

Intuitionistic fuzzy α -semigroup

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Abstract: In this paper we will try to give sense to the notion of intuitionistic fuzzy α -semigroups. Our objective is to solve an intuitionistic fuzzy evolution (differential equation) problem. Since the concept of linear operators is not defined on the set of all intuitionistic fuzzy numbers, we found an obvious inspiration from the nonlinear evolution problem in the classical case.

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1 Introduction

The concept of intuitionistic fuzzy sets is introduced by K. Atanassov in 1983 (see [1, 2]). This concept is a generalization of fuzzy theory introduced by L. Zadeh [15]. Several works were devoted to investigation of Cauchy problems with fuzzy initial conditions [7, 8]. By the metric space defined in [10], we have established a way to study this problem in intuitionistic fuzzy theory.

The central result in the theory of semigroups of linear operators is the characterization, by the Hill–Yosida theorem, of the generators of semigroup of bounded linear operators in general Banach spaces, see [11]. The birth of the α -semigroups of linear operators [4] is it come with the introduce of the new derivative [14]. O. Kaleva in [7] introduced an iteration semigroup of a nonlinear fuzzy-valued function, and showed that the iterates $(i + \frac{f}{n})^n(x)$ denoting the n -fold

composition of $i + \frac{f}{n}$, converge for all $x \in \mathcal{F}$ (with i being the identity function of \mathcal{F}), under some assumptions on the function f and the limit function was called fuzzy exponential function (due to the obvious similarity with the classical exponential function) and was denoted by $e^f(x)$.

This paper is organized as follows. In Section 2, we recall some concepts related to the intuitionistic fuzzy sets. Some properties about measurability, integrability and differentiability are provided in Section 3. Finally, we present the principal goal of this work in Section 4.

2 Preliminaries

In this paper $\alpha \in (0, 1)$.

Definition 1. [10] *The set of all intuitionistic fuzzy numbers is given by*

$$IF_1 = \left\{ \langle u, v \rangle : \mathbb{R} \longrightarrow [0, 1]^2, \quad 0 \leq u + v \leq 1 \right\}$$

with the following conditions:

1. *For each $\langle u, v \rangle \in IF_1$ is normal, i.e., $\exists x_0, x_1 \in \mathbb{R}$, such that $u(x_0) = 1$ and $v(x_1) = 1$.*
2. *For each $\langle u, v \rangle \in IF_1$ is a convex intuitionistic set, i.e., u is fuzzy convex and v is fuzzy concave.*
3. *For each $\langle u, v \rangle \in IF_1$, u is lower continuous and v is upper continuous.*
4. *the closure of $\{x \in \mathbb{R} : v(x) \leq \alpha\}$ is bounded.*

Definition 2. [10] *For $t \in [0, 1]$, we define the upper and lower t -cut by*

$$\begin{aligned} \left[\langle u, v \rangle \right]_t &= \left\{ x \in \mathbb{R}, \quad u(x) \geq t \right\} \\ \left[\langle u, v \rangle \right]^t &= \left\{ x \in \mathbb{R}, \quad v(x) \leq 1 - t \right\} \end{aligned}$$

Definition 3. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0) & x = 0 \\ (0, 1) & x \neq 0 \end{cases}$$

Proposition 1. [6] *We can write*

$$\begin{aligned} \left[\langle u, v \rangle \right]_t &= \left[\left[\langle u, v \rangle \right]_l^+(t), \left[\langle u, v \rangle \right]_r^+(t) \right] \\ \left[\langle u, v \rangle \right]^t &= \left[\left[\langle u, v \rangle \right]_l^-(t), \left[\langle u, v \rangle \right]_r^-(t) \right] \end{aligned}$$

Remark 1. *In the fuzzy case, we can write $\left[\langle u, v \rangle \right]_t = [u]^t$ and $\left[\langle u, v \rangle \right]^t = [1 - v]^t$.*

We define two operations on IF_1 by

$$\begin{aligned}\langle u, v \rangle \oplus \langle u', v' \rangle &= \langle u \vee v, u' \wedge v' \rangle, \quad \forall \langle u, v \rangle, \langle u', v' \rangle \in IF_1 \\ \lambda \langle u, v \rangle &= \langle \lambda u, \lambda v \rangle, \quad \forall \lambda \in \mathbb{R}, \forall \langle u, v \rangle \in IF_1.\end{aligned}$$

