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Convex intuitionistic fuzzy sets

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Abstract

In this paper the concept of convexity for intuitionistic fuzzy sets is introduced. A series of basic results are established.

1 Introduction

Convexity of fuzzy sets is studied and applied in several papers (for example [6], [7], [8]). In [8], M.Mashinchi has introduced a general convexity (concavity) by using the L t-norms and L t-conorms, where L is a complete lattice with the least element and the greatest element. In Section 2, we quote the results obtained in [8], for particular case $L = [0,1]$.

In Section 3, we introduce and study the properties of convex (concave) intuitionistic fuzzy sets (IFS) and in Section 4 we characterize the convexity of IFS by (α, β)-level sets generated by an IFS.

2 Basic definitions and results

In this section we recall some definitions and results (for details, see [1], [3], [4], [5], [8]).

Let $f : [0,1] \times [0,1] \to [0,1]$, satisfying the following conditions:

(1) $f(x,y) = f(x,y)$
(2) $f(x, f(y,z)) = f(f(x,y), z)$
(3) if $x \leq y$, then $f(x,z) \leq f(y,z)$
(4) $f(x,1) = x$ or (4)$'$ $f(x,0) = x$.

f is called a t-norm (written $\top$) iff it satisfies the conditions $(1)-(4)$. f is called a t-conorm (written $\perp$) iff it satisfies the conditions $(1)-(3)$ and $(4)'$.

Theorem 1 *Let f be a t-norm (t-conorm). We define $f^c : [0,1] \times [0,1] \to [0,1]$, by $f^c(x,y) = 1 - f(1-x, 1-y)$. Then f^c is a t-conorm (t-norm), which is called to be associated with f, or the dual of f.*

Definition 2 *([8])Let f, g be t-norms or t-conorms. Then we say*
(1) *g is stronger than f, denoted by $f \leq g$, if and only if $f(x,y) \leq g(x,y)$ for all $x, y \in [0,1]$.*
(2) *g dominate f, denoted by $f \ll g$, if and only if $f(g(x,y), g(z,w)) \leq g(f(x,z), f(y,w))$ for all $x, y, z, w \in [0,1]$.*

Lemma 3 *(see[8])Let $\top$ and $\perp$ be t-norm and t-conorm, respectively. Then*
(1) $\top \leq \wedge \leq \vee \leq \perp$
(2) $\top \leq \wedge \leq \vee \leq \top^c$
(3) $\top \ll \wedge$
(4) $\vee \ll \perp$
(5) $\vee \ll \wedge$
where $\wedge(x,y) = \min(x,y)$ and $\vee = \wedge^c$.

Lemma 4 *(see [8])Let f, g be both t-norms or t-conorms. Then*
(1) *$f \leq g$ if and only if $g^c \leq f^c$*
(2) *$f \ll g$ if and only if $g^c \ll f^c$.*

The Example 2.8 and the Definition 2.6 on [8] implies

Lemma 5 *Every continuous t-norm or t-conorm has the properties*

(1) $\bigwedge_{\alpha\in\Omega,\, \beta\in\Omega'} f(x_\alpha, y_\beta) \leq f(\bigwedge_{\alpha\in\Omega} x_\alpha, \bigwedge_{\beta\in\Omega'} y_\beta)$

(2) $f(\bigvee_{\alpha\in\Omega} x_\alpha, \bigvee_{\beta\in\Omega'} y_\beta) \leq \bigvee_{\alpha\in\Omega,\, \beta\in\Omega'} f(x_\alpha, y_\beta)$

if $\{x_\alpha\}_{\alpha\in\Omega}$, $\{y_\beta\}_{\beta\in\Omega'} \subseteq [0,1]$.

Let X be a non-empty set and fix

$$FS(X) = \{\mu : X \to [0,1]\}.$$

An intuitionistic fuzzy set (IFS) A in X is an object having the form

$$A = \{\langle\, x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\}$$

where μ_A, $\nu_A \in FS(X)$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set $A \subseteq X$, and for every $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

