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RESOLUTION OF COMPOSITE INTUITIONISTIC FUZZY RELATIONAL EQUATIONS

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Abstract. We consider intuitionistic fuzzy relations and their compositions. Attention is paid on the resolution problem for various composite fuzzy relational equations. Analytic expression for the extreme solution is given. The relationship with matrices is studied.

1. Basic notions

In order to make the exposition clear we recall some basic definitions and results and introduce the underlying notions for the fuzzy linear system under study.

We follow [3,5] for definitions of lattice theory. We use the ordinary symbol $\leq$ for the partial order relation on a partially ordered set (poset) P. By a ***greatest element*** of a poset P we mean an element $b \in P$ such that $x \leq b$ for all $x \in P$; the ***least element*** of P being defined dually. The (unique) least and greatest elements of P, when they exist, are called ***universal bounds*** of P and are denoted by 0 and 1 respectively. A ***lattice*** is a poset L any two of whose elements x and y have a ***greatest lower bound*** (g.l.b.) or ***meet*** denoted by $x \wedge y$, and a ***least upper bound*** (l.u.b.) or ***join*** denoted by $x \vee y$. A lattice is ***complete*** when each of its subsets X has a l.u.b., denoted by $\sup X$ or $\vee X$ and a g.l.b. denoted by $\inf X$ or $\wedge X$. A ***Brouwerian lattice*** BL is a lattice L in which for any given elements a and b the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element, denoted $a \alpha b$ (called the *relative pseudocomplement* of a in b). In a ***dually Brouwerian lattice*** for any given elements a and b the set of all $x \in L$ such that $a \vee x \geq b$ contains a least element, denoted $a \varepsilon b$.

Example. Let L be a totally ordered set with universal bounds 0 and 1 and with operations join $\vee$ and meet $\wedge$. L will stay for the bounded chain $\mathsf{L} = (L, \vee, \wedge, 0, 1)$. Obviously L is a complete Brouwerian and a complete dually Brouwerian lattice, if we define

$$a \alpha b = \begin{cases} 1 & \text{if } a \leq b \\ b & a > b \end{cases}, \tag{1.1}$$

$$a \varepsilon b = \begin{cases} b & \text{if } a < b \\ 0 & a \geq b \end{cases}. \tag{1.2}$$

Let BL be a fixed complete Brouwerian lattice (with underlying lattice L). The α-operation has some useful properties, which we shall list now and apply in the proof of the next statements.

If $a, b \in L$ then

i) $a \wedge (a \alpha b) \leq b$ [7] and $a \vee (a \alpha b) \geq b$; (1.3)

ii) $a \alpha (a \wedge b) \geq b$ [7] and $a \alpha (a \vee b) = 1$. (1.4)

If $a, b, d \in L$ then

$a \alpha (b \vee d) \geq a \alpha b$ as well as $a \alpha (b \vee d) \geq a \alpha d$ [7]. (1.5)

Dually, any fixed complete dually Brouwerian lattice (with underlying lattice L) has the following properties concerning the ε-operation, which we shall list now and apply in the proof of the next statements.

If $a, b \in L$ then

i) $a \vee (a\varepsilon b) \geq b$ and $a \wedge (a\varepsilon b) \leq b$; (1.6)

ii) $a\varepsilon(a \vee b) \leq b$ and $a\varepsilon(a \wedge b) = 0$; (1.7)

If $a, b, d \in L$ then

$a\varepsilon(b \wedge d) \leq a\varepsilon b$ as well as $a\varepsilon(b \wedge d) \leq a\varepsilon d$. (1.8)

Definition 1.1. If L is a complete Brouwerian lattice and $E \neq \varnothing$ is a crisp set, $A \subseteq E$, a ***fuzzy set*** $\tilde{A}$ on E is

$$\tilde{A} = \{(x, \mu_A(x)) / x \in E\},$$

where the function $\mu_A : E \to L$ defines the degree of membership of the elements $x \in E$.

Definition 1.2. Let L is a complete Brouwerian and complete dually Brouwerian lattice and $E \neq \varnothing$ is a crisp set, $A \subseteq E$. An ***intuitionistic fuzzy set*** (IFS) $\hat{A}$ on E is

$$\hat{A} = \{(x, \mu_A(x), \nu_A(x)) / x \in E\},$$

where the function $\mu_A : E \to L$ defines the degree of membership and the function $\nu_A : E \to L$ defines the degree of non-membership respectively of the elements $x \in E$ and for any isotonic mapping $\varphi : L \to [0,1]$ we have $0 \leq \varphi(\mu(x)) + \varphi(\nu(x)) \leq 1$ for each $x \in E$.

Remarks 1.3. 1. In what follows we shall write A for the fuzzy set $\tilde{A} = \{(x, \mu_A(x)) / x \in E\}$ and also A for the IFS $\hat{A} = \{(x, \mu_A(x), \nu_A(x)) / x \in E\}$, when there is no danger of confusion.

