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Towards combining two constructions of intuitionistic fuzzy sets

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Abstract

In this paper the authors discuss how *intuitionistic fuzzy sets as defined by Atanassov in [1] can be combined with a construction by Takeuti and Titani ([12]), given by its authors the same name. The resulting object, has the semantics of the former, while satisfying the axioms of the latter. Correctness of the construction is proved in the axiomatic system presented in [12].*

1. Introduction

The first research, devoted to the concept of *"Intuitionistic Fuzzy set" (IFS)* was introduced in 1983 in [1] by the first of the authors (we will refer to this concept as A-IFS). More than a year after this, G. Takeuti and S. Titani introduced in [12] another mathematical concept, using the same name (we will refer to it as T-IFS). When the first author of the present research understood about Takeuti and Titani's paper, he noted this fact in his book [4]. The first bibliography on IFS contains about 400 papers related to A-IFS, prepared in 18 year period (see [11]), while the second bibliography contains new 400 papers only for 3-year period. This fact shows that the interest to A-IFSs increases.

In the present paper the authors will show how the T-IFS construction can be accommodated to an A-IFS representation. Moveover, we will demonstrate how a variety of such constructions can be easily obtained. In [12] Takeuti and Titani suggest that the interval [0;1], viewed as a complete Heyting algebra, can be used to construct a model of the intuitionistic logic. They extend Gentzen's LJ axiomatic of intuitionistic logic with six additional axioms and one derivation rule and prove corectness and strong completeness in the obtained system. We will show how this construction can be trivially modified to use

$$L^* = \{\langle x, y\rangle \in [0;1] \times [0;1] \mid x + y \leq 1\}$$

as the truth value set.

L^* is the truth value set of A-IFS; the two independent components of the pair represent the degree of membership and the degree of non-membership with the condition of non-contradiction. When the second degree depends on the first in the manner

$y = 1-x$, we obtain an ordinary fuzzy set. Therefore, Takeuti and Titani's construction can be modified by introducing an independent degree of non-membership y, which can vary from 0 to $1-x$.

Formulas in intuitionistic logic always satisfy

$$A \to \neg\neg A \tag{1}$$

for every proposition A, but not always

$$\neg\neg A \to A. \tag{2}$$

Whem proposition A satisfies simultaneously (1) and (2), the equality

$$A = \neg\neg A \tag{3}$$

holds, which is valid in classical logic.

Initially, connectives in A-IFS were defined in such manner, that (3) was valid in Intuitionistic Fuzzy Logic (IFL), developed in 1988 in [2] (see, also [4, 5, 8, 9] and others). This fact is the starting point for criticism from some colleagues who argue that the word "intuitionistic" does not apply to A-IFS. Our goal in this paper is to show that one of the possibilities to define connectives in A-IFS, which satisfy the axioms of intuitionistic logic without collapsing to classical logic, is to follow the construction in [12].

2. T-IFS style implication in A-IFS

The T-IFS implication is introduced for $p, q \in [0,1]$ by

$$p \to q = \bigvee\{r \in [0,1] \mid p \wedge r \leq q\} = \begin{cases} 1, & \text{if } p \leq q \\ q, & \text{if } p > q \end{cases}.$$

This implication can be written in an explicit form, using the following auxiliary functions:

$$\mathrm{sg}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}, \mathrm{sg}^*(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0 \end{cases}.$$

Obviously,

$$\mathrm{sg}^*(x) = \mathrm{sg}(1 - \mathrm{sg}(x)).$$

Therefore, the implication in T-IFS sense can be written as

$$p \to q = max(q, \mathrm{sg}^*(p-q)).$$

In the same manner we can define an A-IFS implcation in L^*:

$$\langle p_1, p_2\rangle \to \langle q_1, q_2\rangle = \langle max(q_1, \mathrm{sg}^*(p_1 - q_1)), min(q_2, \mathrm{sg}(p_1 - q_1))\rangle, \tag{4}$$

or equivalently

$$\langle p_1, p_2\rangle \to \langle q_1, q_2\rangle = \begin{cases} \langle 1, 0\rangle, & \text{if } p_1 \leq q_1 \\ \langle q_1, q_2\rangle, & \text{if } p_1 > q_1 \end{cases}. \tag{4'}$$