For every $A \in IFS(X)$, $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \,|x \in X\}$ the following operators are valid (for details, see[1], [3], [4])

$$A^C = \{\langle x,\ \nu_A(x),\ \mu_A(x)\rangle \mid x \in X\}$$

$$\Box A = \{\langle x,\ \mu_A(x),\ 1 - \mu_A(x)\rangle \ \mid x \in X\}$$

$$\Diamond A = \{\langle x,\ 1 - \nu_A(x),\ \nu_A(x)\rangle \ \mid x \in X\}$$

$$CA = \{\langle x,\ K,\ L\rangle \ \mid x \in X\}, \text{ where } K = \bigvee_{x\in X} \mu_A(x),\ L = \bigwedge_{x\in X} \nu_A(x)$$

$$IA = \{\langle x,\ k,\ l\rangle \ \mid x \in X\}, \text{ where } k = \bigwedge_{x\in X} \mu_A(x),\ l = \bigvee_{x\in X} \nu_A(x)$$

$$P_{\alpha,\beta}(A) = \{\langle x,\ \vee(\alpha,\ \mu_A(x)),\ \wedge(\beta,\ \nu_A(x))\rangle \ \mid x \in X\}$$

$$Q_{\alpha,\beta}(A) = \{\langle x,\ \wedge(\alpha,\ \mu_A(x)),\ \vee(\beta,\ \nu_A(x)) \ \mid x \in X\}$$

where α, $\beta \in [0,1]$ and $\alpha + \beta \leq 1$.

Also, if h is a t-norm or a t-conorm and $B \in IFS(X)$, $B = \{\langle x,\ \mu_A(x),\ \upsilon_A(x)\rangle \ \mid x \in X\}$ we can define the expression (see [5])

$$A\tilde{h}B = \Big\{\langle x,\ h(\mu_A(x),\ \mu_A(x)),\ h^C(\nu_A(x),\ \nu_B(x))\rangle \ \mid x \in X\Big\}.$$

For any h, $\tilde{h}$ is an operation over the IFS because

$$h(x,y) + h^C(z,t) \leq h(1-z, 1-t) + 1 - h(1-z, 1-t) = 1$$

for every x, y, z, $t \in [0,1]$ with $x + z \leq 1$ and $y + t \leq 1$.
If $h = \wedge$ or $h = \vee$ we obtain the operations $\cap$ and $\cup$, respectively. If

$h(x,y) = x + y - xy$ or $h(x,y) = xy$ we obtain the operations $+$ and $\cdot$, respectively(see [4]).

In a recent paper ([8]), M. Mashinchi introduced a general fuzzy convexity (concavity) by using the t-norms.
We assume that X and Y are real vector spaces and $I = [0,1]$.

Definition 6 *Let $\mu \in FS(X)$ and f be a t-norm or t-conorm. μ is called to be*

(1) *f-convex in X if* $\mu(kx + (1-k)y) \geq f(\mu(x),\ \mu(y))$, $\forall\ x, y \in X$, $\forall\ k \in I$.

(2) *f-concave in X if* $\mu(kx + (1-k)y) \leq f(\mu(x),\ \mu(y))$, $\forall\ x, y \in X$, $\forall\ k \in I$.

Theorem 7 *([8])Let $\mu \in FS(X)$ and f be a t-norm or t-conorm. μ is f-convex in X if and only if μ^C is f^C-concave in X, where $\mu^C \in FS(X)$, $\mu^C(x) = 1 - \mu(x) \quad \forall\ x \in X$.*

Theorem 8 *([8])Let f, g and h be t-norms or t-conorms and $\mu,\ \eta \in FS(X)$.*

(1) *If μ is f-convex in X, η is g-convex in X, $f \leq g$ and $f \ll h$, then $h(\mu, \eta)$ is f-convex in X.*

(2) *If μ is f-concave in X, η is g-concave in X, $g \leq f$ and $h \ll f$, then $h(\mu, \eta)$ is f-concave in X,*

where $h(\mu, \eta)(x) = h(\mu(x), \eta(x)) \quad \forall\ x \in X$.

Theorem 9 *(see [8])Let $F : X \to Y$ be a linear map and f be a t-norm or t-conorm. If $\mu \in FS(Y)$ is f-convex (f-concave) in Y, then $F^{-1}(\mu) \in FS(X)$ is f-convex (f-concave) in X.*

By Lemma 4.15 from [8] and Lemma 5 we obtain

Theorem 10 *Let $F : X \to Y$ be a linear map and f be a continuous t-norm. If $\lambda \in FS(X)$ is f-convex in X, then $F(\lambda) \in FS(Y)$ is f-convex in Y.*

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Definition 11 *Let $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\} \in IFS(X)$ and f be a t-norm or t-conorm. A is called to be*

(1) *f-convex in X if μ_A is f-convex in X and ν_A is f^C-concave in X.*

(2) *f-concave in X if μ_A is f-concave in X and ν_A is f^C-convex in X.*

We call convex a IFS which is $\wedge$-convex.

