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Fractional differential equations with intuitionistic fuzzy data

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Abstract: The purpose of this paper is to study the existence and uniqueness of solution for fractional differential equation with intuitionistic fuzzy data where the intuitionistic fuzzy fractional derivatives and integral are considered in the Riemann–Liouville sense. Finally we give an example.
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1 Introduction

Fractional differential equations are a powerful tool for modeling many systems in various areas of sciences. There are many systems in nature with a complex behavior and fractional order model capture the properties of these kinds of systems but classical integer order model neglect such properties. Fractional differential equations have played an important role in many fields such as astrophysics, electronics, diffusion, material theory, chemistry, control theory, wave propagation, signal theory, electricity and thermodynamics [4, 6].

The idea of intuitionistic fuzzy set was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy set. Many authors develop the theory of intuitionistic fuzzy set in the different fields. As we give a sense of the concepts of intuitionistic fuzzy fractional integral and derivative in Caputo sense and the existence and uniqueness of mild solution for intuitionistic fuzzy fractional equation are discussed using the concept of semigroup and Banach fixed point method [8]. In this paper we discuss the existence and uniqueness solution of fractional differential equation with intuitionistic fuzzy initial value and we give an example.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

Definition 2.1.

$$IF_1 = IF(\mathbb{R}) = \left\{ \langle u, v \rangle \ : \ \mathbb{R} \to [0,1]^2 , \ |\forall x \in \mathbb{R} \ , \ 0 \leq u(x) + v(x) \leq 1 \right\} .$$

An element $\langle u, v \rangle$ of IF_1 is called intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v \rangle$ *is normal, i.e., there exists* x_0, $x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$*;*

(ii) u *is fuzzy convex and* v *is fuzzy concave;*

(iii) u *is upper semi-continuous and* v *is lower semi-continuous;*

(iv) *supp* $\langle u, v \rangle = cl\{x \in \mathbb{R} : v(x) < 1\}$ *is bounded.*

Thus, we denote the collection of all intuitionistic fuzzy number by IF_1.

For $\alpha \in [0, 1]$ and $\langle u, v \rangle \in \mathrm{IF}^1$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$[\langle u, v \rangle]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u, v \rangle]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\} .$$

Remark 2.1. *If $\langle u, v \rangle$ is an intuitionistic fuzzy number, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{\langle 1,0 \rangle} \in \mathrm{IF}_1$ as

$$0_{\langle 1,0 \rangle}(t) = \begin{cases} \langle 1, 0 \rangle & t = 0 \\ \langle 0, 1 \rangle & t \neq 0 \end{cases} .$$

Let $\langle u, v \rangle$,$\langle u', v' \rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by:

$$\Big(\langle u, v \rangle \oplus \langle u', v' \rangle \Big)(z) = \Big(\sup_{z=x+y} \min \left(u(x), u'(y)\right) , \inf_{z=x+y} \max \left(v(x), v'(y)\right) \Big),$$

$$\lambda \langle u, v \rangle = \begin{cases} \langle \lambda u, \lambda v \rangle \, , & \text{if } \lambda \neq 0 \\ 0_{\langle 1,0 \rangle}, & \text{if } \lambda = 0 \end{cases} .$$

For $\langle u, v \rangle$, $\langle z, w \rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, the addition and scale-multiplication are defined as follows

$$\Big[\langle u, v \rangle \oplus \langle z, w \rangle \Big]^\alpha = \Big[\langle u, v \rangle \Big]^\alpha + \Big[\langle z, w \rangle \Big]^\alpha, \quad \Big[\lambda \langle z, w \rangle \Big]^\alpha = \lambda \Big[\langle z, w \rangle \Big]^\alpha,$$
$$\Big[\langle u, v \rangle \oplus \langle z, w \rangle \Big]_\alpha = \Big[\langle u, v \rangle \Big]_\alpha + \Big[\langle z, w \rangle \Big]_\alpha, \quad \Big[\lambda \langle z, w \rangle \Big]_\alpha = \lambda \Big[\langle z, w \rangle \Big]_\alpha .$$

Definition 2.2. *Let $\langle u,v\rangle$ an element of IF_1 and $\alpha\in[0,1]$, we define the following sets:*

$$\begin{aligned}\Big[\langle u,v\rangle\Big]_l^+(\alpha) &= \inf\{x\in\mathbb{R}\mid u(x)\geq\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^+(\alpha)=\sup\{x\in\mathbb{R}\mid u(x)\geq\alpha\},\\ \Big[\langle u,v\rangle\Big]_l^-(\alpha) &= \inf\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}, \ \Big[\langle u,v\rangle\Big]_r^-(\alpha)=\sup\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}.\end{aligned}$$

Remark 2.2.