Clearly

$$(\langle p_1, p_2\rangle \to \langle q_1, q_2\rangle) = \langle 1, 0\rangle \Longleftrightarrow p_1 \leq q_1.$$

Further, the negation has the form

$$\neg\langle a, b\rangle = \langle a, b\rangle \to \langle 0, 1\rangle = \langle \max(0, \mathrm{sg}^*(a)), \min(1, \mathrm{sg}(a))\rangle$$

$$= \langle \mathrm{sg}^*(a), \mathrm{sg}(a)\rangle = \begin{cases} \langle 0, 1\rangle, & \text{if } a > 0 \\ \langle 1, 0\rangle, & \text{if } a = 0 \end{cases}$$

The other logical connectives - conjuction and disjunction, are to be interpreted in the classical A-IFS manner:

$$\langle p_1, p_2\rangle \vee \langle q_1, q_2\rangle = \langle max(p_1, q_1), min(p_2, q_2)\rangle,$$

$$\langle p_1, p_2\rangle \wedge \langle q_1, q_2\rangle = \langle min(p_1, q_1), max(p_2, q_2)\rangle.$$

It is important to note that a partial ordering has been defined on the truth value set L^* as follows:

$$\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle \mathrm{iff} \mathrm{p_1} \leq \mathrm{p_2} \& \mathrm{q_1} \geq \mathrm{q_2},$$

thus making L^* a complete lattice, as noted by Deschrijver and Kerre in [10]. Then the implication defined above has the following property:

$$\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle \Rightarrow (\langle p_1, p_2\rangle \to \langle q_1, q_2\rangle) = \langle 1, 0\rangle.$$

The implication over [0,1] defined by Takeuti and Titani holds the same property for the standard linear ordering of real numbers.

3. A-IFS as a model of Takeuti and Titani's axiomatic

In [12] Takeuti and Titani extend the axiomatic of intuitionistic logic in order to use a non-discrete truth value set with the power of the continuum. We will show that A-IFSs with the connectives, defined in the previous section can be used as semantics for this axiomatic system. Moreover, this can be done following strictly the construction in [12] by translating the operations in [0,1] to the appropriate operations in L^*.

Let us consider the language $\mathcal{L}$ of first-order predicate logic with propositional variables, in which the axiomatic is presented:

- individual free and bound variables: $a_0, a_1, \ldots, x_0, x_1, \ldots$;
- individual constants: $c_0, c_1, \ldots$;
- propositional variables: $p_0, p_1, \ldots$;
- propositional constants: $z_0, z_1, \ldots$;
- predicate symbols with their arity: $R_0/n_0, R_1/n_1, \ldots$;
- logical symbols: $\vee, \wedge, \to, \neg, \exists, \forall$.

In the A-IFS model, propositional variables and constants are to be interpreted in L^*, while predicates are to be interpreted as functions, ranging over L^*.

Let $\mathcal{A}$ be a non-empty set. Let us consider the language $\mathcal{L}'$, extended with a continuum of propositional constants $\overline{\beta}$ for each $\beta \in L^*$ and individual constants $\overline{\alpha}$ for each

$\alpha \in \mathcal{A}$. We will call $\langle \mathcal{A}, [[\ ]] \rangle$ an *A-IFS model* of Takeuti and Titani's axiomatic if $[[\ ]]$ is an interpretation function over the set of closed terms and formulas in $\mathcal{L}'$ as follows:

$$
\begin{array}{ll}
[[c]] \in \mathcal{A} & \mathrm{foreach individual constant c}, \\
[[\overline{\alpha}]] = \alpha & \mathrm{foreach} \alpha \in \mathcal{A}, \\
[[p]] \in L^*, & \\
[[\overline{\beta}]] = \beta & \mathrm{foreach} \beta \in \mathrm{L}^*, \\
[[R]] : \mathcal{A} \to L^* & \mathrm{foreachn - aritypredicatesymbolR},
\end{array}
$$

