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On properties of some IFS operators and operations

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Abstract

The aim of this paper is to point out and correct some errors of two theorems for IFSs in Atanassov (Two theorems for intuitionistic fuzzy sets, Fuzzy Sets and Systems, 2000) and bring forward a new theorem for IFSs. In the end, some elementary, but non-standard equalities between IFSs were formulated.

Keywords: Fuzzy sets; Intuitionistic fuzzy set (IFS); Operator; Operation

Intuitionistic fuzzy set (IFS) introduced by Atanassov [1] has become a popular topic of investigation in the fuzzy set community [11,12,13]. There exists a large amount of literature involving IFS theory and applications [7-10]. The aim of this paper is to point out and correct some errors in Atanassov [6] and bring forward a new theorem for IFSs. In the end, we formulate some elementary, but non-standard equalities between IFSs.

Definition 1 (see Atanassov [1]). Let E denote a universe of discourse. Then an intuitionistic fuzzy set (IFS) defined on E is given as follows:

$$A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in E\}$$

where the functions $\mu_A(x) : E \to [0,1]$ and $\nu_A(x) : E \to [0,1]$ define, respectively, the degree of membership and the degree of non-membership of the element $x \in E$ to the set A, which is a subset of E, and for every $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

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Furthermore, we call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ the intuitionistic index of x in A. It is a hesitancy degree of x to A. It is obvious that $0 \le \pi_A(x) \le 1$, for each $x \in E$.

Definition 2 (See Atanassov [1]). Let $A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in E\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >| \ x \in E\}$ be two IFSs. We define the following relations:
(1)$A \subset B$ iff $(\forall x \in E)$ $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$;
(2)$A = B$ iff $(\forall x \in E)$ $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$.

Atanassov [1, 2, 3] introduced the following IFS operators. For every IFS A

$$C(A) = \{< x, K, L >| \ x \in E\}$$

where

$$K = \max_{x \in E} \mu_A(x), \ L = \min_{x \in E} \nu_A(x),$$

$$I(A) = \{< x, k, l >| \ x \in E\}$$

where

$$k = \max_{x \in E} \mu_A(x), \ l = \min_{x \in E} \nu_A(x).$$

Let $\alpha, \beta \in [0, 1]$ be fixed numbers. The following operators are defined and their properties are studied in [4]:

$$D_\alpha(A) = \{< x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + (1 - \alpha) \cdot \pi_A(x) >| \ x \in E\};$$

$$F_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha \cdot \pi_A(x), \nu_A(x) + \beta \cdot \pi_A(x) >| \ x \in E\}, for \ \alpha + \beta \le 1;$$

$$H_{\alpha,\beta}(A) = \{< x, \alpha \cdot \mu_A(x), \nu_A(x) + \beta \cdot \pi_A(x) >| \ x \in E\};$$

$$J_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha \cdot \pi_A(x), \beta \cdot \nu_A(x) >| \ x \in E\};$$

$$H^*_{\alpha,\beta}(A) = \{< x, \alpha \cdot \mu_A(x), \nu_A(x) + \beta \cdot (1 - \alpha \cdot \mu_A(x) - \nu_A(x)) >| \ x \in E\};$$

$$J^*_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha \cdot (1 - \mu_A(x) - \beta \cdot \nu_A(x)), \beta \cdot \nu_A(x) >| \ x \in E\};$$

$$P_{\alpha,\beta}(A) = \{< x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x)) >| \ x \in E\}, for \ \alpha + \beta \le 1;$$

$$Q_{\alpha,\beta}(A) = \{< x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x)) >| \ x \in E\}, for \ \alpha + \beta \le 1.$$

Definition 3 (See Atanassov [6]). Let the IFS A over the universe E be called essential, if there exists at least one $x \in E$ for which $\pi_A(x) > 0$.

Theorem 1 (See Atanassov [6]). Let A ,B be two essential IFSs, for which there are $y, z \in E$ such that $\mu_A(y) > 0$ and $\mu_B(z) > 0$. If $C(A) \subset I(B)$, then there are real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$, such that $J_{\alpha,\beta}(A) \subset H_{\gamma,\delta}(B)$.

