4th IFSCOM, 3–7 May 2017, Mersin, Turkey
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 23, 2017, No. 1, 14–20

Intuitionistic fuzzy implication $\rightarrow_{189}$

Lilija Atanassova

Institute of Information and Communication Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2, Sofia 1113, Bulgaria
e-mail: l.c.atanassova@gmail.com

Abstract: In [4], some new intuitionistic fuzzy operations are defined and their properties are studied. On the basis of the third of them, a new intuitionistic fuzzy implication is introduced here, numbered as $\rightarrow_{189}$ and some of its properties will be studied.
Keywords: Implication, Intuitionistic fuzzy implication, Intuitionistic fuzzy logic.
AMS Classification: 03E72.

1 Introduction

In the present paper, we continue our research, related to some new intuitionistic fuzzy implications. Now, we use results from [4], where five new intuitionistic fuzzy operations, including multiplication, were introduced.

As it is discussed in [1], each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity", and "falsity degree" or "degree of non-validity". Thus, to each of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

In [2], the object $\langle \mu(p), \nu(p) \rangle$ is introduced under the name Intuitionistic Fuzzy Pair (IFP). Formula A is an Intuitionistic Fuzzy Tautology (IFT) if and only if (*iff*) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then, $a \geq b$, while it is a (classical) tautology *iff* for every evaluation function V, if $V(A) = \langle a, b \rangle$, then, $a = 1,\ b = 0$.

Following [2], we will mention that, if an IFP is an IFT, we call it Intuitionistic Fuzzy Tautological Pair (IFTP) and if it is a tautology – Tautological Pair (TP).

In [1], different intuitionistic fuzzy operations are introduced, e.g., intuitionistic fuzzy disjunction, conjunction and (classical) negation, by

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle,$$

$$V(\neg p) = \langle \nu(p), \mu(p) \rangle.$$

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" of the intuitionistic fuzzy evaluation of A. Also, for brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A) \rangle$ we use the IFP $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

In [4], for two IFPs $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$ are introduced five novel operations from multiplicative type. Here, we use only one of them:

$$x \times_3 y = \langle ac, bd \rangle$$

and on its basis we introduce a new intuitionistic fuzzy implication. In some sense, it is analogous of implication $\rightarrow_{@}$, that in [1] was denoted by $\rightarrow_{139}$.

For operation $\times_3$, in [4] it was checked that it was defined correctly and for the above x and y and for $z = \langle e, f \rangle$:

$$x \times_3 y = y \times_3 x,$$

$$(x \times_3 y) \times_3 z = x \times_3 (y \times_3 z).$$

We can see that

$$\langle 0, 1 \rangle \times_3 \langle 0, 1 \rangle = \langle 0, 1 \rangle,$$

$$\langle 0, 1 \rangle \times_3 \langle 0, 0 \rangle = \langle 0, 0 \rangle,$$

$$\langle 0, 1 \rangle \times_3 \langle 1, 0 \rangle = \langle 0, 0 \rangle,$$

$$\langle 0, 0 \rangle \times_3 \langle 0, 1 \rangle = \langle 0, 0 \rangle,$$

$$\langle 0, 0 \rangle \times_3 \langle 0, 0 \rangle = \langle 0, 0 \rangle,$$

$$\langle 0, 0 \rangle \times_3 \langle 1, 0 \rangle = \langle 0, 0 \rangle,$$

$$\langle 1, 0 \rangle \times_3 \langle 1, 0 \rangle = \langle 1, 0 \rangle,$$

$$\langle 1, 0 \rangle \times_3 \langle 0, 0 \rangle = \langle 0, 0 \rangle,$$

$$\langle 1, 0 \rangle \times_3 \langle 0, 1 \rangle = \langle 0, 0 \rangle,$$

and for each IFP x:

$$\langle 0, 1 \rangle \times_3 x = \langle 0, b \rangle = x \times_3 \langle 0, 1 \rangle,$$

$$\langle 0, 0 \rangle \times_3 x = \langle 0, 0 \rangle = x \times_3 \langle 0, 0 \rangle,$$

$$\langle 1, 0 \rangle \times_3 x = \langle a, 0 \rangle = x \times_3 \langle 1, 0 \rangle.$$