2. The class of all fuzzy sets over the universe E with L as a complete Brouwerian lattice is denoted by $\mathcal{L}(E)$.

3. The class of all intuitionistic fuzzy sets over the universe E with L as a complete Brouwerian and complete dually Brouwerian lattice is denoted by $\mathcal{L}\mathcal{I}(E)$. For $L = [0,1]$ (cf. [1]) requirement in Definition 1.2 is $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

2. Intuitionistic Fuzzy Relations

The next results are valid for a complete Brouwerian or/and a complete dually Brouwerian lattice. We need completeness of the lattice in order to be able to define various compositions of relations, resp. matrix multiplications. The main part of exposition is organized for the bounded chain $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ because of several different reasons: $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ is both complete Brouwerian and complete dually Brouwerian lattice; computations for fuzzy linear systems of equations and inequalities as well as for relational, resp. matrix equations and inclusions is simple; $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ is a suitable structure for intuitionistic fuzzy sets; $\mathsf{L} = (L, \vee, \wedge, 0, 1)$ is suitable for various applications.

Definition 2.1. A ***fuzzy relation*** between two nonempty sets X and Y is a fuzzy set $\tilde{R}$ of $X \times Y$. An ***intutionistic fuzzy relation*** between two nonempty sets X and Y is an intutionistic fuzzy set $\tilde{R}$ of $X \times Y$. $X \times Y$ is called the ***support*** of the (intutionistic) fuzzy relation $\tilde{R}$.

Definition 2.1 means that the fuzzy relation $\tilde{R} \in \mathcal{L}(X \times Y)$ is described as

$$\tilde{R} = \{((x,y), \mu_R(x,y)) : (x,y) \in X \times Y, \mu_R : X \times Y \to L\},$$

resp. $\tilde{R} \in \mathcal{L}\mathcal{I}(X \times Y)$, is given as

$$\widetilde{R} = \{((x,y), \mu_R(x,y), \nu_R(x,y)) : (x,y) \in X \times Y, \mu_R : X \times Y \to L, \nu_R : X \times Y \to L\}.$$

In what follows we shall study fuzzy relations and intutionistic fuzzy relations. If the notion or statement is valid for intutionistic fuzzy relation and for fuzzy relation, we write this as (intutionistic) fuzzy relation. In order to simplify the exposition we shall omitt the overbar and write R for the (intutionistic) fuzzy relation $\widetilde{R}$. The attention is paid on composite intuitionistic fuzzy relational equations, bearing in mind the classical results for composite fuzzy relational equations [4, 6, 7, 8].

Definition 2.2. The fuzzy relation $R^{-1} \in \mathcal{L}(Y \times X)$ is called ***inverse*** or ***transpose*** of $R \in \mathcal{L}(X \times Y)$, if $R^{-1}(y,x) = R(x,y)$ for any $(y,x) \in Y \times X$. The intutionistic fuzzy relation $R^{-1} \in \mathcal{LI}(Y \times X)$ is called ***inverse*** or ***transpose*** of $R \in \mathcal{LI}(X \times Y)$, if $R^{-1}(y,x) = R(x,y)$ for any $(y,x) \in Y \times X$.

Obviously if $R^{-1} \in \mathcal{L}(Y \times X)$ is the inverse of $R \in \mathcal{L}(X \times Y)$ then

$R^{-1}(y,x) = R(x,y)$ iff $\mu_{R^{-1}}(y,x) = \mu_R(x,y)$ for any $(y,x) \in Y \times X$.

If $R^{-1} \in \mathcal{LI}(Y \times X)$ is the inverse of $R \in \mathcal{LI}(X \times Y)$, then

$R^{-1}(y,x) = R(x,y)$ iff $\mu_{R^{-1}}(y,x) = \mu_R(x,y)$ and $\nu_{R^{-1}}(y,x) = \nu_R(x,y)$ for any $(y,x) \in Y \times X$.

Definition 2.3.

i) Let $R, S \in \mathcal{L}(X \times Y)$ be two fuzzy relations.

$R \subseteq S \Leftrightarrow \mu_R(x,y) \le \mu_S(x,y)$ for any $(x,y) \in X \times Y$; (2.1i)

ii) Let $R, S \in \mathcal{LI}(X \times Y)$ be two intutionistic fuzzy relations.

$R \subseteq S \Leftrightarrow \mu_R(x,y) \le \mu_S(x,y)$ and $\nu_R(x,y) \ge \nu_S(x,y)$ for any $(x,y) \in X \times Y$; (2.1ii)

iii) Let R, S be two (intutionistic) fuzzy relations over the same support $X \times Y$.