In the process of the proof to the theorem in paper [6], the author let

$$a = \max_{x \in E} \pi_A(x) > 0$$

and

$$b = \max_{x \in E} \pi_B(x) > 0,$$

then constructed

$$\alpha = \frac{k-K}{2a},\ \beta = \frac{L+l}{2L},\ \gamma = \frac{K+k}{2k},\ \delta = \frac{L-l}{2b}.$$

It is noted that definition of α and δ in paper [6] is not correct, because it does not ensure that $\alpha, \delta \in [0, 1]$, so the process of the proof does not make sense, too.

Example 1. Let the universe $U = \{x_1, x_2, x_3\}$, the two IFSs

$$A = \{< x_1, 0.5, 0.5 >, < x_2, 0.1, 0.8 >, < x_3, 0.4, 0.5 >\}$$

$$B = \{< x_1, 0.8, 0.2 >, < x_2, 0.8, 0.1 >, < x_3, 0.9, 0.1 >\}$$

thus, we have,

$$K = \max_{x \in E} \mu_A(x) = 0.5,\ k = \min_{x \in E} \mu_B(x) = 0.8,\ l = \max_{x \in E} \nu_B(x) = 0.2,$$

$$L = \min_{x \in E} \nu_A(x) = 0.5,\ a = \max_{x \in E} \pi_A(x) = 0.1,\ b = \max_{x \in E} \pi_B(x) = 0.1.$$

It is obvious that

$$\alpha = \frac{k-K}{2a} = \frac{0.8-0.5}{2 \times 0.1} = 1.5 > 1$$

and

$$\delta = \frac{L-l}{2b} = \frac{0.5-0.2}{2 \times 0.1} = 1.5 > 1.$$

which does not ensure $\alpha, \delta \in [0, 1]$, it shows that the process of the proof is not correct, too.

We modify α, δ as the following forms:

$$\alpha = \frac{k-K}{2},\ \delta = \frac{L-l}{2},$$

then

$$J_{\frac{k-K}{2}, \frac{L+l}{2L}}(A) = \{< x, \mu_A(x) + \frac{k-K}{2}\pi_A(x), \frac{L+l}{2L}\nu_A(x) >| \ x \in E\},$$

$$H_{\frac{k+K}{2k}, \frac{L-l}{2}}(B) = \{< x, \frac{k+K}{2k}\mu_B(x), \nu_B(x) + \frac{L-l}{2}\pi_B(x) >| \ x \in E\}.$$

From

$$\mu_A(x) + \frac{k-K}{2}\pi_A(x) \le K + \frac{k-K}{2} \cdot 1 = \frac{k+K}{2} \le \frac{k+K}{2k}\mu_B(x)$$

and

$$\nu_B(x) + \frac{L-l}{2}\pi_B(x) \le l + \frac{L-l}{2}\cdot 1 = \frac{L+l}{2} \le \frac{L+l}{2L}\nu_A(x)$$

It follows that $J_{\alpha,\beta}(A) \subset H_{\gamma,\delta}(B)$. □

Corollary 1. Let A, B be two essential IFSs, for which there are $y, z \in E$ such that $\mu_A(y) > 0$ and $\mu_B(z) > 0$. If $C(A) \subset I(B)$, then there are real numbers $\alpha, \beta, \gamma, \delta \in [0,1]$, such that $J^*_{\alpha,\beta}(A) \subset H^*_{\gamma,\delta}(B)$.

Proof. Similar to Theorem 1.

Theorem 2. Let A, B be two essential IFSs, for which there are $y, z \in E$ such that $\mu_A(y) > 0$ and $\mu_B(z) > 0$. If $C(A) \subset I(B)$, then there are real numbers $\alpha, \beta, \gamma, \delta \in [0,1]$ such that $\alpha + \beta \le 1$ and $F_{\alpha,\beta}(A) \subset H_{\gamma,\delta}(B)$.