According to Zadeh extension principle, we have

$$\begin{aligned}\left[\langle u, v \rangle \oplus \langle u', v' \rangle \right]_t &= \left[\langle u, v \rangle \right]_t + \left[\langle u', v' \rangle \right]_t, \\ \left[\lambda \langle u, v \rangle \right]_t &= \lambda \left[\langle u, v \rangle \right]_t, \\ \left[\langle u, v \rangle \oplus \langle u', v' \rangle \right]^t &= \left[\langle u, v \rangle \right]^t + \left[\langle u', v' \rangle \right]^t, \\ \left[\lambda \langle u, v \rangle \right]^t &= \lambda \left[\langle u, v \rangle \right]^t.\end{aligned}$$

Definition 4. Let $\langle u, v \rangle, \langle u', v' \rangle \in IF_1$ the H -difference is the $IFN (z, w) \in IF_1$, if it exists, such that

$$\langle u, v \rangle \ominus \langle u', v' \rangle = \langle z, w \rangle \iff \langle u, v \rangle = \langle u', v' \rangle \oplus \langle z, w \rangle$$

Theorem 1. [6] Let $\mathcal{M} = \{M_t, M^t, t \in [0, 1]\}$ be a family of subsets in \mathbb{R} satisfying the following conditions

1. $t \leq s \implies M_s \subset M_t$ and $M^s \subset M^t$;
2. M_t and M_s are nonempty compact convex sets in \mathbb{R} for each $t \in [0, 1]$;
3. For any nondecreasing sequence $t_i \rightarrow t$ on $[0, 1]$, we have $M_t = \bigcap_i M_{t_i}$ and $M^t = \bigcap_i M^{t_i}$.

We define u and v by

$$u(x) = \begin{cases} 0 & x \notin M_0 \\ \sup_{t \in [0, 1]} M_t & x \in M_0 \end{cases}, \quad v(x) = \begin{cases} 1, & x \notin M^0 \\ 1 - \sup_{t \in [0, 1]} M_t & x \in M^0 \end{cases}.$$

Then $\langle u, v \rangle \in IF_1$, with $M_t = [\langle u, v \rangle]_t$ and $M^t = [\langle u, v \rangle]^t$.

Remark 2. [6]

1. The family $\{[\langle u, v \rangle]_t, [\langle u, v \rangle]^t, t \in [0, 1]\}$ satisfies the conditions 1.–3. of the previous theorem.
2. For all $t \in [0, 1]$, $[\langle u, v \rangle]_t \subset [\langle u, v \rangle]^t$.

Theorem 2. [6] On IF_1 , we define the metric

$$\begin{aligned}d_\infty((u, v), (z, w)) &= \frac{1}{4} \left\{ \sup_{0 < \alpha \leq 1} \left\| [\langle u, v \rangle]_r^+(\alpha) - [\langle z, w \rangle]_r^+(\alpha) \right\| \right. \\ &\quad + \sup_{0 < \alpha \leq 1} \left\| [\langle u, v \rangle]_l^+(\alpha) - [\langle z, w \rangle]_l^+(\alpha) \right\| + \sup_{0 < \alpha \leq 1} \left\| [\langle u, v \rangle]_r^-(\alpha) - [\langle z, w \rangle]_r^-(\alpha) \right\| \\ &\quad \left. + \sup_{0 < \alpha \leq 1} \left\| [\langle u, v \rangle]_l^-(\alpha) - [\langle z, w \rangle]_l^-(\alpha) \right\| \right\},\end{aligned}$$

where $\| \cdot \|$ denotes the usual Euclidean norm in \mathbb{R} , and

$$\begin{aligned} d_p(\langle u, v \rangle, \langle u', v' \rangle) &= \left(\frac{1}{4} \int_0^1 |[\langle u, v \rangle]_l^+(t) - [\langle u', v' \rangle]_l^+(t)| dt \right. \\ &\quad + \frac{1}{4} \int_0^1 |[\langle u, v \rangle]_r^+(t) - [\langle u', v' \rangle]_r^+(t)| dt + \frac{1}{4} \int_0^1 |[\langle u, v \rangle]_l^-(t) - [\langle u', v' \rangle]_l^-(t)| dt \\ &\quad \left. + \frac{1}{4} \int_0^1 |[\langle u, v \rangle]_r^-(t) - [\langle u', v' \rangle]_r^-(t)| dt \right)^{\frac{1}{p}}. \end{aligned}$$

For $p \in [1, \infty)$, we have (IF_1, d_p) is a complete metric space.

3 Measurability, integrability and differentiability

3.1 Measurability

The symbol $P_k(\mathbb{R})$ denotes the family of all nonempty compact convex subsets of \mathbb{R} .