$R = S \Leftrightarrow \mu_R(x,y) = \mu_S(x,y)$ (and $\nu_R(x,y) = \nu_S(x,y)$) for any $(x,y) \in X \times Y$. (2.1iii)

Definition 2.4. The (intutionistic) fuzzy relations R over the support $X \times Y$ and S over the support $Y \times Z$ with $pr_2(X \times Y) = pr_1(Y \times Z) = Y$ are called ***conformable*** for composition.

Definition 2.5. Let $R \in \mathcal{L}(X \times Y)$ and $S \in \mathcal{L}(Y \times Z)$ be two fuzzy conformable relations. The fuzzy relation

i) $R \bullet S \in \mathcal{L}(X \times Z)$ is called $\bullet$***-composition*** and is defined by

$$(R \bullet S)(x,z) = \bigvee_{y \in Y} (R(x,y) \wedge S(y,z)), \ (x,z) \in X \times Z. \qquad (2.2i)$$

ii) $R \circ S \in \mathcal{L}(X \times Z)$ is called $\circ$***-composition*** and is defined by

$$(R \circ S)(x,z) = \bigwedge_{y \in Y} (R(x,y) \vee S(y,z)), \ (x,z) \in X \times Z. \qquad (2.2ii)$$

iii) $R \alpha S \in \mathcal{L}(X \times Z)$ is called α***-composition*** and is defined by

$$(R \alpha S)(x,z) = \bigwedge_{y \in Y} (R(x,y) \alpha S(y,z)), \ (x,z) \in X \times Z. \qquad (2.2iii)$$

iv) $R \varepsilon S \in \mathcal{L}(X \times Z)$ is called ε***-composition*** and is defined by

$$(R\varepsilon S)(x,z) = \bigvee_{y\in Y} (R(x,y)\varepsilon S(y,z)),\ (x,z)\in X\times Z. \tag{2.2iv}$$

Definition 2.6. Let $R \in \mathcal{L}\mathcal{I}(X\times Y)$ and $S \in \mathcal{L}\mathcal{I}(Y\times Z)$ be two conformable intuitionistic fuzzy relations. The intuitionistic fuzzy relation

i) $R * S \in \mathcal{L}\mathcal{I}(X\times Z)$ is called $*$*-composition* and is defined as

$R * S = \{((x,z), \mu_{R*S}(x,z), \nu_{R*S}(x,z)) : (x,z)\in R\bullet S \wedge (x,z)\in R\circ S\}$,

with the degree of membership function $\mu_{R*S} = \mu_{R\bullet S}$ and the degree of non-membership function $\nu_{R*S} = \nu_{R\circ S}$ of the elements $(x,z)\in X\times Z$.

ii) $R\otimes S \in \mathcal{L}\mathcal{I}(X\times Z)$ is called $\otimes$*-composition* and is defined as

$R\otimes S = \{((x,z), \mu_{R\otimes S}(x,z), \nu_{R\otimes S}(x,z)) : (x,z)\in R\alpha S \wedge (x,z)\in R\varepsilon S\}$,

with the degree of membership function $\mu_{R\otimes S} = \mu_{R\alpha S}$ and the degree of non-membership function $\nu_{R\otimes S} = \nu_{R\varepsilon S}$ of the elements $(x,z)\in X\times Z$.

According to Definition 2.6 it is more convinient to write

$R * S = (R\bullet S, R\circ S)$ and $R\otimes S = (R\alpha S, R\varepsilon S)$.

For other kind of compositions cf. [1, 2].

Theorem 2.7. *Let* $R \in \mathcal{L}(X\times Y)$ *and* $S \in \mathcal{L}(Y\times Z)$ *be two fuzzy conformable relations. Then:*

i) $S \subseteq R^{-1}\alpha(R\bullet S)$ [7];

ii) $S \supseteq R^{-1}\varepsilon(R\circ S)$;

iii) $R^{-1} \subseteq S\alpha(R\bullet S)^{-1}$ [7];

iv) $R^{-1} \supseteq S\varepsilon(R\circ S)^{-1}$.

Proof. i) [7] Let $T = R\bullet S$, $T\in\mathcal{L}(X\times Z)$ and $P = R^{-1}\alpha T$, $P\in\mathcal{L}(Y\times Z)$. From Definition 2.2, Definition 2.5 (i), (iii) we obtain

$$P(y,z) = \bigwedge_{x\in X} (R^{-1}(y,x)\alpha T(x,z)) = \bigwedge_{x\in X} (R(x,y)\alpha(R\bullet S)(x,z)) =$$

$$= \bigwedge_{x\in X}\left(R(x,y)\alpha \bigvee_{t\in Y} ((R(x,t)\wedge S(t,z)))\right) =$$

$$= \bigwedge_{x\in X}\left(R(x,y)\alpha\left((R(x,y)\wedge S(y,z))\vee\left(\bigvee_{t\in Y,\, t\neq y} (R(x,t)\wedge S(t,z)\right)\right)\right).$$

