for that we first show some result that we need in the sequel.

**Theorem 10.** The groups \((\mathcal{K}, \ast), (\Theta, \circ)\) are isomorphic to each other

*Proof.* Let \(f : \Theta \longrightarrow \mathcal{K}\) defined by \(f(\varphi) = I\) where

\[ I(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & \text{otherwise} \end{cases} \]

It is easy to see that the map \(f\) is one and onto. Let \(\varphi_1, \varphi_2 \in \Theta\) and \(f(\varphi_1) = I_1, f(\varphi_2) = I_2\)

Where \(I_i(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = \tilde{0} \\ \varphi_i(\tilde{y}) & \text{otherwise} \end{cases} \) for \(i = 1, 2\)
Now we have:

\[
(f(\varphi_1) * f(\varphi_2))(\bar{x}, \bar{y}) = (I_1 * I_2)(\bar{x}, \bar{y})
\]

\[
= I_1(\bar{x}, I_2(\bar{x}, \bar{y})
\]

\[
= \begin{cases} 
1 & \text{if } \bar{x} = \bar{0} \\
\varphi_1(\varphi_2(\bar{y})) & \text{otherwise}
\end{cases}
\]

\[
= f(\varphi_1 \circ \varphi_2)(\bar{x}, \bar{y})
\]

Thus \(f\) is an isomorphism. \(\square\)

**Definition 7.** Let \((G, \ast)\) be a group and \(H\) be a nonempty set. A function \(\ast : G \times H \to H\) is called a group action if, for all \(g_1, g_2 \in G\) and \(h \in H\), \(\ast\) satisfies the following two conditions:

1) \(g_1 \ast (g_2 \ast h) = (g_1 \ast g_2) \ast h\)

2) \(e \ast h = h\) where \(e\) is the identity of \(G\).

**Definition 8.** Let \(\ast : \mathcal{K} \times I \to I\) be a map defined by \(T, I \mapsto T \ast I = T \ast I \ast T^{-1}\).

**Lemma 2.** The function \(\ast\) is a group action of \(\mathcal{K}\) on \(I\).

**Proof.** Let \(T_1, T_2 \in \mathcal{K}\) and \(I \in I\).

1) \(T_1 \ast (T_2 \ast I) = T_1 \ast (T_2 \ast I) \ast T_1^{-1}\)

\[
= T_1 \ast T_2 \ast I \ast T_2^{-1} \ast T_1^{-1}
\]

\[
= (T_1 \ast T_2) \ast I \ast (T_1 \ast T_2)^{-1}
\]

\[
= (T_1 \ast T_2) \ast I.
\]

2) Similarly, \(I_D \ast I = I_D \ast I \ast I_D^{-1} = I\), since \(I_D\) is the identity of \((I, \ast)\).

Thus \(\ast\) is a group action of \(\mathcal{K}\) on \(I\). \(\square\)

**Definition 9.** Let \(I, J \in I\). Define \(I \sim J \iff J = T \ast I\) for some \(T \in \mathcal{K}\). In other words, \(I \sim J \iff J = T \ast I \ast T^{-1}\) for some \(T \in \mathcal{K}\).

**Lemma 3.** The relation \(\sim\) is an equivalence relation and it partitions the set \(I\).

**Proof.** We have for \(I, J \in I\)

1 \(I \sim I\) because \(I = I_D \ast I \ast I_D^{-1}\)

2 And we have \(I \sim J \implies J = T \ast I \ast T^{-1}\) this implies that \(I = T^{-1} \ast I \ast T\) then we take \(H = T^{-1}\). Hence \(J \sim I\).

3 for the transitivity let \(I \sim J\) and \(J \sim K\) we can easily show that \(I \sim K\). \(\square\)

**Remark 2.** Let \(I \in \mathcal{K}\). Then the equivalence class containing \(I\) will be of the form \([I]\) = \(\{J \in \mathbb{II} | J = T \ast I \ast T^{-1}\) for some \(T \in \mathcal{K}\}\).

