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Role of clans in the proximities of intuitionistic fuzzy sets

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Abstract : Definitions of filters, grills, clans and proximities are given in intuitionistic fuzzy setting. It is proved that proximities of intuitionistic fuzzy sets is a clan generated structure.

Keywords : Intuitionistic fuzzy sets, filters, grills, clans and proximities of intuitionistic fuzzy sets.

0. Introduction

In [1] K.Atanassov and S.Stoeva defined intuitionistic fuzzy sets. Later on several authors worked on intuitionistic fuzzy sets. Among others mention may be made of Atanassov [2], [3], [4], Burillo and Bustince [5], [6], D.Çoker [8], [9] and Samanta et. el. [10], [11]. Atanassov and Bustince mainly worked on several operators and algebraic properties of intuitionistic fuzzy sets ; where as D.Çoker, Samanta et. el. worked on topological structures of intuitionistic fuzzy sets.
In [7] Chattopadhyay, Samanta and Mukherjee fuzzified an important result of classical proximity by proving that proximities of fuzzy sets are clan generated structer. In this paper we define a pre-proximity and a proximity of intuitionistic fuzzy sets and prove that proximities of intuitionistic fuzzy sets are clan generated structures.

1. Preliminaries and Notations

Definition 1.1 [1] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS in short) A is an object having the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where the functions $\mu_A, \nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of the element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for each $x \in X$.

Example 1.2 [1] Every fuzzy set A on a nonempty set X is obviously an IFS having the form

$$A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}.$$

Notation 1.3 IFSs are denoted by A, B, C, D etc with (or without) suffix. Set of all IFSs on X are denoted by $I(X)$.

Definition 1.4 [1] Let $A, B \in I(X)$. Then
(a) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall\, x \in X$,
(b) $A = B$ iff $A \subset B$ and $B \subset A$,
(c) $A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$,
(d) $A \bigcap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >: x \in X\}$,
(e) $A \bigcup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >: x \in X\}$.

Definition 1.5 [8] $\tilde{0} = \{< x, 0, 1 >: x \in X\}$ and $\tilde{1} = \{< x, 1, 0 >: x \in X\}$.

Corollary 1.6 [8] Let $A, B \in I(X)$. Then
(a) $(A \bigcup B)^c = A^c \bigcap B^c$,
(b) $(A \bigcap B)^c = A^c \bigcup B^c$,
(c) $(\tilde{1})^c = \tilde{0}$,
(d) $(\tilde{0})^c = \tilde{1}$.

Definition 1.7 For $x \in X$, $p \in (0, 1]$, $q \in [0, 1)$ with $p + q \leq 1$, an IFS A s.t.
$\mu_A(x) = p,\ \nu_A(x) = q$
and
$\mu_A(y) = 0,\ \nu_A(y) = 1,\ \forall\, y (\neq x) \in X$
is called an intuitionistic fuzzy point (in short IFP) on X. This is denoted by $(p, q)_x$.

Notation 1.8 We denote J as an indexing set.

2. Filter, Grill, Prime filter of intuitionistic fuzzy sets

Definition 2.1 A stack S of IFSs on X is a subset of $I(X)$ such that $A \supset B \in S \Rightarrow A \in S$.

Definition 2.2 A filter F of IFSs on X is a subset of $I(X)$ satisfying the following :
$F \neq \varphi$
$A \supset B \in F \Rightarrow A \in F$
$A, B \in F \Rightarrow A \bigcap B \in F$.

A filter F of IFSs is called proper if $\tilde{0} \notin F$.

Definition 2.3 A Grill G of IFSs on X is a subset of $I(X)$ satisfying the following :
$\tilde{0} \notin G$
$A \supset B \in G \Rightarrow A \in G$
$A \bigcup B \in G \Rightarrow A \in G$ or $B \in G$.

A grill G of IFSs is called proper if $G \neq \phi$.

Definition 2.4 A stack V of IFSs on X is a prime filter of IFSs on X if it is a filter of IFSs on X and as well as a grill of IFSs on X.

A maximal proper filter U of IFSs is called an ultrafilter of IFSs.

$\phi(X)$ = Set of all filters of IFSs on X.
$\Gamma(X)$ = Set of all grills of IFSs on X.
$\omega(X)$ = Set of all prime filters of IFSs on X.