Definition 5. [6] We say that a mapping $F : [a, b] \longrightarrow IF_1$ is strongly measurable if for all $t \in [0, 1]$, the set-valued mapping $F_t : [a, b] \longrightarrow P_k(\mathbb{R})$ defined by $F_t(x) = [F(x)]_t$ and $F^t : [a, b] \longrightarrow P_k(\mathbb{R})$ defined by $F^t(x) = [F(x)]^t$ are (Lebesgue) measurable, when $P_k(\mathbb{R})$ is endowed with the topology generated the Hausdorff metric d_H .

We have the following remark.

Remark 3. The previous definition is equivalent to the expressions

$$\left\{ (x, y), y \in F_t(x) \right\} \in \mathcal{M}e \times \mathbb{B}(\mathbb{R}),$$

where $\mathcal{M}e$ denotes the σ -algebra of measurable sets and $\mathbb{B}(\mathbb{R})$ denotes the Borel sets of \mathbb{R} .

Definition 6. Let I be an interval of \mathbb{R} . We say that a mapping $F : I \longrightarrow IF_1$ is strongly measurable if for all $t \in [0, 1]$, its restriction on any segment is strongly continuous.

Lemma 1. If F is strongly measurable, then it is measurable with respect to the topology generated by d_∞ , where d_∞ is defined as in [3].

Proof. Let $\epsilon > 0$ and $\langle u, v \rangle \in IF_1$ be arbitrary. Then

$$\begin{aligned} T &= \left\{ x \mid d_\infty(F(x), \langle u, v \rangle) \leq \epsilon \right\} \\ &= \bigcap_{t \in [0, 1]} \{ t \mid d_\infty(F_t(x), [\langle u, v \rangle]_t) \leq \epsilon \} \bigcap_{t \in [0, 1]} \{ t \mid d_\infty(F^t(x), [\langle u, v \rangle]^t) \leq \epsilon \}. \end{aligned}$$

But for all $\langle u, v \rangle \in IF_1$ we have (see [10])

$$\lim_{k \rightarrow \infty} d_H \left([\langle u, v \rangle]^{t_k}, [\langle u, v \rangle]^t \right) = 0$$

and

$$\lim_{k \rightarrow \infty} d_H \left(\left[\langle u, v \rangle \right]_{t_k}, \left[\langle u, v \rangle \right]_t \right) = 0,$$

whenever (t_k) is a nondecreasing sequence converging to t . Thus, by the triangle inequality for the metric d_H we have

$$d_H \left(F_t(x), \left[\langle u, v \rangle \right]_t \right) \leq \overline{\lim} d_H \left(F_{t_k}(x), \left[\langle u, v \rangle \right]_{t_k} \right)$$

and

$$d_H \left(F^t(x), \left[\langle u, v \rangle \right]^t \right) \leq \overline{\lim} d_H \left(F^{t_k}(x), \left[\langle u, v \rangle \right]^{t_k} \right)$$

where $t_k \uparrow t$, and consequently

$$\bigcap_{k \geq 1} \left\{ t \mid d_H \left(F_{t_k}(x), \left[\langle u, v \rangle \right]_{t_k} \right) \leq \epsilon \right\} \subset \left\{ t \mid d_H \left(F_t(x), \left[\langle u, v \rangle \right]_t \right) \leq \epsilon \right\}$$

and

$$\bigcap_{k \geq 1} \left\{ t \mid d_H \left(F^{t_k}(x), \left[\langle u, v \rangle \right]^{t_k} \right) \leq \epsilon \right\} \subset \left\{ t \mid d_H \left(F^t(x), \left[\langle u, v \rangle \right]^t \right) \leq \epsilon \right\}$$

Thus,

$$T = \bigcap_{k \geq 1} \left\{ t \mid d_H \left(F^{t_k}(x), \left[\langle u, v \rangle \right]^{t_k} \right) \leq \epsilon \right\} \bigcap_{k \geq 1} \left\{ t \mid d_H \left(F_{t_k}(x), \left[\langle u, v \rangle \right]_{t_k} \right) \leq \epsilon \right\}$$

where $\{t_k \mid k = 1, 2, \dots\}$ is any denumerable dense subset of $[0, 1]$.

Hence, T is measurable. □

Lemma 2. Let $F : [a, b] \longrightarrow IF_1$ be strongly measurable and denote $F_t(x) = [\mu(t), \nu(t)]$ and $F^t(x) = [\mu'(t), \nu'(t)]$ for $t \in [0, 1]$, Then $\mu(t)$, $\nu(t)$, $\mu'(t)$ and $\nu'(t)$ are measurable.

Proof. Use Remark 1 and apply [9, Lemma 3.3]. □

3.2 Integrability

Definition 7. A mapping $F : [a, b] \longrightarrow IF_1$ is called integrably bounded if there exists an integrable function h such that $\|y\| \leq h(x)$ for all $y \in F^o(x)$.