Since any \(T \in \mathcal{K}\) is of the form

\[
T(\bar{x}, \bar{y}) = \begin{cases} 
1 & \text{if } \bar{x} = \bar{0} \\
\varphi(\bar{y}) & \text{otherwise}
\end{cases}
\]

for some \(\varphi \in \theta\), we have that, if \(J \in [I]\), then \(J(\bar{x}, \bar{y}) = \varphi(I(\bar{x}, \varphi^{-1}(\bar{y})))\) for all \(\bar{x}, \bar{y} \in L^*\).
Now we define a new group action of $\mathcal{K}$ on $\mathbb{I}$.

**Theorem 11.** Let $\sqcup : \mathcal{K} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by $T \sqcup I = T * I$, $T \in \mathcal{K}, I \in \mathbb{I}$. The function $\sqcup$ is a left group action of $\mathcal{K}$ on $\mathbb{I}$.

**Proof.** i) Let $T_1, T_2 \in \mathcal{K}$ and $I \in \mathbb{I}$. Then

\[
T_1 \sqcup (T_2 \sqcup I) = T_1 * (T_2 \sqcup I)
\]

\[
= T_1 * (T_2 * I)
\]

\[
= (T_1 * T_2) * I
\]

\[
= (T_1 * T_2) \sqcup I
\]

ii) $I_D \sqcup I = I_D * I = I$ Thus $\sqcup$ is a left group action of $\mathcal{K}$ on $\mathbb{I}$.

\[\square\]

6 Bijective transformations of intuitionistic fuzzy implications

**Definition 10.** Let $I : L^2 \longrightarrow L^*$ be a function and $\varphi, \psi, \mu \in \Theta$. We define the bijective transformation $J_{\varphi, \psi, \mu} : L^2 \longrightarrow L^*$ of $I$ as follows:

\[
J_{\varphi, \psi, \mu}(\tilde{x}, \tilde{y}) = \varphi(I(\psi(\tilde{x}), \mu(\tilde{y}))
\]

The following result shows that any bijective transformation of the form (1) can also generate intuitionistic fuzzy implications from intuitionistic fuzzy implications.

**Theorem 12.** Let $I : L^2 \longrightarrow L^*$ be a function and $\varphi, \psi, \mu \in \Theta$. Let $J_{\varphi, \psi, \mu}$ be defined as in (1). Then the following statements are equivalent:

i) $I$ is an intuitionistic fuzzy implication

ii) $J_{\varphi, \psi, \mu}$ is an intuitionistic fuzzy implication

**Proof.** $\Rightarrow$ Let $\tilde{x}_1, \tilde{x}_2, \tilde{y} \in L^*$ such that $\tilde{x}_1 \leq \tilde{x}_2$. Then we have $I(\tilde{x}_2, \tilde{y}) \leq I(\tilde{x}_1, \tilde{y})$ using the fact that $\varphi, \psi, \mu \in \Theta$ we defined $\varphi(I(\psi(\tilde{x}_2), \mu(\tilde{y}))) \leq \varphi(I(\psi(\tilde{x}_1), \mu(\tilde{y})))$. This implies that $J_{\varphi, \psi, \mu}$ is decreasing for the first variable.

Similarly for the second variable.

And we have $J_{\varphi, \psi, \mu}(\tilde{0}, \tilde{1}) = \varphi(I(\psi(\tilde{0}), \mu(\tilde{1}))) = \varphi(I(\tilde{0}, \tilde{1})) = \varphi(\tilde{1}) = \tilde{1}$,

$J_{\varphi, \psi, \mu}(\tilde{1}, \tilde{0}) = \varphi(I(\psi(\tilde{1}), \mu(\tilde{0}))) = \varphi(I(\tilde{1}, \tilde{0})) = \varphi(\tilde{0}) = \tilde{0}$,

$J_{\varphi, \psi, \mu}(\tilde{1}, \tilde{1}) = \varphi(I(\psi(\tilde{1}), \mu(\tilde{1}))) = \varphi(I(\tilde{1}, \tilde{1})) = \varphi(\tilde{1}) = \tilde{1}$.

Hence $J_{\varphi, \psi, \mu}$ is an intuitionistic fuzzy implication.