Example 2.5 Let $A \in I(X)$. Define $F \subset I(X)$ by

$$F = \{B \in I(X) : B \supset A\}.$$

Now clearly $F \neq \phi$. Let $C \supset B \in F$. Then $C \supset B \supset A$ and hence $C \in F$. Again let $B, C \in F$ and so $\mu_B(x) \geq \mu_A(x)$, $\nu_B(x) \leq \nu_A(x)$ and $\mu_C(x) \geq \mu_A(x)$, $\nu_C(x) \leq \nu_A(x)$. It follows that $\mu_B(x) \wedge \mu_C(x) \geq \mu_A(x)$ and $\nu_B(x) \vee \nu_C(x) \leq \nu_A(x)$. Thus $B \bigcap C \supset A$ and therefore $B \bigcap C \in F$. Consequently F is a filter of IFSs.

Example 2.6 Let $p > 0$ and X be a nonempty set. Then

$$V_{p_x} = \{A \in I(X) : \mu_A(x) \geq p\}$$

is a prime filter of IFSs on X.

Proof. Let $A \supset B \in V_{p_x}$. Then $\mu_A(x) \geq \mu_B(x) \geq p$ and hence $A \in V_{p_x}$.
Now $\tilde{0} = \{< x, 0, 1 >: x \in X\} \notin V_{p_x}$. Again let $A, B \in V_{p_x}$. Then $A \bigcap B \in V_{p_x}$. Finally suppose

that $A \bigcup B \in V_{p_x}$. Therefore $\mu_{A \bigcup B}(x) \geq p$ that is $\mu_A(x) \geq P$ or $\mu_B(x) \geq p$. It follows that $A \in V_{p_x}$ or $B \in V_{p_x}$.
Hence V_{p_x} is a prime filter of IFS on X.

Remark 2.7 It is to be noted that for IFP $(p,q)_x$, the collection

$$V_{(p,q)_x} = \{A \in I(X) : (p,q)_x \tilde{\in} A\}$$

is a filter but in general not a prime filter. In fact, it may not be a grill. To justify this take an ordinary set $X \neq \phi$. Let p = 0.2, q = 0.3 and a fixed $x \in X$. Let $A, B \in I(X)$ with $\mu_A(x) = 0.25$, $\nu_A(x) = 0.35$, $\mu_B(x) = 0.15$, $\nu_B(x) = 0.25$. Then $A \bigcup B \in V_{(p,q)_x}$ but $A \notin V_{(p,q)_x}$ and $B \notin V_{(p,q)_x}$.

Theorem 2.8 *Let* $F^1,\ F^2 \in \phi(X)$ *and* $G^1,\ G^2 \in \Gamma(X)$. *Then*
(1) $F^1 \bigcap F^2 \subset G^1 \Rightarrow F^1 \subset G^1$ *or* $F^2 \subset G^1$
(2) $F^1 \subset G^1 \bigcup G^2 \Rightarrow F^1 \subset G^1$ *or* $F^1 \subset G^2$.

Proof. (1) If possible suppose that $F^1 \not\subset G^1$ and $F^2 \not\subset G^1$. Then there exists $A_1 \in F^1$ such that $A_1 \notin G^1$ and $A_2 \in F^2$ such that $A_2 \notin G^1$. So $A_1 \bigcup A_2 \notin G^1$ but $A_1 \bigcup A_2 \in F^1 \bigcap F^2$——a contradiction.
(2) If possible suppose that $F^1 \not\subset G^1$ and $F^1 \not\subset G^2$. Then there exists $A_1 \in F^1 - G^1$ and $A_2 \in F^1 - G^2$ and hence $A_1 \bigcap A_2 \in F^1 \subset G^1 \bigcup G^2$. So $A_1 \bigcap A_2 \in G^1$ or $A_1 \bigcap A_2 \in G^2$ which implies $A_1 \in G^1$ or $A_2 \in G^2$——a contradiction.
This completes the proof.

Theorem 2.9 *Intersection of filters of IFSs is a filter of IFSs.*

Proof is straightforward.