Bearing in mind (1.5) and then (1.4) we obtain

$$P(y,z) \ge \bigwedge_{x\in X} (R(x,y)\alpha(R(x,y)\wedge S(y,z))),\ R(x,y)\alpha(R(x,y)\wedge S(y,z)) \ge S(y,z),$$

hence $S \subseteq R^{-1}\alpha(R\bullet S)$.

ii) Let $T = R\circ S$, $T\in\mathcal{L}(X\times Z)$ and $P = R^{-1}\varepsilon T$, $P\in\mathcal{L}(Y\times Z)$. From Definition 2.2, Definition 2.5 (ii), (iv) we obtain

$$P(y,z) = \bigvee_{x\in X} \left(R^{-1}(y,x)\varepsilon T(x,z)\right) = \bigvee_{x\in X} \left(R(x,y)\varepsilon (R\circ S)(x,z)\right) =$$

$$= \bigvee_{x\in X}\left(R(x,y)\varepsilon\left(\bigwedge_{t\in Y} (R(x,t)\vee S(t,z))\right)\right) =$$

$$= \bigvee_{x\in X}\left(R(x,y)\varepsilon\left((R(x,y)\vee S(y,z))\wedge\left(\bigwedge_{t\in Y,\, t\neq y} (R(x,t)\vee S(t,z))\right)\right)\right).$$

According to (1.8) and then (1.7) we have

$P(y,z) \le \bigvee_{x\in X} \left(R(x,y)\varepsilon \left(R(x,y)\vee S(y,z)\right)\right)$, $R(x,y)\varepsilon \left(R(x,y)\vee S(y,z)\right) \le S(y,z)$, i.e.

$S \supseteq R^{-1}\varepsilon (R\circ S)$.

iii) The proof is analogous to the proof of Theorem 2.7 (i).

iv) The proof is analogous to the proof of Theorem 2.7 (ii). □

Corollary 2.8. *Let $R \in \mathcal{L}(X\times Y)$ and $S \in \mathcal{L}(Y\times Z)$ be two fuzzy conformable relations. Then:*

i) $R^{-1}\varepsilon(R\circ S) \subseteq S \subseteq R^{-1}\alpha(R\bullet S)$;

ii) $S\varepsilon(R\circ S)^{-1} \subseteq R^{-1} \subseteq S\alpha(R\bullet S)^{-1}$.

Proof. i) Follows from Theorem 2.7 (i), (ii).

ii) Follows from Theorem 2.7 (iii), (iv). □

Corollary 2.9. *Let $R \in \mathcal{IF}(X\times Y)$ and $S \in \mathcal{IF}(Y\times Z)$ be two conformable intuitionistic fuzzy relations. Then:*

i) $S \subseteq R^{-1}\otimes(R * S)$;

ii) $R^{-1} \subseteq S\otimes(R*S)^{-1}$.

Proof. i) According to Theorem 2.7 (i) we have $\mu_S \le \mu_{R^{-1}\alpha(R\bullet S)}$ and bearing in mind (2.1i), Definition 2.6 (i), $\mu_{R^{-1}\alpha(R\bullet S)} = \mu_{R^{-1}\otimes(R*S)}$ hence $\mu_S \le \mu_{R^{-1}\alpha(R\bullet S)} = \mu_{R^{-1}\otimes(R*S)}$. According to Theorem 2.7 (ii) we have $\nu_S \ge \nu_{R^{-1}\varepsilon(R\circ S)}$ and bearing in mind Definition 2.6 (i) $\nu_{R^{-1}\varepsilon(R\circ S)} = \nu_{R^{-1}\otimes(R*S)}$ hence $\nu_S \ge \nu_{R^{-1}\otimes(R*S)}$. According to (2.1ii) we have $S \subseteq R^{-1}\otimes(R*S)$.

ii) Follows from Theorem 2.7 (iii), (iv) and Definition 2.6, bearing in mind that $\mu_{R^{-1}} \le \mu_{S\alpha(R\circ S)^{-1}} = \mu_{S\otimes(R*S)^{-1}}$ and $\nu_{R^{-1}} \ge \nu_{S\alpha(R\bullet S)^{-1}} = \nu_{S\otimes(R*S)^{-1}}$. Now from Definition 2.3 (ii) we have $R^{-1} \subseteq S\otimes(R*S)^{-1}$. □

Theorem 2.10. *Let $R \in \mathcal{L}(X\times Y)$ and $T \in \mathcal{L}(X\times Z)$ be two fuzzy relations. Then:*

i) $R\bullet(R^{-1}\alpha T) \subseteq T$ [7];

ii) $R\circ(R^{-1}\varepsilon T) \supseteq T$;

iii) $R\bullet(R^{-1}\alpha T) \subseteq T \subseteq R\circ(R^{-1}\varepsilon T)$.