$$\begin{aligned}\Big[\langle u,v\rangle\Big]_\alpha &= \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha),\Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big],\\ \Big[\langle u,v\rangle\Big]^\alpha &= \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha),\Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big].\end{aligned}$$

Proposition 2.1. *For all α, $\beta\in[0,1]$ and $\langle u,v\rangle\in IF_1$*

(i) $\Big[\langle u,v\rangle\Big]_\alpha\subset\Big[\langle u,v\rangle\Big]^\alpha$*;*

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$*;*

(iii) if $\alpha\leq\beta$ *then* $\Big[\langle u,v\rangle\Big]_\beta\subset\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta\subset\Big[\langle u,v\rangle\Big]^\alpha$*;*

(iv) If $\alpha_n\nearrow\alpha$ *then* $\Big[\langle u,v\rangle\Big]_\alpha=\bigcap_n\Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha=\bigcap_n\Big[\langle u,v\rangle\Big]^{\alpha_n}$.

Let M any set and $\alpha\in[0,1]$ we denote by

$$M_\alpha=\{x\in\mathbb{R}\ :\ u(x)\geq\alpha\}\quad\text{and}\quad M^\alpha=\{x\in\mathbb{R}\ :\ v(x)\leq 1-\alpha\}.$$

Lemma 2.1. *[7] Let* $\Big\{M_\alpha,\ \alpha\in[0,1]\Big\}$ *and* $\Big\{M^\alpha,\ \alpha\in[0,1]\Big\}$ *be two families of subsets of* $\mathbb{R}$ *satisfying (i)–(iv) in proposition 2.1, if u and v are defined by*

$$u(x)=\begin{cases}0 & \textit{if } x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & \textit{if } x\in M_0\end{cases};$$

$$v(x)=\begin{cases}1 & \textit{if } x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & \textit{if } x\in M^0\end{cases}.$$

Then $\langle u,v\rangle\in IF_1$.

Let $\langle u,v\rangle\in \mathrm{IF}_1$ such that $\Big[\langle u,v\rangle(t)\Big]_\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t),\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\Big]$, $\Big[\langle u,v\rangle(t)\Big]^\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t),\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big]$ for all $t\in(0,a]$ and $q\in\mathbb{R}_+$ with $k=[q]$ (where $[q]$ is the largest integer less or equal to q).

We define

$$\phi_q(t)=\begin{cases}\frac{t^{q-1}}{\Gamma(q)} & t>0,\\ 0, & t\leq 0\end{cases}$$

and

$$\begin{aligned}\phi_{-q}(t) &= \phi_{1+k-q}(t)*\delta^{1+k}(t) \qquad k=[q]\\ \phi_{-n}(t) &= \delta^{n}(t) \qquad n=0,1,2...\end{aligned}$$

with the property $\phi_q(t)*\phi_p(t)=\phi_{q+p}(t)$ for $p>0$, where $\delta^n(t)$ is the n^{th} derivative of the delta function and $\Gamma(.)$ is the gamma function (for the properties of $\phi_q(t)$ see [3] and [5]). Suppose that $\Big[\langle u,v\rangle\Big]_l^+(\alpha), \Big[\langle u,v\rangle\Big]_r^+(\alpha), \Big[\langle u,v\rangle\Big]_l^-(\alpha), \Big[\langle u,v\rangle\Big]_r^-(\alpha) \in C((0,T],\mathbb{R})\cap L^1((0,a),\mathbb{R})$ for all $\alpha\in[0,1]$ and let

$$\begin{aligned}A_\alpha &:= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)ds\Big]\\ &:= \Big[\phi_q(t)*\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), \phi_q(t)*\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t)\Big]; \qquad (2.1)\end{aligned}$$

$$\begin{aligned}A^\alpha &:= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)ds\Big]\\ &:= \Big[\phi_q(t)*\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), \phi_q(t)*\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big]. \qquad (2.2)\end{aligned}$$

Lemma 2.2. *The family $\{A_\alpha, A^\alpha;\alpha\in[0,1]\}$, given by (2.1) and (2.2), defines an intuitionistic fuzzy number $\langle u,v\rangle\in IF_1$ such that $\Big[\langle u,v\rangle\Big]_\alpha = A_\alpha$ and $\Big[\langle u,v\rangle\Big]^\alpha = A^\alpha$.*