Theorem 2.10 *Union of grills of IFSs is a grill of IFSs.*

Proof. Let $G = \bigcup\{G^j : j \in J,\ G^j \in \Gamma(X)\}$. Since $\tilde{0} \notin G^j$, $\forall j \in J$, $\tilde{0} \notin G$. Also clearly $A \supset B \in G$ implies $A \in G$ and $A \bigcup B \in G$ implies $A \bigcup B \in G^j$ for some $j \in J$ and it follows $A \in G^j$ or $B \in G^j$. Therefore $A \in G$ or $B \in G$. Hence G is a grill of IFSs on X.

Definition 2.11 For each stack S of IFSs, define $dS = \{A : A^c \notin S\}$.

Theorem 2.12 *If* S *(with or without suffixes) is a stack of IFSs,* F *is a filter of IFSs and* G *is a grill of IFSs on* X*, then followings hold :*
(1) $dS^1 \subset dS^2$ *if* $S^1 \supset S^2$,

(2) $d(dS) = S$,
(3) $d(\bigcup S^i) = \bigcap dS^i$,
(4) $d(\bigcap S^i) = \bigcup dS^i$,
(5) dF is a grill of IFSs,
(6) dG is a filter of IFSs.

Proof. (1) Let $A \in dS^1$. That is $A^c \notin S^1$. This implies $A^c \notin S^2$. That is $A \in dS^2$. Thus $dS^1 \subset dS^2$.
(2) Let $A \in I(X)$. Then $A \in d(dS) \Leftrightarrow A^c \notin dS \Leftrightarrow A \in S$. Thus $d(dS) = S$.
(3) Let $A \in d(\bigcup S^i)$. Then $A \in d(\bigcup S^i) \Leftrightarrow A^c \notin \bigcup S^i \Leftrightarrow A^c \notin S^i,\ \forall\ i \in J \Leftrightarrow A \in dS^i,\ \forall\ i \in J$ $\Leftrightarrow A \in \bigcap dS^i$. Thus $d(\bigcup S^i) = \bigcap dS^i$.
(4) Now $A \in d(\bigcap S^i) \Leftrightarrow A^c \notin \bigcap S^i \Leftrightarrow A^c \notin S^{i_o}$, for some $i_o \Leftrightarrow A \in dS^{i_o} \Leftrightarrow A \in \bigcup dS^i$. Thus $d(\bigcap S^i) = \bigcup dS^i$.
(5) Since $\tilde{1} \in F$, $(\tilde{0})^c \in F$ and hence $\tilde{0} \notin dF$. Let $A \supset B \in dF$. Then $B^c \notin F$ and it follows $A^c \notin F$ for $A^c \subset B^c$. Therefore $A \in dF$. Again let $A \bigcup B \in dF$. That is $(A \bigcup B)^c \notin F$. Thus $A^c \bigcap B^c \notin F$ and it follows $A^c \notin F$ or $B^c \notin F$. Therefore $A \in dF$ or $B \in dF$. So dF is a grill of IFSs.
(6) Since $\tilde{0} \notin G$, that is $(\tilde{1})^c \notin G$ and it follows $\tilde{1} \in dG$. Let $A \supset B \in dG$. Thus $B^c \notin G$ and consequently $A^c \notin G$ and it follows $A \in dG$. Further let $A, B \in dG$. That is $A^c \notin G$ and $B^c \notin G$ and it follows $A^c \bigcup B^c \notin G$ and that is $(A \bigcap B)^c \notin G$ and therefore $A \bigcap B \in dG$. Thus dG is a filter of IFSs.

Theorem 2.13 *If F is a filter of IFSs and G is a grill of IFSs such that $F \subset G$ then there exists a prime filter V of IFSs such that $F \subset V \subset G$.*