Theorem 12 *Let $A \in IFS(X)$ and f be a t-norm or t-conorm.*
(1) *A is f-convex if and only if A^C is f^C-concave.*
(2) *If A is f-convex(f-concave) then $\Box A$ and $\Diamond A$ are f-convex (f-concave).*

Proof. Follows from Theorem 7, Definition 11 and the equality $(f^C)^C = f$ ■

We denote by $\alpha - \beta$ the constant intuitionistic fuzzy set $\{\langle x,\ \alpha,\ \beta\rangle \mid x \in X\}$ with $\alpha, \beta \in I$, $\alpha + \beta \leq 1$.

Lemma 13 (1) *If f is a t-norm, then $\alpha - \beta$ is f-convex.*
(2) *If f is a t-conorm, then $\alpha - \beta$ is f-concave.*

Proof. (1) Because f is a t-norm, $f(\alpha, 1) = \alpha$ and $f^C(\beta, 0) = \beta$. Then $\alpha = f(\alpha, 1) \geq f(\alpha, \alpha)$ and $\beta = f^C(\beta, 0) \leq f^C(\beta, \beta)$ therefore $\alpha - \beta$ is f-convex.

(2) Analogous with (1) ■

Theorem 14 *If f is a t-norm (t-conorm), then CA and IA are f-convex (f-concave).*

Proof. Follows from Lemma 13 ■

Theorem 15 *Let f, g and h be three t-norms (or three t-conorms) and $A, B \in IFS(X)$.*
(1) *If $f \leq g$, $f \ll h$, A is f-convex in X and B is g-convex in X, then $A\tilde{h}B$ is f-convex in X.*
(2) *If $g \leq f$, $h \ll f$, A is f-concave in X and B is g-concave in X, then $A\tilde{h}B$ is f-concave in X.*

Proof. Let A, $B \in IFS(X)$, $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x,\ \mu_B(x),\ \nu_B(x)\rangle \mid x \in X\}$.

(1) Because μ_A is f-convex and μ_B is g-convex, $f \leq g$ and $f \ll h$ we obtain (from Theorem 8, (1)) that $h(\mu_A, \mu_B)$ is f-convex. Because ν_A is f^C-concave, ν_B is g^C-concave, $g^C \leq f^C$ and $h^C \ll f^C$ (see Lemma 4) we

obtain (from Theorem 8, (2)) that $h^C(\nu_A, \nu_B)$ is f^C-concave. Therefore $A\tilde{h}B$ is f-convex.

(2) Is similar ■

On Lemma 3 and Theorem 15 the following corollaries are immediate.

Corollary 16 (1) *If f and $\top$ are t-norms and $f \ll \top$, A and B are f-convex, then $A\tilde{\top}B$ is f-convex.*
(2) *If g and $\bot$ are t-conorms and $\bot \ll g$, A and B are f-concave, then $A\tilde{\bot}B$ is f-concave.*

Corollary 17 (1) *If $\top$ is t-norm, $A, B \in IFS(X)$ are $\top$-convex, then $A \cap B$ is $\top$-convex.*
(2) *If $\bot$ is t-conorm, $A, B \in IFS(X)$ are $\bot$-concave, then $A \cup B$ is $\bot$-concave.*

Proof. (1) We take $f = g = \top$ and $h = \wedge$ in Theorem 15, (1).

(2) We take $f = g = \bot$ and $h = \vee$ in Theorem 15, (2) ■

Choosing $\top = \wedge$ in Corollary 17 we obtain immediately

Corollary 18 *An arbitrary intersection of convex IFS is again convex. An arbitrary reunion of $\vee$-concave IFS is $\vee$-concave.*

Because for each $A \in IFS(X)$ and for $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$, $P_{\alpha,\beta}(A) = A \cup (\alpha - \beta)$ and $Q_{\alpha,\beta} = A \cap (\alpha - \beta)$ (see [3]), we have

Theorem 19 (1) *If A is $\top$-convex, then $Q_{\alpha,\beta}(A)$ is $\top$-convex.*
(2) *If A is $\bot$-concave then $P_{\alpha,\beta}(A)$ is $\bot$-concave.*

Proof. Follows from Lemma 13 and Corollary 17 ■

Now, let us consider two sets X and Y and a function $F : X \to Y$.