$$
\begin{array}{rl}
[[R(t_1, \ldots, t_n)]] & = [[R]]([[t_1]], \ldots, [[t_n]]), \\
[[A \wedge B]] & = [[A]] \wedge [[B]], \\
[[A \vee B]] & = [[A]] \vee [[B]], \\
[[\neg A]] & = \neg [[A]], \\
[[A \to B]] & = [[A]] \to [[B]], \\
[[\forall x \varphi(x)]] & = \bigwedge_{\alpha \in \mathcal{A}} [[\varphi(\overline{\alpha})]], \\
[[\exists x \varphi(x)]] & = \bigvee_{\alpha \in \mathcal{A}} [[\varphi(\overline{\alpha})]].
\end{array}
$$

In order to evaluate an open formula in $\mathcal{L}$, we need an assignment of values to the individual and propositional variables $\rho : \{a_0, a_1, \ldots ; z_0, z_1, \ldots\} \to \mathcal{A} \cup L^*$. We will use the notation from [12] for substitution of free variables in an open formula A in $\mathcal{L}$ with their values, given by an assignment ρ to receive the closed formula A_ρ in $\mathcal{L}'$. The truth value of a sequent is defined in a natural way via the implication:

$$[[\Sigma \Rightarrow \Delta]] = \bigvee_{\xi \in \Sigma} [[\xi \to \Delta]].$$

A sequent $\Sigma \Rightarrow \Delta$ is valid in a model $\langle \mathcal{A}, [[\ ]] \rangle$ iff $[[\Sigma \Rightarrow \Delta]]_\rho = \langle 1, 0 \rangle$ for every assignment ρ. A sequent is valid (an A-IFS predicate tautology as defined in [8, 9]) if it is valid in every model.

We should note here that this definition of validity is monotone in L^*, as is the case in [0,1]:

$$[[\Sigma]] \leq [[\Delta]] \Longrightarrow \Sigma \Rightarrow \Delta \mathrm{isvalid}.$$

4. Correctness of the A-IFS model

In [3] Atanassov showed that the axioms and rules of intuitionistic logic are valid in the A-IFS sense. This proof can be translated with the definitions of the validity and implication given above. In this paper we will prove that

(1) the Extra Axiom Schemata are valid in the A-IFS sense

(2) the Extra Inference Rule preserves validity in the A-IFS sense

Theorem 1. The Extra Axiom Schemata for TT-IF are valid.

Proof. In all proofs we will consider an arbitrary model $\langle \mathcal{A}, [[\ ]] \rangle$ with an arbitrary assignment ρ. Furthermore, let

$$
\begin{array}{rl}
[[A]] & = \langle a, b \rangle, \\
[[B]] & = \langle c, d \rangle, \\
[[C]] & = \langle e, f \rangle, \\
[[D]] & = \langle g, h \rangle.
\end{array}
$$

Extra Axiom Schemata of Takeuti and Titani:

$$\begin{array}{ll}
(a) & \Rightarrow (A \to B) \vee ((A \to B) \to B); \\
(b) & (A \to B) \to B \Rightarrow (B \to A) \vee B; \\
(c) & (A \wedge B) \to C \Rightarrow (A \to C) \vee (B \to C); \\
(d) & A \to (B \vee C) \Rightarrow (A \to B) \vee (A \to C); \\
(e) & \forall x(C \vee A(x)) \Rightarrow C \vee \forall x(A(x)), \text{wherexdoesnotoccurinC}; \\
(f) & \forall x A(x) \to C \Rightarrow \exists x(A(x) \to D) \vee (D \to C), \text{wherexdoesnotoccurinD}.
\end{array}$$

(a) Let A and B be formulae in $\mathcal{L}$.