Proof. Let Γ be a collection of subsets of $I(X)$ defined by
$\forall\ \alpha \subset I(X)$, $\alpha \in \Gamma \Leftrightarrow F \subset \alpha$ and $\forall\ A_1, A_2, \ldots\ldots, A_m \in \alpha \Leftrightarrow A_1 \bigcap A_2 \bigcap \ldots\ldots \bigcap A_m \in G$. Clearly $(\Gamma, \subset)$ is a partially ordered set and $F \in \Gamma$. Also if $\alpha \in \Gamma$ then $F \subset \alpha \subset G$. Now one can check (by using Zorn's lemma) that $(\Gamma, \subset)$ has a maximal element. Let V be such element. Obviously $F \subset V \subset G$.
Let $A_1, A_2 \in V$. Then $V \bigcup \{A_1 \bigcap A_2\} \in \Gamma$ and hence by maximality of V, $A_1 \bigcap A_2 \in V$. Let $A \supset B \in V$. Then $V \bigcup \{A\} \in \Gamma$ and again by maximality of V, $A \in V$. Thus V is a filter of IFSs on X. Let $A, B \in I(X)$ such that $A \notin V$ and $B \notin V$. Then both of $V \bigcup \{A\}$ do not belong to Γ. Hence one can find $A_1, \ldots\ldots, A_m \in V$ and $B_1, \ldots\ldots B_n \in V$ such that $A \bigcap A_1 \bigcap \ldots\ldots \bigcap A_m \bigcap B_1 \bigcap \ldots\ldots \bigcap B_n \notin G$ and $B \bigcap A_1 \bigcap \ldots\ldots \bigcap A_m \bigcap B_1 \bigcap \ldots\ldots \bigcap B_n \notin G$ and hence $(A \bigcup B) \bigcap A_1 \bigcap A_2 \bigcap \ldots\ldots \bigcap A_m \bigcap B_1 \bigcap \ldots\ldots \bigcap B_n \notin G$.

This shows that $A \bigcup B \notin V$. Thus V is a prime filter of IFSs. This completes the proof.

Corollary 2.14 Let $G \subset I(X)$. Then G is a grill of IFSs on X iff it is a union of prime filter of

IFSs on X.

Proof. Since an arbitrary union of grills of IFSs is a grill of IFSs, it follows that if G is an union of prime filters of IFSs then it is a grill of IFSs.
Conversely suppose that G is a grill of IFSs. Let $A \in G$. Set

$$F = \{B \in I(X) : A \subset B\}.$$

Then F is a filter of IFSs and $F \subset G$. So by above Theorem, there exists a prime filter V of IFSs on X such that $F \subset V \subset G$ and hence $A \in V \subset G$. Thus G is an union of prime filters of IFSs on X.

3. Proximities of IFSs

Definition 3.1 A binary relation Δ on $I(X)$ is said to be a basic preproximity of IFSs on X if it satisfies the following conditions :
(1) $\tilde{0} \notin \Delta(A)$, $\forall A \in I(X)$,
(2) $\Delta = \Delta^{-1}$.
(3) $A \bigcup B \in \Delta(C) \Leftrightarrow A \in \Delta(C)$ or $B \in \Delta(C)$ where $\Delta(A) = \{B \in I(X) : (A, B) \in \Delta\}$.

A binary relation π on $I(X)$ is said to be a basic proximity of IFSs on X if it is a preproximity of IFSs and X satisfies the condition

$$A \bigcap B \neq \tilde{0} \Rightarrow (A, B) \in \pi.$$

When $\Delta(\pi)$ is a preproximity (proximity) of IFSs on X then X is called the reference set of $\Delta(\pi)$ and is denoted by $X(\Delta)(X(\pi))$.
Set of all basic preproximities (proximities) of IFSs on X is denoted by $m(X)(M(X))$.
In the sequel, we shall, in general, drop the prefix 'basic' and just talk of preproximities of IFSs and proximities of IFSs. The pair $(X, \Delta)((X, \pi))$ is called a preproximity space of IFSs (proximity space of IFSs) whenever $\Delta \in m(X)(\pi \in M(X))$.

Example 3.2 Let $T = \{(A, B) \in I(X) \times I(X) : A \bigcap B \neq \tilde{0}\}$. Then $(A, \tilde{0}) \notin T$ and $(A, B) \in T \Rightarrow (B, A) \in T$. Now

$$\begin{aligned} (A, B \bigcup C) \in T &\Leftrightarrow A \bigcap (B \bigcup C) \neq \tilde{0} \\ &\Leftrightarrow (A \bigcap B) \bigcup (A \bigcap C) \neq \tilde{0} \\ &\Leftrightarrow A \bigcap B \neq \tilde{0} \text{ or } (A \bigcap C) \neq \tilde{0} \\ &\Leftrightarrow (A, B) \in T \text{ or } (A, C) \in T. \end{aligned}$$

Theorem 3.3 *Let Δ^1, $\Delta^2 \in m(X)$ and $A, B \in I(X)$. Then followings hold :*
(1) $\Delta^1(A \bigcup B) = \Delta^1(A) \bigcup \Delta^1(B)$,

(2) $(\Delta^1 \bigcup \Delta^2)(A) = \Delta^1(A) \bigcup \Delta^2(A)$,
(3) $A \subset B \Rightarrow \Delta^1(A) \subset \Delta^2(B)$.