Definition 2.3. *Let $\langle u,v\rangle\in\mathcal{C}((0,a],IF_1)\cap L^1((0,a),IF_1)$. Define the intuitionistic fuzzy fractional primitive of order $q>0$ of $\langle u,v\rangle$ in the Riemann–Liouville sense*

$$I^q\langle u,v\rangle(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\langle u,v\rangle(s)ds, \quad t\in(0,a)$$

by

$$\begin{aligned}\big[I^q\langle u,v\rangle(t)\big]_\alpha &= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)ds\Big], \quad t\in(0,a)\\ &= \Big[\phi_q(t)*\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), \phi_q(t)*\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\Big], \quad t\in(0,a)\\ \big[I^q\langle u,v\rangle(t)\big]^\alpha &= \frac{1}{\Gamma(q)}\Big[\int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)ds, \int_0^t (t-s)^{q-1}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)ds\Big], \quad t\in(0,a)\\ &= \Big[\phi_q(t)*\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), \phi_q(t)*\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big], \quad t\in(0,a)\end{aligned}$$

Definition 2.4. *Let $\langle u,v\rangle\in\mathcal{C}^{1+k}((0,a],IF_1)\cap L^1((0,a),IF_1)$. The intuitionistic fuzzy fractional derivative of order $q>0$ of $\langle u,v\rangle$ in the Riemann–Liouville sense is defined*

$$\begin{aligned}D^q\langle u,v\rangle(t) &= D^{1+k}\langle u,v\rangle(t)*\phi_{1+k-q}(t)\\ &= \frac{1}{\Gamma(1+k-q)}\frac{d^{1+k}}{dt^{1+k}}\int_0^t (t-s)^{k-q}\langle u,v\rangle(s)ds\end{aligned}$$

by

$$\begin{aligned}[D^q\langle u,v\rangle(t)]_\alpha &= \Big[D^{1+k}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t)*\phi_{1+k-q}(t), D^{1+k}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)*\phi_{1+k-q}(t)\Big]\\ &= \frac{1}{\Gamma(1+k-q)}\Big[\frac{d^{1+k}}{dt^{1+k}}\int_0^t (t-s)^{k-q}\Big[\langle u,v\rangle\Big]_l^+(\alpha)(s)ds, \frac{d^{1+k}}{dt^{1+k}}\int_0^t (t-s)^{k-q}\Big[\langle u,v\rangle\Big]_r^+(\alpha)(s)ds\Big]\end{aligned}$$

$$\begin{aligned}[D^q \langle u,v\rangle (t)]^\alpha &= \left[D^{1+k}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t)*\phi_{1+k-q}(t), D^{1+k}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)*\phi_{1+k-q}(t)\right]\\ &= \frac{1}{\Gamma(1+k-q)}\left[\frac{d^{1+k}}{dt^{1+k}}\int_0^t (t-s)^{k-q}\Big[\langle u,v\rangle\Big]_l^-(\alpha)(s)ds, \frac{d^{1+k}}{dt^{1+k}}\int_0^t (t-s)^{k-q}\Big[\langle u,v\rangle\Big]_r^-(\alpha)(s)ds\right]\end{aligned}$$

provided that equations define an intuitionistic fuzzy number $D^q \langle u,v\rangle (t) \in \mathrm{IF}_1$. In fact

$$\Big[D^q \langle u,v\rangle (t)\Big]_\alpha := \left[D^q\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t), D^q\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\right] \text{ for all } t\in(0,a] \text{ and } \alpha\in[0,1]$$

$$\Big[D^q \langle u,v\rangle (t)\Big]^\alpha := \left[D^q\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t), D^q\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\right] \text{ for all } t\in(0,a] \text{ and } \alpha\in[0,1]$$

Example 2.1. *Let $\langle u,v\rangle : (0,a] \to IF_1$ be a constant intuitionistic fuzzy function, i.e., $\langle u,v\rangle (t) = c$ for $t\in(0,a]$ and $0<q\leq 1$. If $[c]_\alpha = [c_1^+(\alpha), c_2^+(\alpha)]$ and $[c]^\alpha = [c_1^-(\alpha), c_2^-(\alpha)]$ then*

$$\begin{aligned}[D^q \langle u,v\rangle (t)]_\alpha &= \frac{1}{\Gamma(1-q)}\left[\frac{d}{dt}\int_0^t (t-s)^{-q}c_1^+(\alpha)ds, \frac{d}{dt}\int_0^t (t-s)^{-q}c_2^+(\alpha)ds\right]\\ &= \frac{t^{-q}}{\Gamma(1-q)}[c_1^+(\alpha), c_2^+(\alpha)],\end{aligned}$$

$$\begin{aligned}[D^q \langle u,v\rangle (t)]^\alpha &= \frac{1}{\Gamma(1-q)}\left[\frac{d}{dt}\int_0^t (t-s)^{-q}c_1^-(\alpha)ds, \frac{d}{dt}\int_0^t (t-s)^{-q}c_2^-(\alpha)ds\right]\\ &= \frac{t^{-q}}{\Gamma(1-q)}[c_1^-(\alpha), c_2^-(\alpha)],\end{aligned}$$

that is $D^q c = \frac{t^{-q}}{\Gamma(1-q)}c$ for every $c\in IF_1$.