Proof. Let $C(A) \subset I(B)$. Therefore,

$$0 < \mu_A(y) \le \max_{x\in E} \mu_A(x) = K \le k = \min_{x\in E} \mu_B(x)$$

and

$$0 < \mu_B(z) \le \max_{x\in E} \nu_B(x) = l \le L = \min_{x\in E} \nu_A(x).$$

Let

$$\alpha = \frac{k-K}{2}, \quad \forall\beta, 0 \le \beta \le 1-\alpha, \quad \gamma = \frac{K+k}{2k}, \quad \delta = \frac{L-l}{2},$$

then

$$F_{\frac{k-K}{2},\beta}(A) = \{< x, \mu_A(x) + \frac{k-K}{2}\pi_A(x), \nu_A(x) + \beta\cdot\pi_A(x) >| \, x \in E\},$$

$$H_{\frac{k+K}{2k},\frac{L-l}{2}}(B) = \{< x, \frac{k+K}{2k}\mu_B(x), \nu_B(x) + \frac{L-l}{2}\pi_B(x) >| \, x \in E\}.$$

From

$$\mu_A(x) + \frac{k-K}{2}\pi_A(x) \le K + \frac{k-K}{2}\cdot 1 = \frac{k+K}{2} \le \frac{k+K}{2k}\mu_B(x)$$

and

$$\nu_B(x) + \frac{L-l}{2}\pi_B(x) \le l + \frac{L-l}{2}\cdot 1 = \frac{L+l}{2} \le \frac{L+l}{2L}\nu_A(x) \le \nu_A(x) \le \nu_A(x) + \beta\cdot\pi_A(x)$$

It follows that $F_{\alpha,\beta}(A) \subset H_{\gamma,\delta}(B)$. □

It is easily seen that with the same hypotheses, if $\beta = 1-\alpha$, then $D_\alpha(A) \subset H_{\gamma,\delta}(B)$.

Corollary 2. Let A, B be two essential IFSs, for which there are $y, z \in E$ such that $\mu_A(y) > 0$ and $\mu_B(z) > 0$. If $C(A) \subset I(B)$, then there are real numbers $\alpha, \beta, \gamma, \delta \in [0,1]$, such that $\alpha + \beta \le 1$ and $J_{\alpha,\beta}(A) \subset F_{\gamma,\delta}(B)$.

Proof. Let

$$\alpha = \frac{k-K}{2}, \quad \beta = \frac{L+l}{2L}, \quad \delta = \frac{L-l}{2}, \quad \forall \gamma, 0 \leq \gamma \leq 1-\delta.$$

Then it is similar to Theorem 2.

It is easily seen that with the same hypotheses, if $\gamma = 1-\delta$, then $J_{\alpha,\beta}(A) \subset D_{\delta}(B)$.

Theorem 3. For every two IFSs A and B, $C(A) \subset I(B)$, iff there exist two real numbers $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$ and $P_{\alpha,\beta}(A) = Q_{\alpha,\beta}(B)$.

It is noted that conclusion in paper [6] is $\alpha+\beta \leq 1$ and $P_{\alpha,\beta}(A) \subset Q_{\alpha,\beta}(B)$, now we illuminate that the conclusion is $\alpha + \beta \leq 1$ and $P_{\alpha,\beta}(A) = Q_{\alpha,\beta}(B)$.

Proof. The necessity of the proof to the proposition is the same as paper [6]. Actually, we just need to construct α and β as any real numbers satisfying $\alpha \in [K, k]$ and $\beta \in [l, L]$, respectively. We demonstrate that $P_{\alpha,\beta}(A) \subset Q_{\alpha,\beta}(B)$ is $P_{\alpha,\beta}(A) = Q_{\alpha,\beta}(B)$ virtually.