Conversely, let $J_{\varphi, \psi, \mu}$ an intuitionistic fuzzy implication. Then for $\tilde{x}_1, \tilde{x}_2, \tilde{y} \in L^*$ such that $\tilde{x}_1 \leq \tilde{x}_2$.

We have $J_{\varphi, \psi, \mu}(\tilde{x}_2, \tilde{y}) \leq J_{\varphi, \psi, \mu}(\tilde{x}_1, \tilde{y})$

$\Rightarrow \varphi(I(\psi(\tilde{x}_2), \mu(\tilde{y}))) \leq \varphi(I(\psi(\tilde{x}_1), \mu(\tilde{y})))$ for some $\varphi, \psi, \mu \in \Theta$

$\Rightarrow I(\psi(\tilde{x}_2), \mu(\tilde{y})) \leq I(\psi(\tilde{x}_1), \mu(\tilde{y}))$ then $I$ is a decreasing function for the first variable because $\varphi, \psi, \mu \in \Theta$. Similarly, $I$ is increasing for the second variable.
Now we have \( J_{\varphi,\psi,\mu}(\tilde{0}, \tilde{1}) = \tilde{1} = \varphi(I(\psi(\tilde{0}), \mu(\tilde{1}))) \) this implies that \( \varphi(I(\tilde{0}, \tilde{1})) = \tilde{1} \).

Hence \( I(\tilde{0}, \tilde{1}) = \tilde{1} \) because \( \varphi(\tilde{1}) = \tilde{1} \) \( \forall \varphi \in \Theta \)

\[ J_{\varphi,\psi,\mu}(\tilde{1}, \tilde{1}) = \tilde{1} = \varphi(I(\psi(\tilde{1}), \mu(\tilde{1}))) \] this implies that \( \varphi(I(\tilde{1}, \tilde{1})) = \tilde{1} \). Hence \( I(\tilde{1}, \tilde{1}) = \tilde{1} \)

\[ J_{\varphi,\psi,\mu}(\tilde{1}, \tilde{0}) = \tilde{0} = \varphi(I(\psi(\tilde{1}), \mu(\tilde{0}))) \] this implies that \( \varphi(I(\tilde{1}, \tilde{0})) = \tilde{0} \). Hence \( I(\tilde{1}, 0) = \tilde{0} \). \( \square \)

From the obvious Theorem, it follows that one can always obtain intuitionistic fuzzy implications from given an intuitionistic fuzzy implication using (1).

Now we define

\[ I_{\sim_{\varphi,\psi,\mu}} J \iff J = I_{\varphi,\psi,\mu} \]

for some \( \varphi, \psi, \mu \in \Theta \). It can be easily seen that \( \sim_{\varphi,\psi,\mu} \) is an equivalence relation, if \( [I]_{\sim_{\varphi,\psi,\mu}} \)

denotes the equivalence class of fuzzy implications containing \( I \) w.r.t. (2), then

\[ [I]_{\sim_{\varphi,\psi,\mu}} = \{ J \in \Pi | J_{\sim_{\varphi,\psi,\mu}} I \} \]

\[ = \{ J \in \Pi | J(x, y) = \varphi(I(\psi(x), \mu(y))) \text{ for some } \varphi, \psi, \mu \in \Theta \} \]

\[ = \{ \varphi(I(\psi(x), \mu(y))) | \varphi, \psi, \mu \in \Theta \}. \]

Now we propose two functions from \( \mathcal{K} \times \Pi \longrightarrow \Pi \). One of these turns out to be a group action of \( \mathcal{K} \) on \( \Pi \), while the other is an anti-group action.

**Definition 11.** Let \( \diamondsuit : \Pi \times \mathcal{K} \longrightarrow \Pi \) be defined by \( I \diamondsuit T = I * T \).

**Theorem 13.** \( \diamondsuit \) is a right group action of \( \mathcal{K} \) on \( \Pi \).

**Proof.** Let \( I \in \Pi \) and \( T_1, T_2 \in \mathcal{K} \).

\[ (I \diamondsuit T_1) \diamondsuit T_2 = (I * T_1) \diamondsuit T_2 = (I * T_1) * T_2 = I * (T_1 * T_2) = I \diamondsuit (T_1 * T_2). \]

\[ I \diamondsuit T_D = I * I_D = I \] for all \( I \in \Pi \).