Definition 8. Let $F : [a, b] \longrightarrow IF_1$. The integral of F over I , denoted $\int_{[a,b]} F(x)dx$, is defined levelwise by the equation

$$\left[\int_{[a,b]} F(x)dx \right]_t = \left\{ \int_I f(x)dx, \quad f : I \longrightarrow \mathbb{R} \text{ is a measurable selection for } F_t \right\}$$

and

$$\left[\int_{[a,b]} F(x)dx \right]^t = \left\{ \int_I f(x)dx, \quad f : I \longrightarrow \mathbb{R} \text{ is a measurable selection for } F^t \right\}$$

for all $0 < t < 1$. A strongly measurable and integrably bounded mapping $F : [a, b] \longrightarrow IF_1$ is said to be integrable over $[a, b]$ if $\int_I F(x)dx \in IF_1$.

Theorem 3. *If $F : [a, b] \longrightarrow IF_1$ is strongly measurable and integrably bounded, then F is integrable.*

Proof. See [12]. □

Corollary 1. [9] *If $F : [a, b] \longrightarrow IF_1$ is continuous, then it is integrable.*

Theorem 4. *Let $F : [a, b] \longrightarrow IF_1$ be integrable and $c \in \mathbb{R}$. Then*

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Corollary 2. *If $F : [a, b] \longrightarrow IF_1$ is continuous, then $G(t) = \int_a^t f(x)dx$ is Lipschitz continuous on $[a, b]$.*

We have the same result as [9] in Theorem 4.2.

Remark 4. *We can extend the concept of integrability on a segment to integrability on an interval of \mathbb{R} . We obtain the same result as improper integral.*

3.3 Differentiability

Let $\langle u, v \rangle, \langle u', v' \rangle \in IF_1$, if there exist $\langle u'', v'' \rangle \in IF_1$ that satisfies

$$\langle u, v \rangle \ominus \langle u', v' \rangle = \langle u'', v'' \rangle \iff \langle u, v \rangle = \langle u', v' \rangle \oplus \langle u'', v'' \rangle$$

then, $\langle u'', v'' \rangle$ is called the H-difference (see [10]).

Definition 9. *A mapping $F : [a, b] \longrightarrow IF_1$ is differentiable at $t_o \in [a, b]$ if there exists a $F'(t_o) \in IF_1$ such that the limits*

$$\lim_{h \rightarrow 0^+} \frac{F(t_o + h) \ominus F(t_o)}{h} \text{ and } \lim_{h \rightarrow 0^+} \frac{F(t_o) \ominus F(t_o - h)}{h}$$

exist and equal to $F'(t_o)$.

Following [9], we have the following properties in the intuitionistic fuzzy case.

Theorem 5. *Let $F : [a, b] \longrightarrow IF_1$ be differentiable. Denote*

$$F_t(x) = [f(x), g(x)] \text{ and } F^t(x) = [h(x), r(x)]$$

then $F'_t(x) = [f'(x), g'(x)]$ and $F'^t(x) = [h'(x), r'(x)]$

Theorem 6. *If $F : [a, b] \longrightarrow IF_1$ is differentiable, then it is continuous.*

Theorem 7. *Let $F : [a, b] \longrightarrow IF_1$ be continuous. Then, for all $t \in [a, b]$ the integral $G(s) = \int_0^s F$ is differentiable and $G'(t) = F(t)$.*

Theorem 8. *Let $F : [a, b] \longrightarrow IF_1$ be differentiable and assume that the derivative F' is integrable over $[a, b]$. Then, for each $s \in T$ we have*

$$F(s) = F(a) \oplus \int_0^s F'(x)dx.$$

4 Intuitionistic fuzzy sets framework

Since the concept of linear mapping is not defined on the sets of all intuitionistic fuzzy numbers IF_1 , it is necessary to answer the following question. *How to solve the evolution problem in the intuitionistic fuzzy case?* Since this problem is a nonlinear problem, thus it is necessary to exploit the results from previous section.

Now, firstly we introduce the connection between the classical case and the intuitionistic fuzzy case.

4.1 Embedding theorem

Since the elements of IF_1 are closed (Hausdorff topology) and convex, we can apply the result of [13]

Theorem 9. *We can extend IF_1 in a normed space*

Proof. Consider the following relation on $IF_1 \times IF_1$ defined by

$$(\langle u, v \rangle, \langle z, w \rangle) \sim (\langle u', v' \rangle, \langle z', w' \rangle) \iff \langle u, v \rangle + \langle z', w' \rangle = \langle z, w \rangle + \langle u', v' \rangle.$$

The relation is a relation of equivalence.