$$\begin{array}{ll}
 & [[(A \to B) \vee ((A \to B) \to B)]]_\rho = \\
= & (\langle a, b\rangle \to \langle c, d\rangle) \vee (\langle a, b\rangle \to \langle c, d\rangle) \to \langle c, d\rangle \\
= & \langle \max(c, \mathrm{sg}^*(a-c)), \min(d, \mathrm{sg}(a-c))\rangle \\
 & \vee \langle \max(c, \mathrm{sg}^*(a-c)), \min(d, \mathrm{sg}(a-c))\rangle \to \langle c, d\rangle \\
= & \langle \max(c, \mathrm{sg}^*(a-c)), \min(d, \mathrm{sg}(a-c))\rangle \\
 & \vee \langle \max(c, \mathrm{sg}^*(\max(c, \mathrm{sg}^*(a-c)) - c)), \min(d, \mathrm{sg}(\max(c, \mathrm{sg}^*(a-c)) - c))\rangle.
\end{array}$$

If $a \le c$, then $\mathrm{sg}^*(a-c) = 1$ and

$$\begin{array}{ll}
[[(A \to B) \vee ((A \to B) \to B)]]_\rho & \\
 = & \langle 1, 0\rangle \vee \langle \max(c, \mathrm{sg}^*(1-c)), \\
 & \quad \min(d, \mathrm{sg}(\max(c, 1-c)))\rangle \\
 = & \langle 1, 0\rangle.
\end{array}$$

If $a > c$, then $\mathrm{sg}^*(a-c) = 0$, $\mathrm{sg}(a-c) = 1$ and

$$\begin{array}{ll}
[[(A \to B) \vee ((A \to B) \to B)]]_\rho & = \langle c, d\rangle \vee \langle \max(c, 1), \min(d, 0)\rangle \\
 & = \langle c, d\rangle \vee \langle 1, 0\rangle = \langle 1, 0\rangle.
\end{array}$$

Therefore, (a) is valid.

(b) Let A and B be formulae in $\mathcal{L}$. From above, we have

$$[[(A \to B) \to B]]_\rho$$

$$= \langle \max(c, \mathrm{sg}^*(\max(c, \mathrm{sg}^*(a-c)) - c)), \min(d, \mathrm{sg}(\max(c, \mathrm{sg}^*(a-c)) - c))\rangle$$

and

$$\begin{array}{ll}
[[(B \to A) \vee B]]_\rho & = \langle \max(a, \mathrm{sg}^*(c-a)), \min(b, \mathrm{sg}(c-a))\rangle \vee \langle c, d\rangle \\
 & = \langle \max(a, c, \mathrm{sg}^*(c-a)), \min(b, d, \mathrm{sg}(c-a))\rangle.
\end{array}$$

If $a \le c = 1$, then $d = 0$ and

$$\max(a, c, \mathrm{sg}^*(c-a)) - \max(c, \mathrm{sg}^*(\max(c, \mathrm{sg}^*(a-c)) - c)) = 1 - 1 = 0$$

and

$$\min(d, \mathrm{sg}(\max(c, \mathrm{sg}^*(a-c)) - c)) - \min(b, d, \mathrm{sg}(c-a)) = 0 - 0 = 0.$$

If $a \le c < 1$, then

$$\max(a, c, \mathrm{sg}^*(c-a)) - \max(c, \mathrm{sg}^*(\max(c, \mathrm{sg}^*(a-c)) - c)c - \max(c, \mathrm{sg}^*(1-c)) = c - c = 0$$

and

$$\begin{array}{c}
\min(d, \mathrm{sg}(\max(c, \mathrm{sg}^*(a-c)) - c)) - \min(b, d, \mathrm{sg}(c-a)) \\
= \min(d, \mathrm{sg}(1-c)) - \min(b, d, \mathrm{sg}(c-a)) \ge d - \min(b, d, \mathrm{sg}(c-a)) \ge 0.
\end{array}$$

If $a > c$, then

$$\max(a, c, \mathrm{sg}^*(c-a)) - \max(c, \mathrm{sg}^*(\max(c, \mathrm{sg}^*(a-c)) - c))1 - \max(c, 0) = 1 - c > 0$$

and

$$\begin{aligned} &\min(d, \mathrm{sg}(\max(c, \mathrm{sg}^*(a-c)) - c)) - \min(b, d, \mathrm{sg}(c-a)) \\ &= \min(d, \mathrm{sg}(\max(c, 0) - c)) - \min(b, d, 0) = 0 - 0 = 0. \end{aligned}$$

Therefore, the value of the left side of (b) is smaller than the value of the right side, hence (b) is valid.