Definition 20 *The direct image map is*

$$F : IFS(X) \to IFS(Y)$$

defined by

$$F(A) = \left\{ \langle y,\ \overline{F}\mu_A(y),\ \underline{F}\nu_A(y) \rangle \ \mid y \in Y \right\},$$

where

$$\overline{F}\mu_A(y) = \begin{cases} \sup\limits_{F(x)=y} \mu_A(x) & if\ y \in F(X) \\ 0 & if\ y \in Y \setminus F(X) \end{cases}$$

$$\underline{F}\nu_A(y) = \begin{cases} \inf\limits_{F(x)=y} \nu_A(x) & if\ y \in F(X) \\ 1 & if\ y \in Y \setminus F(X) \end{cases}$$

if $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\} \in IFS(X)$.

The inverse image map is

$$F^{-1} : IFS(Y) \to IFS(X)$$

defined by

$$F^{-1}(B)(x) = \Big\{\langle x,\ F^{-1}\mu_B(x),\ F^{-1}\nu_B(x)\rangle \mid x \in X\Big\}$$

where

$$F^{-1}\mu_B(x) = \mu_B(F(x))$$

$$F^{-1}\nu_B(x) = \nu_B(F(x))$$

if $B = \{\langle y,\ \mu_B(y),\ \nu_B(y)\rangle \mid y \in Y\} \in IFS(Y)$.

Remark. $F^{-1}(B) \in IFS(X)$
because

$$(F^{-1}\mu_B)(x) + (F^{-1}\nu_B)(x) = \mu_B(F(x)) + \nu_B(F(x)) \leq 1$$

for all $x \in X$.
If $y \notin F(X)$ then $\overline{F}\mu_A(y) + \underline{F}\nu_A(y) = 1$. If $y \in F(X)$ let $\alpha = \inf\limits_{F(x)=y} \nu_A(x)$.
Then

$$\alpha \leq \nu_A(x) \leq 1 - \mu_A(x) \text{ for all } x \in F^{-1}(y),$$

hence

$$\sup_{F(x)=y} \mu_A(x) \leq 1 - \alpha$$

Therefore $F(A) \in IFS(Y)$.
In fact $\overline{F}\mu_A \equiv F\mu_A$(see [9], p.16).

Now we assume that X and Y are real vector spaces.

Theorem 21 *Let $F : X \to Y$ be a linear map and f be a t-norm. If $B \in IFS(Y)$ is f-convex in Y, then $F^{-1}(B) \in IFS(X)$ is f-convex in X.*

Proof. Using Theorem 9 ■

Lemma 22 *If $F : X \to Y$ and $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\} \in IFS(X)$ then*

$$F(A) = \left\{\langle y,\ F\mu_A(y),\ \left(F\nu_A^C\right)^C(y)\rangle \mid y \in Y\right\}.$$

Proof. We prove that $(\underline{F}\nu_A)^C = F(\nu_A^C)$.

$$(\underline{F}\nu_A)^C(y) = 1 - \underline{F}\nu_A(y) = 1 - \begin{cases} \inf\limits_{F(x)=y} \nu_A(x) & \text{if } y \in F(X) \\ 1 & \text{else} \end{cases} =$$

$$\begin{cases} 1- \inf\limits_{F(x)=y} \nu_A(x) & \text{if } y \in F(X) \\ 0 & \text{else} \end{cases} = \begin{cases} \sup\limits_{F(x)=y} (1 - \nu_A(x)) & \text{if } y \in F(X) \\ 0 & \text{else} \end{cases}$$

$$= \overline{F}(\nu_A^C)(y) = F(\nu_A^C)(y) \text{ for all } y \in Y \ ■$$

Theorem 23 *Let $F : X \to Y$ be a linear map and f be a continuous t-norm. If $A \in IFS(X)$ is f-convex, then $F(A)$ is f-convex in Y.*