Let there be $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$ and let $P_{\alpha,\beta}(A) = Q_{\alpha,\beta}(B)$. Then, $(\forall x \in E)$

$$\max(\mu_A(x), \alpha) = \min(\mu_B(x), \alpha) \tag{1}$$

and

$$\min(\nu_A(x), \beta) = \max(\nu_B(x), \beta) \tag{2}$$

Therefore, $(\forall x \in E)$
if $\alpha < \mu_A(x)$, then $\max(\mu_A(x), \alpha) = \mu_A(x)$.

Under the condition of $\alpha < \mu_A(x)$, if $\alpha \leq \mu_B(x)$, then $\min(\mu_B(x), \alpha) = \alpha$, which contradicts formula (1). On the other hand, if $\alpha > \mu_B(x)$, then $\min(\mu_B(x), \alpha) = \mu_B(x)$, which contradicts (1), too. So we have $\alpha \geq \mu_A(x)$, that is $\max(\mu_A(x), \alpha) = \alpha$.

Under the condition of $\alpha \geq \mu_A(x)$, if $\alpha > \mu_B(x)$, then $\min(\mu_B(x), \alpha) = \mu_B(x)$, which contradicts formula (1). So $\alpha \leq \mu_B(x)$, that is $\min(\mu_B(x), \alpha) = \alpha$. Based on the above analysis, it is seen that (1) is equivalent to $\mu_A(x) \leq \alpha \leq \mu_B(x)$, $(\forall x \in E)$.

By the same way, (2) is equivalent to $\nu_B(x) \leq \beta \leq \nu_A(x)$, $(\forall x \in E)$.

Consequently, the conclusion of " There be $\alpha, \beta \in [0,1]$ such that $\alpha+\beta \leq 1$ and $P_{\alpha,\beta}(A) \subset Q_{\alpha,\beta}(B)$ " is equivalent to $\alpha \in [K, k]$ and $\beta \in [l, L]$, which is $C(A) \subset I(B)$ equivalently. Therefore, the conclusion is there be $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$ and $P_{\alpha,\beta}(A) = Q_{\alpha,\beta}(B)$. □

On two IFSs A and B, the following (and other) operations are defined (See Atanassov [1, 5]):

$$A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) >| \; x \in E\};$$
$$A \cup B = \{< x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) >| \; x \in E\};$$
$$A + B = \{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x) >| \; x \in E\};$$
$$A \cdot B = \{< x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x) >| \; x \in E\};$$
$$A@B = \{< x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} >| \; x \in E\};$$
$$A\$B = \{< x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)} >| \; x \in E\};$$
$$A\#B = \{< x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} >| \; x \in E\};$$

for which we shall accept that if $\mu_A(x) = \mu_B(x) = 0$, then

$$\frac{\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)} = 0$$

and if $\nu_A(x) = \nu_B(x) = 0$, then

$$\frac{\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} = 0.$$

Here we shall formulate some elementary, but non-standard equality between IFSs.

Theorem 4. For every two IFSs A and B, then

$$(1)(A@B)\$(A\#B) = A\$B;$$
$$(2)(A + B) \cap (A \cdot B) = A \cdot B, \; (A + B) \cup (A \cdot B) = A + B;$$
$$(3)(A + B) \cap (A@B) = A@B, \; (A + B) \cup (A@B) = A + B;$$
$$(4)(A \cdot B) \cap (A@B) = A \cdot B, \; (A \cdot B) \cup (A@B) = A@B;$$
$$(5)(A + B) \cap (A\$B) = A\$B, \; (A + B) \cup (A\$B) = A + B;$$
$$(6)(A \cdot B) \cap (A\$B) = A \cdot B, \; (A \cdot B) \cup (A\$B) = A\$B;$$
$$(7)(A + B) \cap (A\#B) = A\#B, \; (A + B) \cup (A\#B) = A + B;$$
$$(8)(A \cdot B) \cap (A\#B) = A \cdot B, \; (A \cdot B) \cup (A\#B) = A\#B.$$

Proof.