We denote $IF^* = IF_1 \times IF_1 / \sim$ is a vector space (see [13]).

Now consider that the map

$$j : \begin{cases} IF_1 \longrightarrow IF_1^* \\ \langle u, v \rangle \longrightarrow \overline{(\langle u, v \rangle, \tilde{0})} \end{cases} \quad (1)$$

is an injection, indeed:

$$\begin{aligned} j(\langle u, v \rangle) = j(\langle u', v' \rangle) &\implies \overline{(\langle u, v \rangle, \tilde{0})} = \overline{(\langle u', v' \rangle, \tilde{0})} \\ &\implies (\langle u, v \rangle, \tilde{0}) \sim (\langle u', v' \rangle, \tilde{0}) \implies \langle u, v \rangle = \langle u', v' \rangle \end{aligned}$$

Further we can define the norm on IF_1^* as

$$\| \overline{(\langle u, v \rangle, \langle u', v' \rangle)} \| = d_1(\langle u, v \rangle, \langle u', v' \rangle).$$

This proves that $(IF_1^*, \| \cdot \|)$ is a normed vector space. \square

Theorem 10. *There exists a Banach space X such that IF_1 can be embedded as a convex cone C with vertex 0 in X . Furthermore, the following conditions hold true:*

1. *The embedding j is isometric,*
2. *The addition in X induces the addition in IF_1 ,*
3. *The multiplication by a non-negative real number in X induces the corresponding operation in IF_1 ,*

4. $\mathcal{C} - \mathcal{C} = \{a - b, a, b \in \mathcal{C}\}$ is dense in X ,

5. \mathcal{C} is closed.

Proof. By Theorem 9, IF_1 can be embedded as a convex cone \mathcal{C} in a normed linear space Y such that \mathcal{C} spans Y and the conditions 1. – 3. hold true.

If X is a completion of Y , then also 4 is satisfied. Since (IF_1, d_1) is complete, which follows by combining results in [5] and [13], and the embedding j is isometric we have condition 5. \square

Definition 10. Let $a > 0$ and denote $I = [0, 0 + a]$. A mapping $f : I \longrightarrow \text{IF}_1$ is conformable differentiable of order α if there is a $f^{(\alpha)}(t) \in \text{IF}_1$ such that the limit

$$\lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus f(t)}{\epsilon} \quad \forall t > 0$$

exists and is equal to $f^{(\alpha)}(t)$. Also, $f^\alpha(0) = \lim_{t \rightarrow 0} f^{(\alpha)}(t)$. The α -integral is defined by

$$(I^\alpha f)(t) = \int_0^t \frac{f(s)}{s^{1-\alpha}} ds.$$

Here, the limit is taken in the metric space (IF_1, d_1) .

Theorem 10 is the motivation for the following definition.

Definition 11. A subset A_α of $\text{IF}_1 \times \text{IF}_1$ is in the class \mathcal{A}_ω if for each $0 < \lambda < \alpha\omega^{-1}$ and $\langle u_i, v_i \rangle, \langle u'_i, v'_i \rangle \in A_\alpha, \forall i = 1, 2$ we have

$$d_1\left(\langle u_1, v_1 \rangle + \lambda \langle u'_1, v'_1 \rangle, \langle u_2, v_2 \rangle + \lambda \langle u'_2, v'_2 \rangle\right) \geq \left(1 - \frac{\lambda}{\alpha}\omega\right) d_1\left(\langle u_1, v_1 \rangle, \langle u_2, v_2 \rangle\right)$$

A_α is called intuitionistic fuzzy α -accretive if $A_\alpha \in \mathcal{A}\{0\}$.

Now consider the following initial-value problem

$$\begin{cases} u^{(\alpha)}(t) = A_\alpha u(t) & t \in (0, \infty) \\ u(0) = \langle u_0, v_0 \rangle \in \text{IF}_1 \end{cases} \quad (2)$$

where $A_\alpha \in \mathcal{A}(\omega)$. A function $u(t)$ defined on \mathbb{R}^+ , with values in IF_1 is a solution of (2) if $(u(t), \widetilde{0})$ is absolutely continuous in t , $u(t)$ is IF-differentiable a.e. on $(0, \infty)$, $u(t) \in D(A_\alpha) = \{\langle u, v \rangle \in \text{IF}_1, A_\alpha \langle u, v \rangle \neq \widetilde{0}\}$ and u satisfies (2).