(c) Let A, B and C be formulae in $\mathcal{L}$.

$$\begin{aligned} [[(A \wedge B) \to C]]_\rho &= \langle a, b\rangle \wedge \langle c, d\rangle \to \langle e, f\rangle = \langle \min(a, c), \max(b, d)\rangle \to \langle e, f\rangle \\ &= \langle \max(e, \mathrm{sg}^*(\min(a, c) - e)), \min(f, \mathrm{sg}(\min(a, c) - e))\rangle. \end{aligned}$$

and

$$\begin{aligned} [[(A \to C) \vee (B \to C)]]_\rho = &\quad \langle a, b\rangle \to \langle e, f\rangle \vee \langle c, d\rangle \to \langle e, f\rangle \\ = &\quad \langle \max(e, \mathrm{sg}^*(a-e)), \min(f, \mathrm{sg}(a-e))\rangle \\ &\quad \vee \langle \max(e, \mathrm{sg}^*(c-e)), \min(f, \mathrm{sg}(c-e))\rangle \\ = &\quad \langle \max(e, \mathrm{sg}^*(a-e), \mathrm{sg}^*(c-e)), \\ &\qquad \min(f, \mathrm{sg}(a-e), \mathrm{sg}(c-e))\rangle. \end{aligned}$$

Let

$$X \equiv \max(e, \mathrm{sg}^*(a-e), \mathrm{sg}^*(c-e)) - \max(e, \mathrm{sg}^*(\min(a, c) - e)).$$

If $\min(a, c) \le e$, then $a \le e$ or $c \le e$ and hence

$$X = \max(e, \mathrm{sg}^*(a-e), \mathrm{sg}^*(c-e)) - 1 = 1 - 1 = 0.$$

If $\min(a, c) > e$, then $a > e$ and $c > e$ and hence

$$X = e - e = 0.$$

Therefore, $X = 0$.

For the difference $\min(f, \mathrm{sg}(\min(a, c) - e)) - \min(f, \mathrm{sg}(a-e), \mathrm{sg}(c-e))$ we construct the expression

$$Y \equiv \mathrm{sg}(\min(a, c) - e) - \min(\mathrm{sg}(a-e), \mathrm{sg}(c-e)).$$

If $\min(a, c) \le e$, then $a \le e$ or $c \le e$, and hence

$$Y = 0 - \min(0, 0) = 0,$$

if $\min(a, c) > e$, then $a > e$ and $c > e$, and hence

$$Y = 1 - \min(1, 1) = 0.$$

Therefore, $Y = 0$. Hence, the two sides of (c) have the same values, in particular (c) is valid.

(d) Let A, B and C be formulae in $\mathcal{L}$.

$$\begin{aligned} [[A \to (B \wedge C)]]_\rho &= \langle a, b\rangle \to \langle c, d\rangle \wedge \langle e, f\rangle \\ &= \langle \max(c, e, \mathrm{sg}^*(a - \max(c, e))), \min(d, f, \mathrm{sg}(a - \max(c, e)))\rangle \end{aligned}.$$

and

$$\begin{aligned}[[(A \vee B) \vee (A \to C)]]_\rho = \;& \langle a, b\rangle \to \langle c, d\rangle \vee \langle a, b\rangle \to \langle e, f\rangle \\ = \;& \langle \max(c, \mathrm{sg}^*(a-c)), \min(d, \mathrm{sg}(a-c))\rangle \\ & \vee\langle \max(e, \mathrm{sg}^*(a-e)), \min(f, \mathrm{sg}(a-e))\rangle \\ = \;& \langle \max(c, e, \mathrm{sg}^*(a-c), \mathrm{sg}^*(a-e)), \\ & \quad \min(d, f, \mathrm{sg}(a-c), \mathrm{sg}(a-e))\rangle.\end{aligned}$$

For the difference

$$\max(c, e, \mathrm{sg}^*(a-c), \mathrm{sg}^*(a-e)) - \max(c, e, \mathrm{sg}^*(a - \max(c, e)))$$

let

$$X \equiv \max(\mathrm{sg}^*(a-c), \mathrm{sg}^*(a-e)) - \mathrm{sg}^*(a - \max(c, e)).$$