(1) $(A@B)\$(A\#B)$

$$= \{<, x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} >| \; x \in E\}\$$$

$$\{< x, \frac{2\mu_A(x)\mu_B(x)}{(\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)} >| \, x \in E\}$$
$$= \{< x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)} >| \, x \in E\}$$
$$= A\$B$$

(2) From

$$\mu_A(x)+\mu_B(x) \geq 2\sqrt{\mu_A(x)\mu_B(x)} \geq 2\mu_A(x)\mu_B(x)$$
$$\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x) \geq \mu_A(x)\mu_B(x)$$

Similarly,

$$\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x) \geq \nu_A(x)\nu_B(x)$$

It follows that

$$(A+B)\cap(A\cdot B) = A\cdot B,\ (A+B)\cup(A\cdot B) = A+B.$$

Proofs of (3) - (4) are similar to (2).

(5) From

$$\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)-\sqrt{\mu_A(x)\mu_B(x)}$$
$$\geq 2\sqrt{\mu_A(x)\mu_B(x)}-\mu_A(x)\mu_B(x)-\sqrt{\mu_A(x)\mu_B(x)}$$
$$= \sqrt{\mu_A(x)\mu_B(x)}-\mu_A(x)\mu_B(x) \geq 0$$

and

$$\sqrt{\nu_A(x)\nu_B(x)}-\nu_A(x)\nu_B(x) \geq 0$$

It follows that $(A+B)\cap(A\$B) = A\$B,\ (A+B)\cup(A\$B) = A+B$.

(6) Similar to (5).

(7) $\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)-\frac{2\mu_A(x)\mu_B(x)}{(\mu_A(x)+\mu_B(x)}$

$$= \frac{(\mu_A(x)+\mu_B(x))^2-\mu_A(x)\mu_B(x)(\mu_A(x)+\mu_B(x))-2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}$$
$$= \frac{(\mu_A(x))^2+2\mu_A(x)\mu_B(x)+(\mu_B(x))^2-(\mu_A(x))^2\mu_B(x)-\mu_A(x)(\mu_B(x))^2-2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}$$
$$= \frac{(\mu_A(x))^2(1-\mu_B(x))+(\mu_B(x))^2(1-\mu_A(x))}{\mu_A(x)+\mu_B(x)} \geq 0;$$
$$\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}-\nu_A(x)\nu_B(x)$$
$$= \frac{\nu_A(x)\nu_B(x)[2-(\nu_A(x)+\nu_B(x))]}{\nu_A(x)+\nu_B(x)} \geq 0.$$

(8) Similar to (7). □

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986) 87-96.

[2] K. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33(1) (1989) 37-46.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61(2) (1994) 137-142.

[4] K. Atanassov, Remarks on the intuitionistic fuzzy sets-III, Fuzzy Sets and Systems 75(1995) 401-402.

[5] K. Atanassov, An equality between intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996) 257-258.

[6] K. Atanassov, Two theorems for intuitionistic fuzzy sets, Fuzzy Sets and Systems 110 (2000) 267-269.

[7] Supriya Kumar De, Ranjit Biswas, Akhil Ranjan Roy, Some operations on intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (2000) 477-484.

[8] Dengfeng, Li, Chuntian, Chen, New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognition Letters. 23 (2002) 221-225.

[9] Wen-Liang Hung, Miinshen Yang. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance, Pattern Recognition Letters 25 (2004) 1603-1611.

[10] Weiqiong Wang, Xiaolong Xin. Distance measure between intuitionistic fuzzy sets, Pattern Recognition Letters 26 (2005) 2063-2069.

[11] P. Grzegorzewski, E. Mrowka. Some notes on (Atanassov's) intuitionistic fuzzy sets, Fuzzy Sets and Systems 156 (2005) 492-495.

[12] K. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H.Prade's paper " Terminological difficulties in fuzzy set theory-the case of " Intuitionistic Fuzzy Sets " ", Fuzzy Sets and Systems 156 (2005) 496-499.

[13] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade, Are (Atanassov's) intuitionistic fuzzy sets intuitionistic, Fuzzy Sets and Systems, to appear.