3 Fractional differential equations with intuitionistic fuzzy data

Let $0<q\leq 1$, We consider the initial value problem

$$\begin{cases} D^q r(t) = f(t,r(t)) \\ \lim\limits_{t\to 0^+} t^{1-q} r(t) = \langle u_0,v_0\rangle \end{cases}, \tag{3.1}$$

where f is a continuous mapping from $[0,a]\times\mathbb{R}$ into $\mathbb{R}$ and $\langle u_0,v_0\rangle = r_0$ is an intuitionistic fuzzy number with α-level intervals $\big[\langle u_0,v_0\rangle\big]_\alpha = \Big[[\langle u_0,v_0\rangle]_l^+(\alpha), [\langle u_0,v_0\rangle]_r^+(\alpha)\Big]$ and $\Big[\langle u_0,v_0\rangle\Big]^\alpha = \Big[[\langle u_0,v_0\rangle]_l^-(\alpha), [\langle u_0,v_0\rangle]_r^-(\alpha)\Big]$, $0<\alpha\leq 1$.

The extension principle of Zadeh leads to the following definition of $f(t,r)$ when r is an intuitionistic fuzzy number, i.e., $r = \langle u,v\rangle \in \mathrm{IF}_1$

$$f(t,r)(y) = \left(\sup\Big\{u(x), y=f(t,x)\ ,\ \forall x\in\mathbb{R}\Big\}, \inf\Big\{v(x), y=f(t,x)\ ,\ \forall x\in\mathbb{R}\Big\}\right).$$

It follows that:

$$\Big[f(t,r)\Big]_\alpha = \left[\min\Big\{f(t,x)\ :\ x\in\Big[[\langle u,v\rangle]_l^+(\alpha), [\langle u,v\rangle]_r^+(\alpha)\Big]\Big\}, \max\Big\{f(t,x)\ :\ x\in\Big[[\langle u,v\rangle]_l^+(\alpha), [\langle u,v\rangle]_r^+(\alpha)\Big]\Big\}\right]$$

$$\Big[f(t,r)\Big]^\alpha = \left[\min\Big\{f(t,x)\ :\ x\in\Big[[\langle u,v\rangle]_l^-(\alpha), [\langle u,v\rangle]_r^-(\alpha)\Big]\Big\}, \max\Big\{f(t,x)\ :\ x\in\Big[[\langle u,v\rangle]_l^-(\alpha), [\langle u,v\rangle]_r^-(\alpha)\Big]\Big\}\right]$$

for $r \in \mathrm{IF}_1$, with α-level intervals $[r]_\alpha = [\langle u, v\rangle]_\alpha$ and $[r]^\alpha = [\langle u, v\rangle]^\alpha$.

Definition 3.1. *A function $\langle u, v\rangle : [0, a] \to IF_1$ is called an integral solution for* (3.1) *if $f \in C\Big((0, a], IF_1\Big) \cap L^1\Big((0, a), IF_1\Big)$ and the following equation holds on $[0, a]$*

$$\langle u, v\rangle (t) = t^{q-1} \langle u_0, v_0\rangle + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\Big(s, \langle u, v\rangle (s)\Big) ds.$$

We call $\langle u, v\rangle : (0, a] \to \mathrm{IF}_1$ an intuitionistic fuzzy solution of (3.1) if:

$$\begin{array}{ll}
D^q[\langle u,v\rangle]_l^+(\alpha)(t) = \min\big\{f(t,x) \ : \ x \in \big[[\langle u,v\rangle]_l^+(\alpha)(t), [\langle u,v\rangle]_r^+(\alpha)(t)\big]\big\}, & \lim\limits_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^+(\alpha)(t) = [\langle u_0,v_0\rangle]_l^+(\alpha) \\
D^q[\langle u,v\rangle]_r^+(\alpha)(t) = \max\big\{f(t,x) \ : \ x \in \big[[\langle u,v\rangle]_l^+(\alpha)(t), [\langle u,v\rangle]_r^+(\alpha)(t)\big]\big\}, & \lim\limits_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^+(\alpha)(t) = [\langle u_0,v_0\rangle]_r^+(\alpha) \\
D^q[\langle u,v\rangle]_l^-(\alpha)(t) = \min\big\{f(t,x) \ : \ x \in \big[[\langle u,v\rangle]_l^-(\alpha)(t), [\langle u,v\rangle]_r^-(\alpha)(t)\big]\big\}, & \lim\limits_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^-(\alpha)(t) = [\langle u_0,v_0\rangle]_l^-(\alpha) \\
D^q[\langle u,v\rangle]_r^-(\alpha)(t) = \max\big\{f(t,x) \ : \ x \in \big[[\langle u,v\rangle]_l^-(\alpha)(t), [\langle u,v\rangle]_r^-(\alpha)(t)\big]\big\}, & \lim\limits_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^-(\alpha)(t) = [\langle u_0,v_0\rangle]_r^-(\alpha)
\end{array}$$

for $t \in (0, a]$ and $0 < \alpha \le 1$.