Proof. i) [7] Let $R\bullet(R^{-1}\alpha T) = S$, $S \in \mathcal{L}(X\times Z)$. Then

$$S(x,z)=\bigvee_{y\in Y}\left(R(x,y)\wedge(R^{-1}\alpha T)(y,z)\right)=\bigvee_{y\in Y}\left(R(x,y)\wedge(\bigwedge_{t\in X}R(t,y)\alpha T(t,z))\right)=$$

$$=\bigvee_{y\in Y}\left(R(x,y)\wedge\left(R(x,y)\alpha T(x,z)\right)\wedge\bigwedge_{t\in X,\,t\neq x}\left(R(t,y)\alpha T(t,z)\right)\right).$$

The last expression means that

$$S(x,z)\leq\bigvee_{y\in Y}\left(R(x,y)\wedge\left(R(x,y)\alpha T(x,z)\right)\right),$$

hence from (1.3) we have

$S(x,z)\leq T(x,z)$, i.e. $R\bullet(R^{-1}\alpha T)\subseteq T$.

ii) Let $R\circ(R^{-1}\varepsilon T)=S$, $S\in\mathcal{L}(X\times Z)$. Then

$$S(x,z)=\bigwedge_{y\in Y}\left(R(x,y)\vee(R^{-1}\varepsilon T)(y,z)\right)=\bigwedge_{y\in Y}\left(R(x,y)\vee(\bigvee_{t\in X}R(t,y)\varepsilon T(t,z))\right)=$$

$$=\bigwedge_{y\in Y}\left(R(x,y)\vee\left(R(x,y)\varepsilon T(x,z)\right)\vee\bigvee_{t\in X,\,t\neq x}\left(R(t,y)\varepsilon T(t,z)\right)\right).$$

The last expression means that

$$S(x,z)\geq\bigwedge_{y\in Y}\left(R(x,y)\vee\left(R(x,y)\varepsilon T(x,z)\right)\right),$$

hence from (1.6) we have

$S(x,z)\geq T(x,z)$, i.e. $R\circ(R^{-1}\varepsilon T)\supseteq T$.

iii) Follows from Theorem 2.10 (i) and (ii). □

Corrolary 2.11. *Let* $R\in\mathcal{LI}(X\times Y)$ *and* $T\in\mathcal{LI}(X\times Z)$ *be two intuitionistic fuzzy relations. Then* $T\supseteq R*(R^{-1}\otimes T)$.

Proof. Follows from Theorem 2.10, bearing in mind that $\mu_T\geq\mu_{R\bullet(R^{-1}\alpha T)}=\mu_{R*(R^{-1}\otimes T)}$ and $\nu_T\leq\nu_{R\circ(R^{-1}\varepsilon T)}=\nu_{R*(R^{-1}\otimes T)}$. □

Theorem 2.12. *Let* $Q\in\mathcal{L}(Y\times Z)$ *and* $T\in\mathcal{L}(X\times Z)$ *be two fuzzy relations. Then:*

i) $(Q\alpha T^{-1})\bullet Q\subseteq T$ [7];

ii) $(Q\varepsilon T^{-1})\circ Q\supseteq T$. □

The proof is analogous to the proof of Theorem 2.9.

Corollary 2.13. *Let* $Q\in\mathcal{LI}(Y\times Z)$ *and* $T\in\mathcal{LI}(X\times Z)$ *be two intuitionistic fuzzy relations. Then* $T\supseteq(Q\otimes T^{-1})*Q$.

Proof. Follows from the facts that

$\mu_T\geq\mu_{(Q\alpha T^{-1})\bullet Q}=\mu_{(Q\otimes T^{-1})*Q}$ and $\nu_T\leq\mu_{(R\varepsilon T^{-1})\circ R}=\nu_{(Q\otimes T^{-1})*Q}$. □

Theorem 2.14. [7] *Let* $R\in\mathcal{L}(X\times Y)$ *and* $T\in\mathcal{L}(X\times Z)$ *be two fuzzy relations,* $\mathcal{Q}_\bullet$ *be the set of fuzzy relations* $Q\in\mathcal{L}(Y\times Z)$ *such that* $R\bullet Q=T$. *Then:*

i) $\mathcal{Q}_\bullet \neq \varnothing$ *iff* $R^{-1}\alpha T \in \mathcal{Q}_\bullet$;

ii) *if* $\mathcal{Q}_\bullet \neq \varnothing$ *then* $R^{-1}\alpha T$ *is the greatest element in* $\mathcal{Q}_\bullet$.