In [4] it is proved that if

$$\mathcal{L} = \{\langle a, b\rangle | a, b \in [0, 1] \ \&\ a + b \leq 1\}$$

is the set of all IFPs, then $\langle \mathcal{L}, \times_3\rangle$ is a commutative semi-group and if x and y are IFTPs, then $x \times_3 y$, is an IFTP, while if x and y are TPs, then $x \times_3 y$ is a TP.

Using the classical negation, defined in intuitionistic fuzzy propositional logic, for $\times_3$ we obtain for every two IFPs x and y:

$$\neg(\neg x \times_3 \neg y) = \neg(\neg\langle a, b\rangle \times_3 \neg\langle c, d\rangle)$$

$$\neg(\langle b, a\rangle \times_3 \langle d, c\rangle) = \neg\langle bd, ac\rangle$$

$$\langle ac, bd\rangle = x \times_3 y.$$

Therefore, operation $\times_3$, similarly to operation @ (see [1]), simultaneously has the behaviour of both the operations disjunction and conjunction.

2 Intuitionistic fuzzy implication $\rightarrow_{189}$ and its properties

In the paper, using the standard logical formula

$$x \rightarrow y = \neg x \vee y,$$

we obtain the new intuitionistic fuzzy implication

$$x \rightarrow_{189} y = \neg x \vee y = \langle bc, ad\rangle.$$

First, we see that

$$0 \leq bc + ad \leq b + a \leq 1,$$

i.e., implication $\rightarrow_{189}$ is defined correctly.

Second, we see that

$$\langle 0, 1\rangle \rightarrow_{189} \langle 0, 1\rangle = \langle 0, 0\rangle,$$

$$\langle 0, 1\rangle \rightarrow_{189} \langle 1, 0\rangle = \langle 1, 0\rangle,$$

$$\langle 1, 0\rangle \rightarrow_{189} \langle 0, 1\rangle = \langle 0, 1\rangle,$$

$$\langle 1, 0\rangle \rightarrow_{189} \langle 1, 0\rangle = \langle 0, 0\rangle,$$

i.e., this operation, similarly to operation $\rightarrow_{188}$ from [3], satisfies only a part of the basic properties of an implication. So, operation $\rightarrow_{189}$ can be classified as a semi-implication.

Third, semi-implication $\rightarrow_{189}$ generates the following negation

$$\neg^*\langle a, b\rangle = \langle a, b\rangle \rightarrow_{189} \langle 0, 1\rangle = \langle 0, a\rangle,$$

i.e., the same negation as the one generated by implication $\rightarrow_{188}$ from [3].

Therefore,

$$\neg^*\neg^*\langle a,b\rangle = \neg^*\langle 0,a\rangle = \langle 0,0\rangle. \tag{1}$$

Fourth, we see that

$$\begin{aligned}&(x \to_{189} y) \vee (y \to_{189} x)\\ &= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \vee (\langle c,d\rangle \to_{189} \langle a,b\rangle)\\ &= \langle bc, ad\rangle \vee \langle ad, bc\rangle\\ &= \langle ad, bc\rangle.\end{aligned}$$

Therefore, for this operation the above expression can not be an IFTP.

On the other hand, if we use the analogue of operation $\vee$ in its new form $\times_3$, we will obtain

$$(x \to_{189} y) \times_3 (y \to_{189} x)$$

$$\begin{aligned}&= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \times_2 (\langle c,d\rangle \to_{189} \langle a,b\rangle)\\ &= \langle bc, ad\rangle \times_2 \langle ad, bc\rangle\\ &= \langle ad, b^2c^2\rangle,\end{aligned}$$

i.e., the situation is similar to above one.

Following [1], we check G. F. Rose's formula [6, 7], that has the form:

$$((\neg\neg x \to_{189} x) \to_{189} (\neg\neg x \vee \neg x)) \to_{189} (\neg\neg x \vee \neg x),$$

but compared to [3], here we use $\neg^*$ instead of classical negation and will prove the following theorem.