Denote $\tilde{f} = (f_1, f_2, f_3, f_4)$ where

$$\begin{aligned}
f_1(t,u) =&\ \min\big\{f(t,x) \ : \ x \in [u_1, u_2]\big\} \\
f_2(t,u) =&\ \max\big\{f(t,x) \ : \ x \in [u_1, u_2]\big\} \\
f_3(t,u) =&\ \min\big\{f(t,x) \ : \ x \in [v_1, v_2]\big\} \\
f_4(t,u) =&\ \max\big\{f(t,x) \ : \ x \in [v_1, v_2]\big\}
\end{aligned}$$

where $u = (u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ with $v_1 \le u_1 \le u_2 \le v_2$.

Thus for fixed α, we have initial value problems in $\mathbb{R}^4$:

$$\begin{aligned}
D^q[\langle u,v\rangle]_l^+(\alpha)(t) &= \tilde{f}\Big(t, [\langle u,v\rangle]_l^+(\alpha)(t), [\langle u,v\rangle]_r^+(\alpha)(t)\Big), \ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^+(\alpha)(t) = [\langle u_0,v_0\rangle]_l^+(\alpha) \\
D^q[\langle u,v\rangle]_r^+(\alpha)(t) &= \tilde{f}\Big(t, [\langle u,v\rangle]_l^+(\alpha)(t), [\langle u,v\rangle]_r^+(\alpha)(t)\Big), \ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^+(\alpha)(t) = [\langle u_0,v_0\rangle]_r^+(\alpha) \\
D^q[\langle u,v\rangle]_l^-(\alpha)(t) &= \tilde{f}\Big(t, [\langle u,v\rangle]_l^-(\alpha)(t), [\langle u,v\rangle]_r^-(\alpha)(t)\Big), \ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^-(\alpha)(t) = [\langle u_0,v_0\rangle]_l^-(\alpha) \\
D^q[\langle u,v\rangle]_r^-(\alpha)(t) &= \tilde{f}\Big(t, [\langle u,v\rangle]_l^-(\alpha)(t), [\langle u,v\rangle]_r^-(\alpha)(t)\Big), \ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^-(\alpha)(t) = [\langle u_0,v_0\rangle]_r^-(\alpha)
\end{aligned} \tag{3.2}$$

If we can solve their (uniquely), we have only to verify that the intervals

$$\Big[[\langle u,v\rangle]_l^+(\alpha)(t), [\langle u,v\rangle]_r^+(\alpha)(t)\Big]$$

and

$$\Big[[\langle u,v\rangle]_l^-(\alpha)(t), [\langle u,v\rangle]_r^-(\alpha)(t)\Big],$$

$0 < \alpha \le 1$, define an intuitionistic fuzzy number $\langle u, v\rangle (t) \in \mathrm{IF}_1$.

Moreover, as f is assumed continuous, the initial value problem (3.2) is equivalent to the following fractional integral equation

$$r(t) = r_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \tilde{f}(s, r(s)) ds, \ 0 \le t \le a, \tag{3.3}$$

where

$$r_0(t) = \frac{t^{q-1} r_0}{\Gamma(q)}.$$

Theorem 3.1. *Assume that*

a) $f \in \mathcal{C}([0,a] \times \mathbb{R} \to \mathbb{R})$ *and* $\mid f(t,u) \mid \leq M_0$ *on* $[0,a] \times [0,b]$;

b) $g \in \mathcal{C}([0,a] \times [0,b] \to \mathbb{R}_+)$, $g(t,m) \leq M_1$ *on* $[0,a] \times [0,b]$, $g(t,0) \equiv 0$, $g(t,m)$ *is nondecreasing in* m *for each* t *and* $m(t) \equiv 0$ *is the only solution of*

$$D^q m(t) = g(t, m(t)),\ t \in (0,a] \tag{3.4}$$

with the initial condition $\lim\limits_{t \to 0^+} t^{1-q} m(t) = 0$;

c)

$$\mid f(t,u) - f(t,\overline{u}) \mid \leq g(t, \mid u - \overline{u} \mid), t \geq 0,\ u, \overline{u} \in \mathbb{R}; \tag{3.5}$$

d) *solutions* $m(t, m_0)$ *of (3.4) are continuous with respect to the initial condition* $m_0 = \lim\limits_{t \to 0^+} t^{1-q} m(t)$.