Proof. i) If $\mathcal{Q}_\bullet \neq \varnothing$ then there exists at least one fuzzy relation $Q \in \mathcal{L}(Y \times Z)$ such that $R \bullet Q = T$. From Theorem 2.7 (i) we have $Q \subseteq R^{-1}\alpha(R \bullet Q) = R^{-1}\alpha T$. Since $Q \subseteq R^{-1}\alpha T$ we have $R \bullet Q \subseteq R \bullet (R^{-1}\alpha T)$, i. e. $T \subseteq R \bullet (R^{-1}\alpha T)$. But from Theorem 2.10 (i) we have $R \bullet (R^{-1}\alpha T) \subseteq T$, hence $R \bullet (R^{-1}\alpha T) = T$ and thus $R^{-1}\alpha T$ belongs to $\mathcal{Q}_\bullet$.

ii) If $\mathcal{Q}_\bullet \neq \varnothing$ then according to the proof of Theorem 2.14 (i), $Q \subseteq R^{-1}\alpha(R \bullet Q) = R^{-1}\alpha T$. Since $R^{-1}\alpha T$ belongs to $\mathcal{Q}_\bullet$, it is the greatest element in $\mathcal{Q}_\bullet$. □

Theorem 2.15. [7] *Let* $R \in \mathcal{L}(X \times Y)$ *and* $T \in \mathcal{L}(X \times Z)$ *be two fuzzy relations,* $\mathcal{Q}_\circ$ *be the set of fuzzy relations* $Q \in \mathcal{L}(Y \times Z)$ *such that* $R \circ Q = T$. *Then:*

i) $\mathcal{Q}_\circ \neq \varnothing$ *iff* $R^{-1}\varepsilon T \in \mathcal{Q}_\circ$;

ii) *if* $\mathcal{Q}_\circ \neq \varnothing$ *then* $R^{-1}\varepsilon T$ *is the least element in* $\mathcal{Q}_\circ$. □

The proof is analogous to those of Theorem 2.14.

Theorem 2.14 gives an easy way to check whether the relational equation $R \bullet Q = T$ is solvable for the unknown relation Q and if it is solvable – to find the greatest solution. Theorem 2.15 gives an easy way to check whether the relational equation $R \circ Q = T$ is solvable for the unknown relation Q and if it is solvable – to find the least solution.

The next theorem is the main result for composite intuitionistic fuzzy relational equations. As mentioned in [9], this was an open problem up to now.

Theorem 2.16. *Let* $R \in \mathcal{LI}(X \times Y)$ *and* $T \in \mathcal{LI}(X \times Z)$ *be two intuitionistic fuzzy relations,* $\mathcal{Q}_*$ *be the set of intuitionistic fuzzy relations* $Q \in \mathcal{LI}(Y \times Z)$ *such that* $R * Q = T$. *Then:*

i) $\mathcal{Q}_* \neq \varnothing$ *iff* $R^{-1} \otimes T \in \mathcal{Q}_*$;

ii) *if* $\mathcal{Q}_* \neq \varnothing$ *then* $R^{-1} \otimes T \in \mathcal{Q}_*$ *is the greatest element in* $\mathcal{Q}_*$.

Proof. Follows from Theorem 2.14 and 2.15. □

3. Intuitionistic Fuzzy Relations and Matrices

If the relation is over finite support, we can present it by a finite matrix and vice versa.

Let $I, J \neq \varnothing$ be finite sets of indices, $a : I \times J \to L$ be a map and $L^{I \times J} = \{ a : I \times J \to L \}$ be the set of all maps from $I \times J$ to L. Any map from the set $L^{I \times J}$ defines a matrix over L as follows:

Definition 3.1. $A = (a_{ij})_{m \times n}$, $m = |I|$, $n = |J|$ is called a ***matrix over*** L if there exists a map $a \in L^{I \times J}$, such that $a_{ij} = a(i,j)$ for each $i \in I$ and each $j \in J$.

Definition 3.2. The matrix $A^t = (a_{ji})_{n \times m}$ is the ***transpose*** of $A = (a_{ij})_{m \times n}$.

First we consider various kinds of matrix multiplication.

Definition 3.3. Let $A = (a_{ij}) \in L^{I \times K}$ and $B = (b_{ij}) \in L^{K \times J}$ be given conformable finite matrices over L.

i) The matrix $C = A \bullet B = (c_{ij}) \in L^{I \times J}$ is called $\bullet$*-product* (or only *product*, if there is no danger of confusion) of the matrices A and B if

$$c_{ij} = \bigvee_{k=1}^{|K|} (a_{ik} \wedge b_{kj}) \text{ for each } i \in I,\ j \in J. \quad (3.3i)$$

ii) The matrix $C = A \circ B = (c_{ij}) \in L^{I \times J}$ is called $\circ$*-product* of A and B if

$$c_{ij} = \bigwedge_{k=1}^{|K|} (a_{ik} \vee b_{kj}) \text{ for each } i \in I,\ j \in J. \quad (3.3ii)$$

iii) The matrix $C = A \,\alpha\, B = (c_{ij}) \in L^{I \times J}$ is called α*-product* of the matrices A and B if

$$c_{ij} = \bigwedge_{k=1}^{|K|} (a_{ik} \,\alpha\, b_{kj}) \text{ for each } i \in I,\ j \in J. \quad (3.3iii)$$

iv) The matrix $C = A \,\varepsilon\, B = (c_{ij}) \in L^{I \times J}$ is called ε*-product* of the matrices A and B if

$$c_{ij} = \bigvee_{k=1}^{|K|} (a_{ik} \,\varepsilon\, b_{kj}) \text{ for each } i \in I,\ j \in J. \quad (3.3iv)$$

v) The pair $(A \bullet B, A \circ B) = A * B$ is called $*$*-product* of the matrices A and B.