Proof is straightforward.

Theorem 3.4 *Let Δ be a binary relation on $I(X)$. Then Δ is a preproximity of IFSs on X if and only if $\Delta = \Delta^{-1}$ and $\Delta(A) \in \Gamma(X)$, $\forall\ A \in I(X)$.*

Proof is straightforward.

Definition 3.5 Let $\Delta \in m(X)$, $A \in I(X)$. Then $B \in I(X)$ is called a neighbourhood (in short nbd) of A with respect to Δ if $B^c \notin \Delta(A)$.
The collection of all nbds of A w.r.t. Δ is denoted by $N(\Delta, A)$.

Theorem 3.6 *Let $\Delta, \Delta^1, \Delta^2 \in m(X)$. Then followings hold :*
(1) $N(\delta, \tilde{0}) = I(X)$,
(2) if $B \in N(\Delta, A)$, $B^1 \in N(\Delta, A^1)$, *then* $B \bigcup B^1 \in N(\Delta, A \bigcup A^1)$,
(3) $N(\Delta, A \bigcup B) = N(\Delta, A) \bigcap N(\Delta, B)$,
(4) $N(\Delta, A) \subset N(\Delta, A^1)$ *if* $A^1 \subset A$.
(5) $N(\Delta^1 \bigcup \Delta^2, A) = N(\Delta^1, A) \bigcap N(\Delta^2, A)$,
(6) $N(\Delta^1, A) \subset N(\Delta^2, A)$ *if* $\Delta^2 \subset \Delta^1$.

Proof. (1) Since $A \notin \Delta(\tilde{0})$, $\forall\ A \in I(X)$, $N(\Delta, \tilde{0}) = I(X)$.
(2) Since $B \in N(\Delta, A)$ and $B^1 \in N(\Delta, A^1)$, $B^c \notin \Delta(A)$ and $(B^1)^c \notin \Delta(A^1)$. It follows $(B \bigcup B^1)^c \notin \Delta(A)$ and $(B \bigcup B^1)^c \notin \Delta(A^1)$. That is $(B \bigcup B^1)^c \notin \Delta(A) \bigcup \Delta(A^1) = \Delta(A \bigcup A^1)$ and thus $B \bigcup B^1 \in N(\Delta, A \bigcup A^1)$.
(3) For

$$\begin{aligned} D \in I(X),\ D \in N(\Delta, A \bigcup B) &\Leftrightarrow D^c \notin \Delta(A \bigcup B) = \Delta(A) \bigcup \Delta(B) \\ &\Leftrightarrow D^c \notin \Delta(A) \text{ and } D^c \notin \Delta(B) \\ &\Leftrightarrow D \in N(\Delta, A) \bigcap N(\Delta, B) \end{aligned}$$

Thus $N(\Delta, A \bigcup B) = N(\Delta, A) \bigcap N(\Delta, B)$.
(4) This follows from (3).
(5) Let $B \in I(X)$. Now

$$\begin{aligned} B \in N(\Delta^1 \bigcup \Delta^2, A) &\Leftrightarrow B^c \notin (\Delta^1 \bigcup \Delta^2)(A) = \Delta^1(A) \bigcup \Delta^2(A) \\ &\Leftrightarrow B^c \notin \Delta^1(A) \text{ and } B^c \notin \Delta^2(A) \\ &\Leftrightarrow B \in N(\Delta^1, A) \bigcap N(\Delta^2, A). \end{aligned}$$

Thus $N(\Delta^1 \bigcup \Delta^2, A) = N(\Delta^1, A) \bigcap N(\Delta^2, A)$.

(6) This follows from (5).

Definition 3.7 Let $\Delta \in m(X)$, $A \in I(X)$. We define $C_\Delta : I(X) \to I(X)$ by

$$C_\Delta A = A \bigcup (\bigcup \{(p,q)_x : (p,q)_x \in \Delta(A)\})$$

where $(p,q)_x = \{< x,p,q >: x \in X\}$, $0 < p$, $0 \leq q$ and $p + q \leq 1$.
C_Δ is called the closure operator induced by Δ on X.