Theorem 1. Rose's formula is an IFT.

Proof. Having in mind (1), we obtain sequentially:

$$((\neg^*\neg^* x \to_{189} x) \to_{189} (\neg^*\neg^* x \vee \neg^* x)) \to_{189} (\neg^*\neg^* x \vee \neg^* x)$$

$$\begin{aligned}&= ((\neg^*\neg^*\langle a,b\rangle \to_{189} \langle a,b\rangle) \to_{189} (\neg^*\neg^*\langle a,b\rangle \vee \neg^*\langle a,b\rangle)) \to_{189} (\neg^*\neg^*\langle a,b\rangle \vee \neg^*\langle a,b\rangle)\\ &= (((\langle 0,0\rangle \to_{189} \langle a,b\rangle) \to_{189} (\langle 0,0\rangle \vee \langle 0,a\rangle)) \to_{189} (\langle 0,0\rangle \vee \langle 0,a\rangle)\\ &= (\langle 0,0\rangle \to_{189} \langle 0,0\rangle) \to_{189} \langle 0,0\rangle\\ &= \langle 0,0\rangle \to_{189} \langle 0,0\rangle = \langle 0,0\rangle,\end{aligned}$$

which is an IFT. □

We obtain the same result, if we change operation $\vee$ with operation $\times_3$.

Fifth, following [1], we discuss the well-known Contraposition Law

$$(x \to_{189} y) \to_{189} (\neg y \to_{189} \neg x).$$

Theorem 2. The Contraposition Law is an IFT, but not a tautology as for classical negation, as well as for the new negation $\neg^*$.

Proof. Sequentially, we obtain:

$$\begin{aligned}
&(x \to_{189} y) \to_{189} (\neg y \to_{189} \neg x)\\
&= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \to_{189} (\neg\langle c,d\rangle \to_{189} \neg\langle a,b\rangle)\\
&= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \to_{189} (\langle d,c\rangle \to_{189} \langle b,a\rangle)\\
&= \langle bc, ad\rangle \to_{189} \langle bc, ad\rangle\\
&= \langle abcd, abcd\rangle,
\end{aligned}$$

which is an IFT.

$$\begin{aligned}
&(x \to_{189} y) \to_{189} (\neg^* y \to_{189} \neg^* x)\\
&= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \to_{189} (\neg^*\langle c,d\rangle \to_{189} \neg^*\langle a,b\rangle)\\
&= (\langle a,b\rangle \to_{189} \langle c,d\rangle) \to_{189} (\langle 0,c\rangle \to_{189} \langle 0,a\rangle)\\
&= \langle bc, ad\rangle \to_{189} \langle 0,0\rangle\\
&= \langle 0,0\rangle,
\end{aligned}$$

which is an IFT. □

Now, we check the validity of Klir and Yuan's axioms for fuzzy implications (marked by $I(x,y)$) [5], but in the intuitionistic fuzzy version from [1]:

Axiom $A1$ $(\forall x,y)(x \le y \to (\forall z)(I(x,z) \ge I(y,z)))$,
Axiom $A2$ $(\forall x,y)(x \le y \to (\forall z)(I(z,x) \le I(z,y)))$,
Axiom $A3$ $(\forall y)(I(0,y) = 1)$,
Axiom $A4$ $(\forall y)(I(1,y) = y)$,
Axiom $A5$ $(\forall x)(I(x,x) = 1)$,
Axiom $A6$ $(\forall x,y,z)(I(x,I(y,z)) = I(y,I(x,z)))$,
Axiom $A7$ $(\forall x,y)(I(x,y) = 1$ iff $x \le y)$,
Axiom $A8$ $(\forall x,y)(I(x,y) = I(N(y),N(x)))$,
Axiom $A9$ I is a continuous function.

For our research, having in mind the specific forms of the intuitionistic fuzzy implication $\to_{189}$ and following [1], we modify three of these axioms, as follows.