vi) The pair $(A \alpha B, A \varepsilon B) = A \otimes B$ is called $\otimes$*-product* of the matrices A and B.

Example. Let A and B be the conformable finite matrices:

$$A = \begin{pmatrix} 0.2 & 0.5 & 0.1 \\ 0.7 & 0.6 & 0.9 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & 0.1 \\ 0.5 & 0.8 \\ 0.9 & 0.4 \end{pmatrix}$$

i) The $\bullet$-product of the matrices A and B is the matrix $C = A \bullet B$, with elements computed according to (3.3i):

$$C = \begin{pmatrix} (0.2 \wedge 0.7) \vee (0.5 \wedge 0.5) \vee (0.1 \wedge 0.9) & (0.2 \wedge 0.1) \vee (0.5 \wedge 0.8) \vee (0.1 \wedge 0.4) \\ (0.7 \wedge 0.7) \vee (0.6 \wedge 0.5) \vee (0.9 \wedge 0.9) & (0.7 \wedge 0.1) \vee (0.6 \wedge 0.8) \vee (0.9 \wedge 0.4) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.9 & 0.6 \end{pmatrix}.$$

ii) The $\circ$-product of the matrices A and B is the matrix $C = A \circ B$, with elements computed according to (3.3ii):

$$C = \begin{pmatrix} (0.2 \vee 0.7) \wedge (0.5 \vee 0.5) \wedge (0.1 \vee 0.9) & (0.2 \vee 0.1) \wedge (0.5 \vee 0.8) \wedge (0.1 \vee 0.4) \\ (0.7 \vee 0.7) \wedge (0.6 \vee 0.5) \wedge (0.9 \vee 0.9) & (0.7 \vee 0.1) \wedge (0.6 \vee 0.8) \wedge (0.9 \vee 0.4) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 \\ 0.6 & 0.7 \end{pmatrix}.$$

iii) The α-product of the matrices A and B is the matrix $C = A \,\alpha\, B$ with elements computed according to (3.3iii):

$$C = \begin{pmatrix} (0.2 \alpha 0.7) \wedge (0.5 \alpha 0.5) \wedge (0.1 \alpha 0.9) & (0.2 \alpha 0.1) \wedge (0.5 \alpha 0.8) \wedge (0.1 \alpha 0.4) \\ (0.7 \alpha 0.7) \wedge (0.6 \alpha 0.5) \wedge (0.9 \alpha 0.9) & (0.7 \alpha 0.1) \wedge (0.6 \alpha 0.8) \wedge (0.9 \alpha 0.4) \end{pmatrix} = \begin{pmatrix} 1 & 0.1 \\ 0.5 & 0.1 \end{pmatrix}.$$

iv) The ε-product of the matrices A and B is the matrix $C = A \,\varepsilon\, B$ with elements computed according to (3.3iv):

$$C = \begin{pmatrix} (0.2 \varepsilon 0.7) \vee (0.5 \varepsilon 0.5) \vee (0.1 \varepsilon 0.9) & (0.2 \varepsilon 0.1) \vee (0.5 \varepsilon 0.8) \vee (0.1 \varepsilon 0.4) \\ (0.7 \varepsilon 0.7) \vee (0.6 \varepsilon 0.5) \vee (0.9 \varepsilon 0.9) & (0.7 \varepsilon 0.1) \vee (0.6 \varepsilon 0.8) \vee (0.9 \varepsilon 0.4) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.8 \\ 0 & 0.8 \end{pmatrix}.$$

v) The $*$-product of the matrices A and B is the pair $(A \bullet B, A \circ B)$, cf. (3.3v):

$$A * B = \left(\begin{pmatrix} 0.5 & 0.5 \\ 0.9 & 0.6 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.2 \\ 0.6 & 0.7 \end{pmatrix} \right).$$

vi) The $\otimes$-product of the matrices A and B is, cf. (3.3vi):

$$A \otimes B = \left(\begin{pmatrix} 1 & 0.1 \\ 0.5 & 0.1 \end{pmatrix}, \begin{pmatrix} 0.9 & 0.8 \\ 0 & 0.8 \end{pmatrix} \right).$$

Definition 3.4. Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ be finite fuzzy matrices of the same type. We say that

i) $A \le B$ if the relation $a_{ij} \le b_{ij}$ holds for any i, j, $1 \le i \le m$, $1 \le j \le n$;

ii) $A = B$ if the relation $a_{ij} = b_{ij}$ holds for any i, j, $1 \le i \le m$, $1 \le j \le n$

The next properties concern conformable matrices.