Theorem 3.8 *Let $\Delta, \Delta^1 \in m(X)$. $A, B \in I(X)$. Then C_Δ satisfies the following conditins :*
(1) $C_\Delta \tilde{0} = \tilde{0}$,
(2) $A \subset C_\Delta A$,
(3) $C_\Delta(A \bigcup B) = C_\Delta A \bigcup C_\Delta B$,
(4) $C_\Delta A \subset C_{\Delta^1}(A)$ if $\Delta \subset \Delta^1$.

Proof follows from above definition and Theorem 3.3.

From the above Theorem it is mentioned that C_Δ is a Čech closure operator and it is a Kuratowski closure operator if $C_\Delta(C_\Delta A) = C_\Delta A$, $\forall\ A \in I(X)$.

Theorem 3.9 *Let Δ^1, $\Delta^2 \in m(X)$. Then $\forall\ p \in (0,1], \forall\ q \in [0,1)$ with $p + q \leq 1$, $\forall\ x \in X$, $\Delta^1((p,q)_x) = \Delta^2((p,q)_x)$ implies $C_{\Delta^1} A = C_{\Delta^2} A$, $\forall\ A \in I(X)$.*

Proof.

$$\begin{aligned} C_{\Delta^1} A &= A \bigcup (\bigcup \{(p,q)_x : (p,q)_x \in \Delta^1(A)\}) \\ &= A \bigcup (\bigcup \{(p,q)_x : A \in \Delta^1((p,q)_x)\}) \\ &= A \bigcup (\bigcup \{(p,q)_x : A \in \Delta^2((p,q)_x)\}) \\ &= C_{\Delta^2} A. \end{aligned}$$

Theorem 3.10 *For a proximity π of IFSs on X, C_π is a Kuratowski closure operator iff*

$$\tilde{1} \in \pi(C_\pi B) \Rightarrow \tilde{1} \in \pi(B).$$

Proof. Suppose that C_π is a Kuratowski closure operator and let $\tilde{1} \in \pi(C_\pi B)$. Then $C_\pi(C_\pi B) = \tilde{1} = C_\pi B$ and $\tilde{1} \in \pi(B)$. Conversely, suppose that $\tilde{1} \in \pi(C_\pi B) \Rightarrow \tilde{1} \in \pi(B)$. We have to show that C_π is a Kuratowski closure operator. Note that for each $x \in X$, $0 < p \leq 1$, $0 \leq q < 1$ with $p + q \leq 1$, $(p,q)_x \in \pi(C_\pi B)$ implies $(1,0)_x \in \pi(C_\pi B)$ and hence $(1,0)_x \in \pi(B)$. Consequently, $C_\pi B = \tilde{1}$. Thus $C_\pi(C_\pi B) \subset C_\pi B$ and therefore $C_\pi(C_\pi B) = C_\pi B$.

Definition 3.11 Let $\Delta \in m(X)$ and $F \in \phi(X)$. Then we define

$$\Delta(F) = \bigcap\{\Delta(A) : A \in F\}.$$

Theorem 3.12 *For $\Delta, \Delta^1, \Delta^2 \in m(X)$ and $F, F^1, F^2 \in \phi(X)$ followings hold :*
(1) $\Delta(F) \in \Gamma(X)$.
(2) $\Delta(A) = \bigcup\{\Delta(V) : V \in \omega(X),\ A \in V\}$,
(3) $F^1 \subset \Delta(F^2) \Rightarrow F^2 \subset \Delta(F^1)$,
(4) $(\Delta^1 \bigcup \Delta^2)(F) = \Delta^1(F) \bigcup \Delta^2(F)$,
(5) $\Delta(F^1 \bigcap F^2) = \Delta(F^1) \bigcup \Delta(F^2)$.