If $a \le c$ or $a \le e$, then $a \le \max(c, e)$ and

$$X = 1 - 1 = 0,$$

if $a > c$ and $a > e$, then $a > \max(c, e)$ and

$$X = \max(0, 0) - 0 = 0,$$

i.e., $X = 0$. Therefore,

$$\max(c, e, \mathrm{sg}^*(a-c), \mathrm{sg}^*(a-e)) = \max(c, e, \mathrm{sg}^*(a - \max(c, e))).$$

Let

$$Y \equiv \min(d, f, \mathrm{sg}(a - \max(c, e))) - \min(d, f, \mathrm{sg}(a-c), \mathrm{sg}(a-e)).$$

If $a \le c$ or $a \le e$, then $a \le \max(c, e)$ and

$$Y = \min(d, f, 0) - \min(d, f, 0, 0) = 0,$$

if $a > c$ and $a > e$, then $a > \max(c, e)$ and

$$X = \min(d, f, 1) - \min(d, f, 1) = \min(d, f) - \min(d, f) = 0,$$

i.e., $Y = 0$. Therefore, the two sides of (d) have the same value and (d) is valid.

(e) Let $A(x)$ and C be formulae in $\mathcal{L}$ and let x do not occur in C. Let

$$[[A(x)]]_\rho = \langle a(\rho(x)), b(\rho(x))\rangle \mathrm{forsomeassignment}\rho.$$

Then

$$\begin{aligned}[[\forall x(C \vee A(x))]]_\rho &= \textstyle\bigwedge_{\alpha\in\mathcal{A}}(\langle e, f\rangle \vee \langle a(\alpha), b(\alpha)\rangle) \\ &= \textstyle\bigwedge_{\alpha\in\mathcal{A}}\langle \max(e, a(\alpha)), \min(f, b(\alpha))\rangle \\ &= \langle \min_\alpha \max(e, a(\alpha)), \max_\alpha \min(f, b(\alpha))\rangle\end{aligned}$$

and

$$\begin{aligned}[[C \vee \forall x A(x)]]_\rho &= \langle e, f\rangle \vee \textstyle\bigwedge_{\alpha\in\mathcal{A}}\langle a(\alpha), b(\alpha)\rangle \\ &= \langle e, f\rangle \vee \langle \min_\alpha a(\alpha), \max_\alpha b(\alpha)\rangle \\ &= \langle \max(e, \min_\alpha a(\alpha)), \min(f, \max_\alpha b(\alpha))\rangle.\end{aligned}$$

Let

$$X \equiv \max(e, \min_\alpha a(\alpha)) - \min_\alpha \max(e, a(\alpha)).$$

If $e < \min_\alpha a(\alpha)$, then for all $\alpha \in \mathcal{A}$: $e < a(\alpha)$ and

$$X = \min_\alpha a(\alpha) - \min_\alpha a(\alpha) = 0,$$

if $e \geq \min_\alpha a(\alpha)$, then there exists $\alpha \in \mathcal{A}$ for which $e \geq a(\alpha)$ and

$$X = e - \min_\alpha \max(e, a(\alpha)) \geq e - e = 0.$$

Let

$$Y \equiv \max_\alpha \min(f, b(\alpha)) - \min(f, \max_\alpha b(\alpha)).$$

If $f \leq \max_\alpha b(\alpha)$, then there exists $\alpha \in \mathcal{A}$ for which $f \leq b(\alpha)$ and

$$Y = \max_\alpha \min(f, b(\alpha)) - f \geq f - f = 0,$$

if $f > \max_\alpha b(\alpha)$, then for all $\alpha \in \mathcal{A}$: $f > b(\alpha)$ and

$$Y = f - \max_\alpha b(\alpha) > 0,$$

i.e., $Y \geq 0$. Thereforfe, (e) is valid.