Theorem 3.5. *For every pair of conformable finite fuzzy matrices* $A = (a_{ij})_{m \times p}$ *and* $B = (b_{ij})_{p \times n}$ *we have:*

i) $A^t \,\varepsilon\, (A \circ B) \le B \le A^t \,\alpha\, (A \bullet B)$;

ii) $B \varepsilon (A \circ B)^t \le A^t \le B \alpha (A \bullet B)^t$.

Proof. i) Follows from Corollary 2.8 (i):

Let us denote by $D = A^t \alpha (A \bullet B)$, $D = (d_{ij})$. Bearing in mind the matrix multiplication operations (cf. Definition 3.3), from $A^t \alpha (A \bullet B) = D = (d_{ij})$ we obtain for any i, j, $1 \le i \le p$, $1 \le j \le n$:

$$d_{ij} = \bigwedge_k \left(a_{ki} \,\alpha \bigvee_t (a_{kt} \wedge b_{tj}) \right) = \bigwedge_k \left(a_{ki} \,\alpha \left[(a_{ki} \wedge b_{ij}) \vee \left(\bigvee_{t, t \ne i} (a_{kt} \wedge b_{tj}) \right) \right] \right).$$

According to (1.4) we have

$$d_{ij} \ge \bigwedge_k (a_{ki} \,\alpha\, (a_{ki} \wedge b_{ij})),$$

following (1.5) we obtain

$$a_{ki} \,\alpha\, (a_{ki} \wedge b_{ij}) \ge b_{ij},$$

hence $d_{ij} \ge \bigwedge_k (a_{ki} \,\alpha\, (a_{ki} \wedge b_{tj})) \ge b_{ij}$, i.e. $B \le A^t \alpha (A \bullet B)$.

Now let us denote by $D = (d_{ij}) = A^t \varepsilon (A \circ B)$. Bearing in mind the matrix multiplication operations (cf. Definition. 3.3), from $A^t \varepsilon (A \circ B) = D = (d_{ij})$ we obtain for any i, j, $1 \le i \le p$, $1 \le j \le n$:

$$d_{ij} = \bigvee_k \left(a_{ki} \,\varepsilon \bigwedge_t (a_{kt} \vee b_{tj}) \right) = \bigvee_k \left(a_{ki} \,\varepsilon \left((a_{ki} \vee b_{ij}) \wedge \left(\bigwedge_{t, t \ne i} (a_{kt} \vee b_{tj}) \right) \right) \right).$$

According to (1.8) and then (1.7) we have

$d_{ij} \le \bigvee_k (a_{ki} \,\varepsilon\, (a_{ki} \vee b_{ij}))$, $a_{ki} \,\varepsilon\, (a_{ki} \vee b_{ij}) \le b_{ij}$, i.e. $A^t \varepsilon (A \circ B) \le B$. □

Theorem 3.6. *For every pair of fuzzy matrices* $A = (a_{ij})_{m \times p}$ *and* $C = (c_{ij})_{m \times n}$ *we have:*

i) *If* $\mathbf{B}_\bullet$ *is the set of the matrices such that* $A \bullet B = C$ *then* $\mathbf{B}_\bullet \neq \varnothing$ *iff* $A^t \alpha C \in B_\bullet$;

ii) *if* $\mathbf{B}_\bullet \neq \varnothing$ *then* $A^t \alpha C$ *is the greatest element in* $\mathbf{B}_\bullet$;

iii) *If* $\mathbf{B}_\circ$ *is the set of the matrices such that* $A \circ B = C$ *then* $\mathbf{B}_\circ \neq \varnothing$ *iff* $A^t \varepsilon C \in B_\circ$;

iv) *if* $\mathbf{B}_\circ \neq \varnothing$ *then* $A^t \varepsilon C$ *is the least element in* $\mathbf{B}_\circ$.

Proof. i), ii) follow from Theorem 2.14; iii), iv) follow from Theorem 2.15. □

Theorem 3.7. *Let* $A = (a_{ij})_{m \times p}$ *and* $C = (c_{ij})_{m \times n}$ *be two fuzzy matrices,* $\mathbf{B}_*$ *be the set of the matrices such that* $A * B = C$. *Then:*

iii) $\mathbf{B}_* \neq \varnothing$ *iff* $A^{-1} \otimes C \in \mathbf{B}_*$;

iv) *if* $\mathbf{B}_* \neq \varnothing$ *then* $A^{-1} \otimes C$ *is the greatest element in* $\mathbf{B}_*$.

Proof. Follows from Theorem 2.16. □

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