Thus \( \diamondsuit \) is a right group action. \( \square \)

**Definition 12.** Define \( \sim_{\diamondsuit} \) on \( \Pi \) by \( I \sim_{\diamondsuit} J \iff J = I \diamondsuit T = I * T \) for some \( T \in \mathcal{K} \).

It is easy to verify that \( \sim_{\diamondsuit} \) is an equivalence relation.

**Remark 3.** Let \( I \in \Pi \). If \([I]_{\diamondsuit} \) denotes the equivalence class containing \( I \), then

\[ [I]_{\diamondsuit} = \{ J \in \Pi | J \sim_{\diamondsuit} I \} \]

\[ = \{ J \in \Pi | J = I \diamondsuit T \text{ for some } T \in \mathcal{K} \} \]

\[ = \{ J \in \Pi | J(x, y) = I(\varphi(x), \psi(y)) \text{ for some } \varphi, \psi \in \Theta \} \]

\[ = \{ I(x, \varphi(y)) \text{ for some } \varphi \in \Theta \}. \]

**Definition 13. (See [6])** Let \( (G, \ast) \) be a group with identity \( e \) and \( S \) being a nonempty set. A map \( \circ : G \times S \longrightarrow S \) is called anti-group action if for all \( g_1, g_2 \in G \), \( s \in S \) the map \( \circ \) satisfies the following:

(i) \( g_1 \circ (g_2 \circ s) = (g_2 \circ g_1) \circ s. \)

(ii) \( e \circ s = s. \)
**Theorem 14.** Let $\triangledown : \mathcal{K} \times \mathbb{I} \rightarrow \mathbb{I}$ be defined by $T \triangledown I = (I \triangleright T) * T^{-1}$, $T \in \mathcal{K}$, $I \in \mathbb{I}$. Then $\triangledown$ is an anti-group action of $\mathcal{K}$ on $\mathbb{I}$.

**Proof.** i) Let $I \in \mathbb{I}$ and $T_1, T_2 \in \mathcal{K}$. Then

$$T_1 \triangledown (T_2 \triangledown I) = T_1(\triangledown (I \triangleright T_2) * T_2^{-1})$$

$$= ((I \triangleright T_2) * T_2^{-1} \triangleright T_1) * T_1^{-1}$$

Since $T_1, T_2 \in \mathcal{K}$. Then $T_1, T_2$ are of the following form

$$T_i(x, y) = \begin{cases} \tilde{I} & \text{if } \tilde{x} = 0 \\ \varphi_i(y) & \text{otherwise} \end{cases} \quad i = 1, 2$$

for some $\varphi_i \in \theta$, if $\tilde{x} = 0$. Then

$$(T_1 \triangledown (T_2 \triangledown I))(\tilde{x}, y) = ((I \triangleright T_2) * T_2^{-1} \triangleright T_1) * T_1^{-1}(\tilde{x}, y)$$

$$= ((I \triangleright T_2) * T_2^{-1} \triangleright T_1)(\tilde{x}, T_1^{-1}(\tilde{x}, y))$$

$$= ((I \triangleright T_2) * T_2^{-1} \triangleright T_1)(\tilde{x}, \varphi_1^{-1}(\tilde{y}))$$

$$= ((I \triangleright T_2)(\varphi_1(\tilde{x}), T_2^{-1}(\varphi_1(\tilde{x}), \tilde{y})))$$

$$= I(T_2(\tilde{I}, \varphi_1(\tilde{x})), T_2(\varphi_1(\tilde{x}), \varphi_2^{-1}(\tilde{y})))$$

$$= I(\varphi_2(\varphi_1(\tilde{x})), \tilde{y}),$$

While

$$(T_2 * T_1) \triangledown I(\tilde{x}, y) = (I \triangleright (T_2 * T_1) * (T_2 * T_1)^{-1})(\tilde{x}, y)$$

$$= (I \triangleright (T_2 * T_1) * (T_2^{-1} * T_2^{-1}))(\tilde{x}, y)$$

$$= (I \triangleright (T_2 * T_1) * (T_2^{-1}))(\tilde{x}, \varphi_2^{-1}(\tilde{y}))$$

$$= I((T_2 * T_1)(\tilde{I}, \tilde{x}), (T_2 * T_1)(\tilde{x}, \varphi_2^{-1}(\varphi_2^{-1}(\tilde{y}))))$$

$$= I(\varphi_2(\varphi_1(\tilde{x})), \tilde{y}).$$

Thus in all cases we have shown that $T_1 \triangledown (T_2 \triangledown I) = (T_2 * T_1) \triangledown I$, for all $T_2, T_1 \in \mathcal{K}$ and $I \in \mathbb{I}$.