If (2) has a solution $u(t)$, for every $\langle u_0, v_0 \rangle \in D(A_\alpha)$, we define $S_\alpha(t)\langle u_0, v_0 \rangle = u(t)$, $S_\alpha(t)$ is a continuous operator on $D(A_\alpha)$ and extending it by continuity to $\overline{D(A_\alpha)}$ we obtain a family $\{S_\alpha(t), t \geq 0\}$ of operators $S_\alpha : \overline{D(A_\alpha)} \longrightarrow \overline{D(A_\alpha)}$ satisfying the conditions

1. $S_\alpha(0) = i = i_{\text{IF}_1}$;
2. $S_\alpha\left(\left((t+s)^{\frac{1}{\alpha}}\right)\right) = S_\alpha(t^{\frac{1}{\alpha}})S_\alpha(s^{\frac{1}{\alpha}})$;
3. $\lim_{t \rightarrow 0} S_\alpha(t)x = x$ for $x \in \overline{D(A_\alpha)}$;

4. For all $\langle u, v \rangle, \langle u', v' \rangle \in IF_1$,

$$d_1(S_\alpha(t)\langle u, v \rangle, S_\alpha(t)\langle u', v' \rangle) \leq e^{\omega t} d_1(\langle u, v \rangle, \langle u', v' \rangle).$$

This and the previous section are the motivation for the following definition.

Definition 12. A continuous one-parameter intuitionistic fuzzy α -semigroup $\{T_\alpha(t), t \geq 0\}$ of operators on IF_1 is defined by the following conditions:

1. For any fixed $t \geq 0$, $T_\alpha(t)$ is a continuous operator defined on IF_1 into IF_1 ;
2. For any $\langle u, v \rangle \in IF_1$, $T_\alpha(t)\langle u, v \rangle$ is strongly continuous in t with the metric d_1 ;
3. $T_\alpha\left((t+s)^{\frac{1}{\alpha}}\right) = T_\alpha\left(t^{\frac{1}{\alpha}}\right) T_\alpha\left(s^{\frac{1}{\alpha}}\right)$;
4. For all $\langle u, v \rangle, \langle u', v' \rangle \in IF_1$

$$d_1(T_\alpha(t)\langle u, v \rangle, T_\alpha(t)\langle u', v' \rangle) \leq e^{\omega t^\alpha} d_1(x, y) \quad \forall t \geq 0.$$

We call such a family $\{T_\alpha(t)\}$ simply intuitionistic fuzzy α -semigroup of type ω . The strict α -infinitesimal generator A_α of an intuitionistic fuzzy α -semigroup $\{T_\alpha(t)\}$ is defined by

$$A_\alpha x = \lim_{t \rightarrow 0} T_\alpha^{(\alpha)}(t)\langle u, v \rangle, \quad \langle u, v \rangle \in IF_1.$$

The right-hand side exists in IF_1 .

We define the domain of A_α , by

$$D(A_\alpha) = \left\{ \langle u, v \rangle \in IF_1, \lim_{t \rightarrow 0} T_\alpha^{(\alpha)}(t)\langle u, v \rangle \text{ exists} \right\}.$$

Lemma 3. If the family $\{T_\alpha(t), t \geq 0\}$ is an intuitionistic fuzzy α -semigroup of type ω , then $jT_\alpha(t)j^{-1}$ is a nonlinear α -semigroup of type ω on \mathcal{C} .

Proof. By [7], $jT_\alpha(t)j^{-1} : \mathcal{C} \longrightarrow \mathcal{C}$, since j is isometric, which implies that $jT_\alpha(t)j^{-1}$ is a nonlinear α -semigroup of type ω on \mathcal{C} . \square

Lemma 4. If A_α is an intuitionistic fuzzy infinitesimal generator of an intuitionistic fuzzy α -semigroup of type ω $\{T_\alpha(t)\}_{t \geq 0}$, then $jA_\alpha j^{-1}$ is the infinitesimal generator of $jT_\alpha(t)j^{-1}$.

Proof. Let $x \in \mathcal{C}$ and put $R_\alpha(t) = jT_\alpha(t)j^{-1}$. We have $T(t) : \mathcal{C} \longrightarrow \mathcal{C}$, and $\langle u, v \rangle = j^{-1}x$

$$\lim_{t \rightarrow 0} \left\| \frac{R_\alpha(t + \epsilon t^{1-\alpha})x - R_\alpha(t)x}{\epsilon} - R_\alpha^{(\alpha)}(t)x \right\| = 0,$$

which implies

$$\lim_{t \rightarrow 0} \left\| \frac{jT_\alpha(t + \epsilon t^{1-\alpha})j^{-1}x - jT_\alpha(t)j^{-1}x}{\epsilon} - jT_\alpha^{(\alpha)}(t)j^{-1}x \right\| = 0$$

and

$$\lim_{t \rightarrow 0} d_1\left(\frac{T_\alpha(t + \epsilon t^{1-\alpha})j^{-1}x \ominus T_\alpha(t)j^{-1}x}{\epsilon}, T_\alpha^{(\alpha)}(t)j^{-1}x\right) = 0,$$

which implies

$$\lim_{t \rightarrow 0} d_1 \left(\frac{T_\alpha(t + \epsilon t^{1-\alpha}) \langle u, v \rangle \ominus T_\alpha(t) \langle u, v \rangle}{\epsilon}, T_\alpha^{(\alpha)}(t) \langle u, v \rangle \right) = 0.$$