Axiom $A3^*$ $(\forall y)(I(0,y)$ is an IFT,
Axiom $A5^*$ $(\forall x)(I(x,x)$ is an IFT),
Axiom $A7^*$ $(\forall x,y)($ if $x \le y$, then, $I(x,y)$ is an IFT).

Theorem 5. Intuitionistic fuzzy implication $\to_{189}$ satisfies axioms $A1$, $A2$, $A3^*$, $A5^*$, $A6$, $A7^*$, $A8$ (for the classical negation N) and $A9$.

Proof. Let $x = \langle a, b\rangle, y = \langle c, d\rangle, z = \langle e, f\rangle$. We obtain sequentially. Let $x \leq y$. Then for $A1$ is valid:

$$I(x, z) = \langle be, af\rangle \geq \langle de, cf\rangle = I(y, z).$$

The checks for $A2$ is similar. For $A3^*$ we have

$$I(0, y) = \langle c, 0\rangle,$$

i.e., $A3$ is not valid, but $A3^*$ is valid, while $A4$ is not valid, because

$$I(1, y) = \langle 0, d\rangle$$

is not an IFT. For $A5^*$ we obtain

$$I(x, x) = \langle a, b\rangle \rightarrow_{189} \langle a, b\rangle = \langle ab, ab\rangle.$$

Therefore, $A5^*$ is valid, while $A5$ is not. For $A6$ we have:

$$I(x, I(y, z)) = \langle a, b\rangle \rightarrow_{189} (\langle c, d\rangle \rightarrow_{189} \langle e, f\rangle)$$

$$= \langle a, b\rangle \rightarrow_{189} \langle de, cf\rangle$$

$$= \langle bde, acf\rangle$$

$$= \langle c, d\rangle \rightarrow_{189} \langle be, af\rangle$$

$$= \langle c, d\rangle \rightarrow_{189} (\langle a, b\rangle \rightarrow_{189} \langle e, f\rangle) = I(y, I(x, z)),$$

i.e., this axiom is valid.

Let $x \leq y$, i.e., $a \leq c$ and $b \geq d$. Then,

$$I(x, y) = \langle bc, ad\rangle,$$

which is an IFT. Therefore, $A7^*$ is valid, but $A7$ is not valid.

From

$$I(N(y), N(x)) = \neg^*\langle c, d\rangle \rightarrow_{189} \neg^*\langle a, b\rangle$$

$$= \langle 0, c\rangle \rightarrow_{189} \langle 0, a\rangle = \langle 0, 0\rangle \neq I(x, y),$$

it follows that $A8$ is not valid for the new negation, while,

$$I(N(y), N(x)) = \neg\langle c, d\rangle \rightarrow_{189} \neg\langle a, b\rangle$$

$$= \langle d, c\rangle \rightarrow_{189} \langle b, a\rangle = \langle bc, ad\rangle = I(x, y),$$

i.e., $A8$ is valid.

Finally, obviously, $A9$ is valid. □

3 Conclusion

In a next step of this leg of research, other properties of the implication $\rightarrow_{189}$ will be introduced and studied. For example, we will check the validity of axioms of intuitionistic logic, Kolmogorov's axioms and others.

References

[1] Atanassov, K. (2017). *Intuitionistic Fuzzy Logics*. Springer, Cham.

[2] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[3] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017). Intuitionistic fuzzy implication $\rightarrow_{189}$. *Notes on Intuitionistic Fuzzy Sets*, 23(1), 6–13.

[4] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017). Multiplicative type of operations over intuitionistic fuzzy pairs. *Proceedings of FQAS'17*, London, 21–22 June 2017, Springer (in press).

[5] Klir, G., & Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic*. Prentice Hall, New Jersey.

[6] Plisko, V. (2009). A survey of propositional realizability logic. *The Bulletin of Symbolic Logic*, 15(1), 1–42.

[7] Rose, G. F. (1953). Propositional calculus and realizability. *Transactions of the American Mathematical Society*, 75, 1–19.