Then the initial value problem (3.1) has a unique intuitionistic fuzzy solution.

Proof 3.1. *It can be shown that (3.5) implies*

$$\parallel \tilde{f}(t,u) - \tilde{f}(t,\overline{u}) \parallel \leq g(t, \parallel u - \overline{u} \parallel), t \geq\ u, \overline{u} \in \mathbb{R}^4, \tag{3.6}$$

where the $\parallel . \parallel$ *is defined by* $\parallel u \parallel = max\{|u_1|, |u_2|, |u_3|, |u_4|\}$. *It is well known that (3.6) and the assumptions on* g *[[10] Theorems 2.1 and 2.2] guarantee the existence, uniqueness and continuous dependence on initial value of the solution to*

$$D^q r(t) = \tilde{f}(t, r(t)),\ t \in (0,a],\ \ \lim_{t \to 0^+} t^{1-q} r(t) = r_0 \in \mathbb{R}^4 \tag{3.7}$$

and that for any continuous function $r_0 : \mathbb{R}^+ \to \mathbb{R}^4$ *the successive approximations*

$$r_{n+1}(t) = r_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \tilde{f}(s, r_n(s)) ds, n = 0, 1, \ldots \tag{3.8}$$

converge uniformly on closed subintervals of $\mathbb{R}^+$ *to the solutions of (3.7). By choosing* $r_0 = \left(u^\alpha_{01}, u^\alpha_{02}, v^\alpha_{01}, v^\alpha_{02}\right)$ *in (3.7) with* $v^\alpha_{01} \leq u^\alpha_{01} \leq u^\alpha_{02} \leq v^\alpha_{02}$ *we get a unique solution* $r^\alpha(t) = \left(u^\alpha_1(t), u^\alpha_2(t), v^\alpha_1(t), v^\alpha_2(t)\right)$ *to (3.2) for each* $\alpha \in (0,1]$.

In the sequel we will prove that the intervals $[u^\alpha_1, u^\alpha_2]$ *and* $[v^\alpha_1, v^\alpha_2]$ *define an intuitionistic fuzzy number* $r(t) \in$ ***IF***$_1$ *for each* $t \geq 0$ *with* $[r_0]_\alpha = [u^\alpha_{01}, u^\alpha_{02}] \subseteq [r_0]^\alpha = [v^\alpha_{01}, v^\alpha_{02}]$, *i.e.,* $r(t)$ *is an intuitionistic fuzzy solution to* (3.2).

The successive approximation $r_0(t) = r_0 \in$ ***IF***$_1$

$$r_{n+1}(t) = r_0(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \tilde{f}(s, r_n(s)) ds, n = 0, 1, \ldots$$

for all $t \geq 0$ *define a sequence of intuitionistic fuzzy numbers* $\{r_n(t)\} \in$ ***IF***$_1$.

Then, if $0 \leq \alpha \leq \beta \leq 1$ *then* $\left[r_n(t)\right]_\beta \subseteq \left[r_n(t)\right]_\alpha$ *and* $\left[r_n(t)\right]^\beta \subseteq \left[r_n(t)\right]^\alpha$ *since by the convergence of sequence* (3.8) *the end points of* $\left[r_n(t)\right]_\alpha$ *and* $\left[r_n(t)\right]^\alpha$ *converge to* $u^\alpha_1(t), u^\alpha_2(t), v^\alpha_1(t)$

and $v_2^\alpha(t)$ *respectively. Then* $\left[u_1^\beta(t), u_2^\beta(t)\right] \subseteq \left[u_1^\alpha(t), u_2^\alpha(t)\right]$ *and* $\left[v_1^\beta(t), v_2^\beta(t)\right] \subseteq \left[v_1^\alpha(t), v_2^\alpha(t)\right]$, *i.e., the property (iii) of Lemma* (2.1) *holds. Now, let consider a nondecreasing sequence* $\{\alpha_k\}$ *in* $(0,1]$ *converging to* α *as,* $r_0(t) \in IF_1$ *then,* $u_{01}^{\alpha_k}, u_{02}^{\alpha_k}$, $v_{01}^{\alpha_k}$ *and* $v_{02}^{\alpha_k}$ *converge respectively to* $u_{01}^\alpha, u_{02}^\alpha$, v_{01}^α *and* v_{02}^α *and by the the continuous dependence on the initial value of the solution of* (3.7), $u_1^{\alpha_k}(t), u_2^{\alpha_k}(t)$, $v_1^{\alpha_k}(t)$ *and* $v_2^{\alpha_k}(t)$ *converge respectively to* $u_1^\alpha(t), u_2^\alpha(t)$, $v_1^\alpha(t)$ *and* $v_2^\alpha(t)$, *i.e.,* $[u_1^\alpha, u_2^\alpha] = \lim_{k\to\infty}[u_1^{\alpha_k}, u_2^{\alpha_k}] = \bigcap_{k\geq 0}[u_1^{\alpha_k}, u_2^{\alpha_k}]$ *and* $[v_1^\alpha, v_2^\alpha] = \lim_{k\to\infty}[v_1^{\alpha_k}, v_2^{\alpha_k}] = \bigcap_{k\geq 0}[v_1^{\alpha_k}, v_2^{\alpha_k}]$. *Thus the property (iv) holds.*