Proof. $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\}$ f-convex implies that μ_A is f-convex so that $F\mu_A$ is f-convex in Y (from Theorem 10) and ν_A^C is f-convex in X (from Theorem 7) so that $F\nu_A^C$ is f-convex in Y (also Theorem 10). Using Lemma 22 and Theorem 7 we have that $F(A)$ is f-convex ■

4 The characterized of the convexity of IFS by (α, β)-level sets

In this section X is a real vector space and $I = [0, 1]$. In the following we will denote also by μ the restriction of $\mu \in FS(X)$ at $Y \subseteq X$. Its immediate the next lemma

Lemma 24 *Let f be a t-norm or t-conorm and Y a convex subset of X.*

(1) *If $\mu \in FS(X)$ is f-convex (f-concave) in X, then μ is f-convex (f-concave) in Y.*

(2) *If $A \in IFS(X)$ is f-convex (f-concave) in X, then A is f-convex (f-concave) in Y.*

Following the idea of a fuzzy set from α-level, in [2] is introduced

$$N_{\alpha,\beta}(A) = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X,\ \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\}$$

a set from (α, β)-level generated by the $A \in IFS(X)$, where $\alpha, \beta \in I$ are fixed numbers for which $\alpha + \beta \leq 1$.

Also we call the set

$$N_\alpha(A) = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X,\ \mu_A(x) \geq \alpha\}$$

a set of level of membership α, generated by A, and the set

$$N^\alpha(A) = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X, \nu_A(x) \leq \alpha\}$$

a set of level of nonmembership α, generated by A.

Theorem 25 *([2])$N_\alpha(A)$, $N^\alpha(A) \in IFS(X)$ for every $A \in IFS(X)$, $\alpha \in I$ and for every $\alpha, \beta \in I$,*

$$N_{\alpha,\beta}(A) = N_\alpha(A) \cap N^\beta(A).$$

We obtain the next result for convexity of these IFS.

Lemma 26 (1) *Let $\perp$ be a t-conorm. If $A \in IFS(X)$ is $\perp$-convex, then $N_\alpha(A)$ and $N^\alpha(A)$ are $\perp$-convex for all $\alpha \in I$.*

(2) *If A is convex, then $N_\alpha(A)$ and $N^\alpha(A)$ are convex for all $\alpha \in I$.*

Proof. We suppose that $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\}$ and we denote

$$Y = \{x \in X \mid \mu_A(x) \geq \alpha\},\ Z = \{x \in X \mid \nu_A(x) \leq \alpha\}$$

for a certain $\alpha \in I$.

(1) If $x, y \in Y$ then

$$\mu_A(kx + (1-k)y) \geq \perp(\mu_A(x), \mu_A(y)) \geq \perp(\alpha, \alpha) \geq \perp(\alpha, 0) = \alpha,$$

for all $k \in I$.
If $x, y \in Z$ then

$$\nu_A(kx + (1-k)y) \leq \perp^C (\nu_A(x), \nu_A(y)) \leq \perp^C(\alpha, \alpha) \leq \perp^C(\alpha, 1) = \alpha,$$

for all $k \in I$.

Therefore Y and Z are convex subsets of X and Lemma 24, (2) implies that $\{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in Y\} = N_\alpha(A)$ and $\{\langle x, \mu_A(x), \nu_A(x) \mid x \in Z\} = N^\alpha(A)$ are f-convex.

(2) Is similar because

$$\mu_A(kx + (1-k)y) \geq \wedge (\mu_A(x), \mu_A(y)) \geq \alpha$$

and

$$\nu_A(kx + (1-k)y) \leq \vee (\nu_A(x), \nu_A(y)) \leq \alpha$$

for all $k \in I$ if $x, y \in Y$ or Z, respectively ■

The convexity can be characterized in a simple way by (α, β)-level sets

Theorem 27 *For $A \in IFS(X)$ the following statements are equivalent:*
(1) *A is convex*
(2) *$N_{\alpha,\,\beta}(A)$ is convex for all $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$.*

Proof. The afirmation (2) implies (1) because $N_{0,1}(A) = A$.

If A is convex, then $N_\alpha(A)$ and $N^\beta(A)$ are convex for all $\alpha, \beta \in I$ (from Lemma 26, (2)). Using the Corollary 18 we have that $N_\alpha(A) \cap N^\beta(A) = N_{\alpha,\beta}(A)$ is convex for all $\alpha, \beta \in I$ (see and Theorem 25), hence (1) implies (2) ■

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