ii) Let $I \in \mathbb{I}$. Then $I_D \triangledown I = (I \triangleright I_D) * I_D^{-1} = I \triangleright I_D = I$, hence $\triangledown$ is an anti-group action. $\square$

**Definition 14.** Let $I, J \in \mathbb{I}$. Then the relation defined as follows is an equivalence relation: $I \sim I$ if and only if $J = T_1 \triangledown ((T_3 \triangledown I) \triangledown T_2)$ for some $T_1, T_2, T_3 \in \mathcal{K}$.

In fact, by expanding the above $J$ as follows

$$J = T_1 \triangledown ((T_3 \triangledown I) \triangledown T_2) = T_1 * ((T_3 \triangledown I) \triangledown T_2)$$

$$= T_1 * ((T_3 \triangledown I) * T_2) = T_1 * ((I \triangleright T_3) * T_3^{-1} * T_2)$$

Then $I \sim J$ if and only if $J = T_1 * ((I \triangleright T_3) * T_3^{-1} * T_2)$ for some $T_1, T_2, T_3 \in \mathcal{K}$.
Theorem 15. The equivalence classes of fuzzy implications as given in (16) are exactly the equivalence classes obtained from the relation ∼□, i.e., for any $I \in \mathbb{I}$, $[I]_{\sim \varphi, \psi, \mu} = [I]_{\sim \Box}$.

Proof. Let $I \in \mathbb{I}$. Then

$$[I]_{\sim \Box} = \{ J \in \mathbb{I} | J \sim \Box I \} = \{ J \in \mathbb{I} | J = T_1 \ast ((I \triangleright T_3) \ast T_3^{-1} \ast T_2) \text{ for some } T_1, T_2, T_3 \in K \} = \{ J \in \mathbb{I} | J(x, y) = (T_1 \ast ((I \triangleright T_3) \ast T_3^{-1} \ast T_2)(x, y) \text{ for all } x, y \in L^* \} = \{ J \in \mathbb{I} | J(x, y) = T_1(x, ((I \triangleright T_3)(x, (T_3^{-1} \ast T_2)(x, y)))) \text{ for all } x, y \in L^* \} = \{ J \in \mathbb{I} | J(x, y) = T_1(x, I(T_3(\tilde{x}, x), T_3^{-1} \ast T_2)(x, y)))) \text{ for all } x, y \in L^* \} = \{ J \in \mathbb{I} | J(x, y) = \begin{cases} \tilde{1} & \text{if } \tilde{x} = 0 \\ \varphi(I(\psi(\tilde{x}), \mu(\tilde{y}))) & \text{otherwise} \end{cases} \text{ for some } \varphi, \psi, \mu \in \Theta \} = [I]_{\sim \varphi, \psi, \mu}$$

In other words, this result shows that any bijective transformation can be represented by a composition of group actions and an anti-group action of $K$ on $\mathbb{I}$.

7 Conclusion

Our motivation for this study was to propose a binary operation $\ast$ on the set $\mathbb{I}$ of all intuitionistic fuzzy implications that would give a rich enough algebraic structure to glean newer and better perspectives on intuitionistic fuzzy implications. The operation $\ast$ proposed in this work not only gave a novel way of generating newer intuitionistic fuzzy implications from given ones, but also, for the first time, imposed a monoid structure on $\mathbb{I}$. By defining a suitable group action on $\mathbb{I}$ and the equivalence classes obtained therefrom. And we have shown that the bijective transformations given in (1) can be seen as a composition of group actions $\diamondsuit, \square$ and $\Box$.

References