To prove that $r(t) = \langle u, v\rangle(t)$ *is an intuitionistic fuzzy solution it remains to show that* $[\langle u, v\rangle(t)]_\alpha \subseteq [\langle u, v\rangle(t)]^\alpha$.

We have

$$\lim_{t\to 0^+} t^{1-q} v_1^\alpha(t) = v_{01}^\alpha \leq \lim_{t\to 0^+} t^{1-q} u_1^\alpha(t) = u_{01}^\alpha$$

and

$$\lim_{t\to 0^+} t^{1-q} u_2^\alpha(t) = u_{02}^\alpha \leq \lim_{t\to 0^+} t^{1-q} v_2^\alpha(t) = v_{02}^\alpha,$$

then

$$t^{1-q} v_1^\alpha(t) \leq t^{1-q} u_1^\alpha(t)$$

and

$$t^{1-q} u_2^\alpha(t) \leq t^{1-q} v_2^\alpha(t)$$

for all t *belonging to the neighborhood of* 0^+.

So $[\langle u, v\rangle(t)]_\alpha \subseteq [\langle u, v\rangle(t)]^\alpha$.

Hence, by Lemma (2.1), $\langle u, v\rangle(t) \in IF_1$ *and so* $r(t) = \langle u, v\rangle(t)$ *is an intuitionistic fuzzy solution of (3.1). The uniqueness follows from the uniqueness of the solution of (3.7).* □

4 Example

Consider the crisp differential equation

$$D^q r(t) = -r(t) \tag{4.1}$$

with the intuitionistic fuzzy initial condition

$$\lim_{t\to 0^+} t^{1-q} r(t) = \langle 1, 2, 3; 0, 2, 5\rangle, \tag{4.2}$$

where $t \in (0, a], 0 < q \leq 1$, and $\langle u_0, v_0\rangle = \langle 1, 2, 3; 0, 2, 5\rangle \in \mathrm{IF}_1$ is an intuitionistic fuzzy triangular number, that is, $[\langle u_0, v_0\rangle]_\alpha = [1+\alpha, 3-\alpha]$ and $[\langle u_0, v_0\rangle]^\alpha = [2\alpha, 5-3\alpha]$ for $\alpha \in (0, 1]$.

If we put

$$\left[r(t)\right]_\alpha = \left[[\ \langle u, v\rangle]_l^+(\alpha)(t), [\langle u, v\rangle]_r^+(\alpha)(t)\right]$$

and

$$[r(t)]^\alpha = \left[[\langle u, v\rangle]_l^-(\alpha)(t), [\langle u, v\rangle]_r^-(\alpha)(t)\right],$$

then

$$\left[D^q r(t)\right]_\alpha = \left[D^q[\langle u, v\rangle]_l^+(\alpha)(t), D^q[\langle u, v\rangle]_r^+(\alpha)(t)\right]$$

and

$$\left[D^q r(t)\right]^\alpha = \left[D^q[\langle u,v\rangle]_l^-(\alpha)(t), D^q[\langle u,v\rangle]_r^-(\alpha)(t)\right].$$

We obtain the system

$$\begin{aligned}
D^q[\langle u,v\rangle]_l^+(\alpha)(t) &= -[\langle u,v\rangle]_r^+(\alpha)(t)\,,\ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^+(\alpha)(t) = 1+\alpha\\
D^q[\langle u,v\rangle]_r^+(\alpha)(t) &= -[\langle u,v\rangle]_l^+(\alpha)(t),\ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^+(\alpha)(t) = 3-\alpha\\
D^q[\langle u,v\rangle]_l^-(\alpha)(t) &= -[\langle u,v\rangle]_r^-(\alpha)(t),\ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_l^-(\alpha)(t) = 2\alpha\\
D^q[\langle u,v\rangle]_r^-(\alpha)(t) &= -[\langle u,v\rangle]_l^-(\alpha)(t),\ \lim_{t\to 0^+} t^{1-q}[\langle u,v\rangle]_r^-(\alpha)(t) = 5-3\alpha
\end{aligned}$$

or

$$D^q x(t) = Ax(t),\ \lim_{t\to 0^+} t^{1-q}x(t) = c, \tag{4.3}$$

where $x(t) = \begin{pmatrix} [\langle u,v\rangle]_l^+(\alpha)(t)\\ [\langle u,v\rangle]_r^+(\alpha)(t)\\ [\langle u,v\rangle]_l^-(\alpha)(t)\\ [\langle u,v\rangle]_r^-(\alpha)(t)\end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{pmatrix}$, $c = \begin{pmatrix} 1+\alpha\\ 3-\alpha\\ 2\alpha\\ 5-3\alpha\end{pmatrix}$.