□

Theorem 11. *The family of $T_\alpha(t)$ is an intuitionistic fuzzy α -semigroup if and only if the family $T_\alpha(t^{\frac{1}{\alpha}})$ is an intuitionistic fuzzy semigroup.*

Proof. Just use Lemmas 3 and 4. □

Lemma 5. *If $t \rightarrow T_\alpha(t)$ is intuitionistic fuzzy differentiable and $t \rightarrow T(t)$ is differentiable, then*

$$T_\alpha^{(\alpha)}(t) = \alpha \frac{d}{dt} T(t^\alpha).$$

Proof. Just use Lemmas 3 and 4. □

Proposition 2. *If $t \rightarrow T_\alpha(t)$ is intuitionistic fuzzy differentiable, then*

$$D(A_\alpha) = \left\{ \langle u, v \rangle \in IF_1, \lim_{t \rightarrow 0} T(t^\alpha)x \right\}$$

and

$$A_\alpha x = \alpha A x, \quad \forall x \in D(A_\alpha),$$

where A is the infinitesimal generator of $T(t)$.

Proof. Just use Lemmas 3 and 4. □

5 Intuitionistic fuzzy conformable problem

In this section, we consider the problem

$$\begin{cases} u^{(\alpha)}(t) = A_\alpha u(t) + f(t, u(t)) & 0 \leq t \leq T \\ u(0) \in IF_1 \end{cases}, \quad (3)$$

where $f : [0, T] \times IF_1 \rightarrow IF_1$, $A_\alpha : D(A_\alpha) \subset IF_1 \rightarrow IF_1$ is the infinitesimal generator of an intuitionistic fuzzy α -semigroups $T_\alpha(t)$.

Lemma 6. *The Problem (3) is equivalent to the integral equation*

$$u(t) = u(0) + \left(I^\alpha A_\alpha u(\cdot) \right)(t) + \left(I^\alpha f(\cdot, u(\cdot)) \right)(t).$$

Proof. Just use 4 and Theorems 6, 7 and 8. □

Lemma 7. *The space $(C^0([0, T], IF_1), d)$ is a complete metric space, with the metric*

$$d(u, v) = \sup_{0 \leq s \leq T} d_1(u(s), v(s))$$

Proof. Let (u_n) be a Cauchy sequence in $(\mathcal{C}^0([0, T], \mathbf{IF}_1), d)$.

$$\forall \epsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \text{such that} \quad \sup_{0 \leq s \leq T} d_1(u_n(s), u_p(s)) \leq \epsilon \quad n \geq p \geq n_0$$

(\mathbf{IF}_1, d_1) is a complete metric space, so $(u_n(t))$ converge to a limit $u(t)$, for all $t \in [0, T]$.
When n tends to $+\infty$, we get

$$d_1(u_n(s), u(s)) \leq \epsilon \quad \forall s \in [0, T]$$

which implies $\sup_{0 \leq s \leq T} d_1(u(s), u_p(s)) \leq \epsilon$ for all $p \geq n_0$, and (u_n) converges uniformly to u on $[0, T]$. \square

Definition 13. An intuitionistic fuzzy solution of Problem 3 is a mapping $u : [0, T] \rightarrow \mathbf{IF}_1$ α -differentiable, satisfying the condition

$$u(t) = T_\alpha(t)u(0) \oplus (I^\alpha T_\alpha(\cdot))(t) \oplus (I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot)))(t),$$

where $\{T_\alpha(t)\}$ is an intuitionistic fuzzy α -semigroups and A_α is an intuitionistic fuzzy α -infinite-simal generator.

We assume that there exists $M > 0$ such that $\forall t \in [0, T], \forall \langle u, v \rangle, \langle u', v' \rangle \in \mathbf{IF}_1$,

$$d_1\left(f\left(t, \langle u, v \rangle\right), f\left(t, \langle u', v' \rangle\right)\right) \leq M d_1\left(\langle u, v \rangle, \langle u', v' \rangle\right).$$

Theorem 12. By the previous condition on f , the Problem 3 has a unique solution.