(f) Let A, C and D be formulae in $\mathcal{L}$ and let x do not occur in D. Similarly, let

$$[[A(x)]]_\rho = \langle a(\rho(x)), b(\rho(x))\rangle forsomeassignment \rho.$$

Then

$$[[\forall x A(x) \to C]]_\rho = \langle \max(e, \mathrm{sg}^*(\min_\alpha a(\alpha) - e)), \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e))\rangle$$

and

$$\begin{array}{rcl}
[[\exists x(A(x) \to D) \vee (D \to C)]]_\rho & = & \bigvee_{\alpha \in \mathcal{A}} \langle \max(g, \mathrm{sg}^*(a(\alpha) - g)), \\
& & \min(h, \mathrm{sg}(a(\alpha) - g))\rangle \\
& & \vee \langle \max(e, \mathrm{sg}^*(g - e)), \min(f, \mathrm{sg}(g - e))\rangle \\
& = & \langle \max_\alpha(\max(g, \mathrm{sg}^*(a(\alpha) - g))), \\
& & \min_\alpha(\min(h, \mathrm{sg}(a(\alpha) - g)))\rangle \\
& & \vee \langle \max(e, \mathrm{sg}^*(g - e)), \min(f, \mathrm{sg}(g - e))\rangle \\
& = & \langle \max(\max_\alpha(\max(g, \mathrm{sg}^*(a(\alpha) - g))), \\
& & \max(e, \mathrm{sg}^*(g - e))), \\
& & \min(\min_\alpha(\min(h, \mathrm{sg}(a(\alpha) - g))), \\
& & \min(f, \mathrm{sg}(g - e)))\rangle \\
& = & \langle \max(g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g)), e, \mathrm{sg}^*(g - e), \\
& & \min(f, h, \min_\alpha \mathrm{sg}(a(\alpha) - g)), \mathrm{sg}(g - e)\rangle.
\end{array}$$

Let

$$X \equiv \max(g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g)), e, \mathrm{sg}^*(g - e) - \max(e, \mathrm{sg}^*(\min_\alpha a(\alpha) - e)).$$

If there is $\alpha \in \mathcal{A}$: $e \geq a(\alpha)$, then

$$\begin{array}{rl}
X & = \max(e, g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g), \mathrm{sg}^*(g - e)) - \max(e, 1) \\
& = \max(e, g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g), \mathrm{sg}^*(g - e)) - 1.
\end{array}$$

If $g \le e$, then

$$X = \max(e, g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g), 1) - 1 = 0,$$

if $g > e$, then there exists $\alpha \in \mathcal{A}$: $g > e > a(\alpha)$ and

$$X = \max(e, g, 1, 0) - 1 = 0,$$

if for all $\alpha \in \mathcal{A}$: $e < a(\alpha)$, then

$$X = \max(e, g, \max_\alpha \mathrm{sg}^*(a(\alpha) - g), \mathrm{sg}^*(g - e)) - \max(e, 0) \ge 0.$$

Therefore, always $X \ge 0$.

Let

$$Y \equiv \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - \min(f, h, \min_\alpha \mathrm{sg}(a(\alpha) - g), \mathrm{sg}(g - e)).$$

If there is $\alpha \in \mathcal{A}$: $g \ge a(\alpha)$, then

$$\begin{aligned} Y &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - \min(f, h, 0, \mathrm{sg}(g - e)) \\ &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - 0 \ge 0, \end{aligned}$$

if for all $\alpha \in \mathcal{A}$: $g < a(\alpha)$, then

$$\begin{aligned} Y &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - \min(f, h, 1, \mathrm{sg}(g - e)) \\ &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - \min(f, h, \mathrm{sg}(g - e)). \end{aligned}$$

If $e \ge g$, then

$$\begin{aligned} X &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - \min(f, h, 0) \\ &= \min(f, \mathrm{sg}(\min_\alpha a(\alpha) - e)) - 0 \ge 0, \end{aligned}$$

if $e < g$, then for all $\alpha \in \mathcal{A}$: $e < g < a(\alpha)$. i.e.,

$$\min_\alpha a(\alpha) > e.$$

Hence

$$Y = \min(f, 1) - \min(f, h, 1, \mathrm{sg}(g - e)) = f - \min(f, h, 1) = f - \min(f, h) \ge 0.$$

Therefore, always $Y \ge 0$ and (f) is valid. 2

Theorem 2. The Extra Inference Rule TT-IF preserves validity in the A-IFS model.