proof. (1) Clearly $\tilde{0} \notin \Delta(F)$. Again $A \bigcup B \in \Delta(F) \Rightarrow F \subset \Delta(A \bigcup B) = \Delta(A) \bigcup \Delta(B)$. It follows that $F \subset \Delta(A)$ or $F \subset \Delta(B)$. Thus $A \in \Delta(F)$ or $B \in \Delta(F)$. Clearly $A \supset B \in \Delta(F) \Rightarrow A \in \Delta(F)$. Hence $\Delta(F) \in \Gamma(X)$.
(2) Let $B \in \bigcup\{\Delta(V) : V \in \omega(X),\ A \in V\}$.That is there exists $V \in \omega(X)$ such that $B \in \Delta(V)$ and $A \in V$. By above definition, $B \in \Delta(A)$. Thus $\Delta(A) \supset \bigcup\{\Delta(V) : A \in V\}$.
Next, let $B \in \Delta(A)$. Then $A \in \Delta(B)$ and it follows by corollary 2.13, $A \in V \subset \Delta(B)$ for some $V \in \omega(X)$. That is $A \in V$ and $B \in \Delta(V)$. Therefore $\Delta(A) \subset \bigcup\{\Delta(V) : A \in V\}$.
Thus $\Delta(A) = \bigcup\{\Delta(V) : V \in \omega(X),\ A \in V\}$.
(3) Let $A \in F^2$. Since $F^1 \subset \Delta(F^2)$, $F^1 \subset \Delta(A)$. That is $A \in \Delta(F^1)$. Therefore $F^2 \subset \Delta(F^1)$.
(4) For $A \in I(X)$,

$$\begin{aligned} A \in (\Delta^1 \bigcup \Delta^2)(F) &\Leftrightarrow F \subset (\Delta^1 \bigcup \Delta^2)(A) \\ &\Leftrightarrow F \subset \Delta^1(A) \bigcup \Delta^2(A) \\ &\Leftrightarrow F \subset \Delta^1(A) \text{ or } F \subset \Delta^2(A) \text{ (by Theorem 2.8)} \\ &\Leftrightarrow A \in \Delta^1(F) \text{ or } A \in \Delta^2(F) \\ &\Leftrightarrow A \in \Delta^1(F) \bigcup \Delta^2(F) \end{aligned}$$

Thus $(\Delta^1 \bigcup \Delta^2)(F) = \Delta^1(F) \bigcup \Delta^2(F)$.
(5) For $A \in I(X)$,

$$\begin{aligned} A \in \Delta(F^1 \bigcap F^2) &\Leftrightarrow F^1 \bigcap F^2 \subset \Delta(A) \\ &\Leftrightarrow F^1 \subset \Delta(A) \text{ or } F^2 \subset \Delta(A) \text{ (by Theorem 2.7(1))} \\ &\Leftrightarrow A \in \Delta(F^1) \text{ or } A \in \Delta(F^2) \\ &\Leftrightarrow A \in \Delta(F^1) \bigcup \Delta(F^2). \end{aligned}$$

Thus $\Delta(F^1 \bigcap F^2) = \Delta(F^1) \bigcup \Delta(F^2)$.

Theorem 3.13 *For a proximity π of IFSs on X, $F \subset \pi(F)$, $\forall$ proper filter F of IFSs on X.*

Proof. Let $A \in F$. Now $A \bigcap B \neq \tilde{0}$ for all $B \in F$. It follows that $A \in \pi(B)$ for all $B \in F$. Hence $A \in \pi(F)$. Therefore $F \subset \pi(F)$ for all proper filter F of IFSs on X.

Definition 3.14 Let $\Delta \in m(X)$. A subfamily T of $I(X)$ is said to be Δ-compatible if

$$A, B \in T \Rightarrow A \in \Delta(B).$$

A Δ-compatible grill is called a Δ-clan.

Theorem 3.15 *For $\Delta \in m(X)$, $G \in \Gamma(X)$, the followings are equivalent :*
(1) G is a Δ-clan.
(2) If $V \in \omega(X)$ such that $V \subset G$ then $G \subset \Delta(V)$,
(3) $G \subset \bigcap\{\Delta(V) : V \in \omega(X), V \subset G\}$,
(4) If $V^1, V^2 \in \omega(X)$ such that $V^1, V^2 \subset G$ then $V^1 \subset \Delta(V^2)$.