Proof. Let $\mathcal{C}^0 = \mathcal{C}([0, T], X)$ and consider the following mapping

$$\begin{aligned} P : \mathcal{C}^0 &\longrightarrow \mathcal{C}^0 \\ u &\longrightarrow (Pu)(t) = T_\alpha(t)u(0) \oplus \left(I^\alpha T_\alpha u(\cdot)\right)(t) \oplus \left(I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot))\right)(t) \end{aligned} \quad (4)$$

and the following metric

$$d(u, v) = \sup_{0 \leq s \leq T} d_1(u(s), v(s)), \quad \forall u, v \in \mathcal{C}^0.$$

Step 1. Let $h > 0$ be small, then we have

$$\begin{aligned} d_1\left((Pu)(t+h), (Pu)(t)\right) &= d_1\left(T_\alpha(t)u(0) \oplus \left(I^\alpha T_\alpha u(\cdot)\right)(t+h) \right. \\ &\quad \left. \oplus \left(I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot))\right)(t+h), T_\alpha(t)u(0) \oplus \left(I^\alpha T_\alpha(t)u_\alpha(\cdot)\right)(t) \right. \\ &\quad \left. \oplus \left(I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot))\right)(t)\right) \\ d_1\left((Pu)(t+h), (Pu)(t)\right) &= d_1\left(\left(I^\alpha T_\alpha(t)u(\cdot)\right)(t+h) \oplus \left(I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot))\right)(t+h), \right. \\ &\quad \left. \left(I^\alpha T_\alpha(t)u_\alpha(\cdot)\right)(t) \oplus \left(I^\alpha T_\alpha(\cdot - t)f(\cdot, u(\cdot))\right)(t)\right) \end{aligned}$$

$$d_1\left((Pu)(t+h), (Pu)(t)\right) = d_1\left(\left(I^\alpha T_\alpha(t)u(\cdot)\right)(t+h), \left(I^\alpha T_\alpha(t)u(\cdot)\right)(t)\right) \\ + d_1\left(\left(I^\alpha T_\alpha(\cdot-t)f(\cdot, u(\cdot))\right)(t+h), \left(I^\alpha T_\alpha(\cdot-t)f(\cdot, u(\cdot))\right)(t)\right)$$

We have

$$d_1\left(\left(I^\alpha T_\alpha(t)u(\cdot)\right)(t+h), \left(I^\alpha T_\alpha(t)u(\cdot)\right)(t)\right) = d_1\left(\int_t^{t+h} \frac{T_\alpha(t)u(s)}{s^{1-\alpha}} ds, \tilde{0}\right) \\ \leq \int_t^{t+h} \frac{1}{s^{1-\alpha}} e^{\omega t^\alpha} d_1(u(s), \tilde{0}) \\ \leq \frac{d(u, \tilde{0})}{\alpha} \left((t+h)^\alpha - t^\alpha\right) e^{\omega t^\alpha} \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

Also

$$d_1\left(\left(I^\alpha T_\alpha(\cdot-t)f(\cdot, u(\cdot))\right)(t+h), \left(I^\alpha T_\alpha(\cdot-t)f(\cdot, u(\cdot))\right)(t)\right) \\ \leq \int_t^{t+h} \frac{1}{s^{1-\alpha}} e^{\omega t^\alpha} d_1(f(s, u(s)), \tilde{0}) \\ \leq \frac{Me^{\omega T^\alpha} d(u, \tilde{0})}{\alpha} \left((t+h)^\alpha - t^\alpha\right) \longrightarrow 0, \text{ as } h \longrightarrow 0,$$

which implies

$$d_1\left((Pu)(t+h), (Pu)(t)\right) \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

Step 2. Let $u, v \in \mathcal{C}^0$, from the first part we have

$$d_1\left((Pu)(t), (Pv)(t)\right) \leq \int_0^t \frac{e^{\omega t^\alpha}}{s^{1-\alpha}} d_1(u(s), v(s)) + \int_0^t \frac{Me^{\omega t^\alpha}}{s^{1-\alpha}} d_1(u(s), v(s)) \\ \leq t \frac{(1+M)e^{\omega T^\alpha}}{\alpha} d(u, v).$$

It follows easily that

$$d_1\left((P^n u)(t), (P^n v)(t)\right) \leq \frac{\left(T \frac{(1+M)e^{\omega T^\alpha}}{\alpha}\right)^n}{n!} d(u, v).$$

By Lemma 7 and using the result of [11, p. 184], P has a unique fixed point $u \in \mathcal{C}^0$. This fixed point is the desired solution of Problem 3.

This completes the proof. □

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