Proof. We will prove that in the Extra Inference Rule

$$\frac{\Gamma \Rightarrow A \vee (C \to z) \vee (z \to B)}{\Gamma \Rightarrow A \vee (C \to B)}$$

whenever the upper sequent is valid then the lower sequent is also valid, where Γ is a finite set of propositional formulas and z is a propositional variable, which does not occur in the lower sequent.

The proof for the TT-IFS case, as done in [12] can easily be translated in the A-IFS model in the following manner.

We will prove the contraposition of the rule, i.e. whenever the lower sequent is not valid, the upper sequent is not valid either. Let $\langle \mathcal{A}, [[\ ]] \rangle$ be an A-IFS model, and ρ be an assignment in this model for which:

$$[[\Gamma \Rightarrow A \vee (C \to B)]]_\rho \ne \langle 1, 0 \rangle.$$

As $[[\Gamma \Rightarrow F]]_\rho = \bigvee_{\xi\in\Gamma}[[\xi \Rightarrow F]]_\rho$, then for every $\xi \in \Gamma$ it is true that

$$[[\xi \Rightarrow A \vee (C \to B)]]_\rho \neq \langle 1, 0\rangle.$$

Let $[[A]]_\rho = \langle a_1, a_2\rangle, [[B]]_\rho = \langle b_1, b_2\rangle, [[C]]_\rho = \langle c_1, c_2\rangle, [[\xi]]_\rho = \langle \xi_1, \xi_2\rangle, [[C \to B]]_\rho = \langle d_1, d_2\rangle$. Let us fix a formula $\xi \in \Gamma$. From the properties of the implication we have

$$\xi_1 > max(a_1, d_1),$$

or equivalently

$$\xi_1 > a_1, \xi_1 > d_1.$$

Therefore $d_1 < 1$ and from the properties of the implication again we have

$$c_1 > b_1, \langle d_1, d_2\rangle = \langle b_1, b_2\rangle.$$

Finally, for each $\xi \in \Gamma$ we have

$$\xi_1 > a_1, \underbrace{\xi_1 > b_1, c_1 > b_1}_{min(\xi_1, c_1) > b_1},$$

Since Γ is finite, we can find a number z_1 for which:

$$min(min_{\xi\in\Gamma}(\xi_1), c_1) > z_1 > b_1.$$

Therefore if we take a propositional variable z for which $[[z]]_\rho = \langle z_1, z_2\rangle$ for any $z_2 \in [0; 1 - z_1]$, then we shall have

$$[[C \to z]]_\rho \neq \langle 1, 0\rangle,$$

$$[[z \to B]]_\rho \neq \langle 1, 0\rangle,$$

$$[[\xi \Rightarrow A \vee (C \to z) \vee (z \to B)]]_\rho \neq \langle 1, 0\rangle,$$

for each $\xi \in \Gamma$. Finally,

$$[[\Gamma \Rightarrow A \vee (C \to z) \vee (z \to B)]]_\rho \neq \langle 1, 0\rangle. \ 2$$

Theorems 1 and 2 complete the proof of the A-IFS model correctness in Takeuti and Titani's axiomatic.

6. Conclusion

The present arctile is the first step in our research. In future we will prove strong completeness of this construct with respect to the axiomatic. We will also show that this construction is not the only possibility for building an A-IFS model. In this paper we have only discussed the straightforward generalization of Takeuti and Titani's implication. It should be noted that in [2] there has been defined a sg−implication, which is already more general than (4). However, an even more stronger implication can be defined and we plan to show this in our future work. Other A-IFS implications have been discussed in [6] and A-IFS negations – in [7].

This paper is an illustration for an object, which can be constructed to follow the intuitionistic idea but in two senses - TT-IFS sense and A-IFS sense. How should we name such kind of object? From A-IFS point of view it is a natural extension by introducing two new opearations. The authors' opinion is that it is not justified that for every set of logical connectives on A-IFS we should come up with a new name.

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