Using the same method that in [9], we obtain the solution of (4.3). It is given by

$$x(t) = t^{q-1}E_{q,q}(At^q)c = t^{q-1}E_{q,q}(At^q)\begin{pmatrix} 1+\alpha\\ 3-\alpha\\ 2\alpha\\ 5-3\alpha\end{pmatrix},$$

where $E_{q,q}(At^q)$ is Mittag–Leffler function of matric argument given by

$$E_{q,q}(At^q) = \sum_{k=0}^{\infty}\frac{(At^q)^k}{\Gamma(q(k+1))} = \begin{pmatrix} \sum_{n=0}^\infty \frac{t^{2nq}}{\Gamma(q(2n+1))} & 0 & 0 & 0\\ 0 & \sum_{n=0}^\infty \frac{t^{2nq}}{\Gamma(q(2n+1))} & 0 & 0\\ 0 & 0 & \sum_{n=0}^\infty \frac{t^{2nq}}{\Gamma(q(2n+1))} & 0\\ 0 & 0 & 0 & \sum_{n=0}^\infty \frac{t^{2nq}}{\Gamma(q(2n+1))}\end{pmatrix}$$

$$+\begin{pmatrix} 0 & -\sum_{n=0}^\infty \frac{t^{(2n+1)q}}{\Gamma(q(2n+2))} & 0 & 0\\ -\sum_{n=0}^\infty \frac{t^{(2n+1)q}}{\Gamma(q(2n+2))} & 0 & 0 & 0\\ 0 & 0 & 0 & -\sum_{n=0}^\infty \frac{t^{(2n+1)q}}{\Gamma(q(2n+2))}\\ 0 & 0 & -\sum_{n=0}^\infty \frac{t^{(2n+1)q}}{\Gamma(q(2n+2))} & 0\end{pmatrix}$$

Then we obtain

$$\begin{aligned}
[\langle u,v\rangle]_l^+(\alpha)(t) &= \sum_{n=0}^{\infty}\frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(1+\alpha)-\sum_{n=0}^{\infty}\frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(3-\alpha)\\
[\langle u,v\rangle]_r^+(\alpha)(t) &= \sum_{n=0}^{\infty}\frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(3-\alpha)-\sum_{n=0}^{\infty}\frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(1+\alpha)\\
[\langle u,v\rangle]_l^-(\alpha)(t) &= \sum_{n=0}^{\infty}\frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}2\alpha-\sum_{n=0}^{\infty}\frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(5-3\alpha)\\
[\langle u,v\rangle]_r^-(\alpha)(t) &= \sum_{n=0}^{\infty}\frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(5-3\alpha)-\sum_{n=0}^{\infty}\frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}2\alpha.
\end{aligned}$$

We observe that $[\langle u,v\rangle]_l^+(\alpha)(t)$ and $[\langle u,v\rangle]_l^-(\alpha)(t)$ are nondecreasing with respect to α and $[\langle u,v\rangle]_r^+(\alpha)(t)$ and $[\langle u,v\rangle]_r^-(\alpha)(t)$ are nonincreasing with respect to α.

Moreover,

$$[\langle u,v\rangle]_l^+(\alpha)(t)\le[\langle u,v\rangle]_r^+(\alpha)(t)$$

and

$$[\langle u,v\rangle]_l^-(\alpha)(t)\le[\langle u,v\rangle]_r^-(\alpha)(t),$$

also

$$[\langle u,v\rangle]_l^-(\alpha)(t)\le[\langle u,v\rangle]_l^+(\alpha)(t)$$

and

$$[\langle u,v\rangle]_r^+(\alpha)(t)\le[\langle u,v\rangle]_r^-(\alpha)(t)$$

for all $\alpha\in[0,1]$.

Thus, due to Lemma (2.1)

$$\Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha)(t),\Big[\langle u,v\rangle\Big]_r^+(\alpha)(t)\Big]$$

and

$$\Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha)(t),\Big[\langle u,v\rangle\Big]_r^-(\alpha)(t)\Big]$$

define the α-level intervals of an intuitionistic fuzzy number. So $\Big[r(t)\Big]_\alpha$ and $\Big[r(t)\Big]^\alpha$ are the lower and upper α-cut of the intuitionistic fuzzy solution of (4.1), (4.2).

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