Proof. ((1)$\Rightarrow$(2)) : Suppose that (1) holds. Let $V \in \omega(X)$ such that $V \subset G$ and $A \in G$. It follows that $A \in \Delta(B)$. $\forall\ B \in V$. That is $A \in \Delta(V)$. Therefore $G \subset \Delta(V)$. Hence (1)$\Rightarrow$(2).
((2)$\Rightarrow$(3)) : It is clear from the conditions of (2) and (3).
((3)$\Rightarrow$(4)) : Suppose that (3) holds. Let $V^1, V^2 \in \omega(X)$ such that $V^1, V^2 \subset G$. Since $V^2 \subset G$, $G \subset \Delta(V^2)$. Consequently, $V^1 \subset \Delta(V^2)$. Thus (3) implies (4).
((4)$\Rightarrow$(1)) : Suppose that (4) holds. Let $A, B \in G$. So by Corollary 2.13, there exists $V^1, V^2 \in \omega(X)$ such that $A \in V^1$, $B \in V^2$ where $V^1 \subset G$ and $V^2 \subset G$. Therefore, $A \in V^1 \subset \Delta(v^2) \subset \Delta(B)$. Hence (4)$\Rightarrow$(1).

Theorem 3.16 *Let $\Delta \in m(X)$. Then every Δ-clan is contained in a maximal Δ-clan.*

Proof. By Theorem 2.9, union of grills of IFSs is a grill of IFSs. Further for a family $\{G^j : j \in J\}$ Δ-clans with $G^j \subset G^{j'}$, $j \leq j'$, $\bigcup\{G^j : j \in J\}$ is a Δ-clan. Hence appling Zorn's lemma on the collection of all Δ-clans containing a Δ-clan G, proof of the Theorem follows.

Lemma 3.17 Let $\Delta \in m(X)$. If $A \in \Delta(B)$, then there exists $V^1, V^2 \in \omega(X)$ such that $A \in V^1$. $B \in V^2$ and $V^1 \subset \Delta(V^2)$.

Proof. Since $\Delta(B)$ is a grill of IFSs on X, by Corollary 2.13, there exists a prime filter V^1 of IFSs such that $A \in V^1 \subset \Delta(B)$. By symmetry of Δ, $B \in \Delta(V^1)$. Now $\Delta(V^1) \in \Gamma(X)$. Again by Corollary 2.13, there exists $V^2 \in \omega(X)$ such that $B \in V^2 \subset \Delta(V^1)$. Therefore $A \in V^1$, $B \in V^2$ and $V^1 \subset \Delta(V^2)$.

Theorem 3.18 *Let $\pi \in M(X)$. If $A \in \pi(B)$, then there is a π-clan of the form $V^1 \bigcup V^2$ where $V^1, V^2 \in \omega(X)$ such that $A \in V^1$ and $B \in V^2$.*

Proof. Let $A \in \pi(B)$. So by the above Lemma, there exists V^1, $V^2 \in \omega(X)$ such that $A \in V^1$, $B \in V^2$ and $V^1 \subset \pi(V^2)$. Since $\pi \in M(X)$ and V^1, $V^2 \in \omega(X)$ then for any C, $D \in V^1$ or V^2, $(C, D) \in \pi$ and hence $V^1 \bigcup V^2$ is a π-clan such that $A \in V^1$ and $B \in V^2$.

Corollary 3.19 Let $\pi \in M(X)$. If $A \in \pi(B)$, then there exists a maximal π-clan containing $\{A, B\}$.

Proof. From the above Theorem, there exists a π-clan $V^1 \bigcup V^2$ where V^1, $V^2 \in \omega(X)$, such that $\{A, B\} \subset V^1 \bigcup V^2$. By Theorem 3.16 every π-clan is contained in a maximal π-clan and hence the proof follows.

Corollary 3.20 Let $\pi \in M(X)$. Then
$\pi = \bigcup\{G \times G : G \text{ is maximal } \pi\text{-clan }\}$
$= \bigcup\{G \times G : G \text{ is a } \pi\text{-clan }\}$.

Proof. Let $(A, B) \in \pi$. Then by above Corollary 3.19, there exists a maximal π-clan G such that $(A, B) \in G \times G$. It follows that $\pi \subset \bigcup\{G \times G : G \text{ is maximal } \pi\text{-clan }\}$. By the definition of π-clan, $G \times G \subset \pi$, for any π-clan G. Hence
$\pi = \bigcup\{G \times G : G \text{ is maximal } \pi\text{-clan }\}$
$= \bigcup\{G \times G : G \text{ is a } \pi\text{-clan }\}$.

Remark 3.21 It is known that one of the most fundamental results in the area of proximities of fuzzy sets is that they are clan generated structure [3]. Because of the above representation, it follows that proximities of IFSs are also clan generated structures in the sense of their description as in the above Theorem.

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