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Intuitionistic fuzzy measures and intuitionistic entropies

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Abstract: With the help of an abstract integral we first give a family of intuitionistic entropies introduced by Burillo and Bustince in [4]. Second we prove that certain intuitionistic entropies are intuitionistic fuzzy measures.

Keywords: Intuitionistic entropy, intuitionistic norm function, triangular norm, intuitionistic measure.

1 Introduction

Similar with [10] for measures of fuzziness, in Section 2 we give an existence theorem of intuitionistic entropies defined by an abstract integral and we prove some properties of intuitionistic entropies introduced in this way.

In Section 3 we recall the concept of intuitionistic (fuzzy) measure (see [2]), we prove a characterization theorem for intuitionistic entropies more general than one of [4] and using this a result which make a connection between intuitionistic (fuzzy) measures and intuitionistic entropies.

2 Intuitionistic entropies defined by an integral

Let $X \neq \emptyset$ be a given set and we will denote with $IFS(X)$ the set of all the intuitionistic fuzzy sets on X.

The following expressions are defined in [1] for all $A, B \in IFS(X)$,
$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$

$A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

$A = B$ if and only if $A \leq B$ and $B \leq A$

$A \preceq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$

$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$

$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), 1 - \mu_A(x) - \alpha\pi_A(x)\rangle : x \in X\}$

where $\alpha \in [0,1]$ and $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \forall x \in X$.

$A\tilde{f}B = \{\langle x, f(\mu_A(x), \mu_B(x)), f^c(\nu_A(x), \nu_B(x))\rangle : x \in X\}$

where f is a triangular norm or a triangular conorm and $f(x,y) = 1 - f(1-x, 1-y)$ $\forall x, y \in [0,1]$.

In [4], P. Burillo and H. Bustince considered the following definition in the case of finite X.

Definition 1 *A real function $I : IFS(X) \to \mathbf{R}_+$ is called an entropy on $IFS(X)$, if I has the following properties:*
$(IP1)$ $I(A) = 0$ *if and only if* $\mu_A(x) + \nu_A(x) = 1$ *for all* $x \in X$
$(IP2)$ $I(A) =$*maximum if and only if* $\mu_A(x) = \nu_A(x) = 0$ *for all* $x \in X$
$(IP3)$ $I(A) = I(A^c)$ *for all* $A \in IFS(X)$
$(IP4)$ *If* $A \preceq B$ *then* $I(A) \geq I(B)$

Now let $(X, \mathcal{A}, m)$ denote an arbitrary measure space with $0 < m(X) < \infty$ (All measure-theoretic terms and results used here may be found in Halmos's book [8]).

We will say that $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$ is $\mathcal{A}$-measurable if and only if the real functions μ_A and ν_A are $\mathcal{A}$-measurable. We denote by $\mathcal{I}_\mathcal{A}(X)$ the family of all $\mathcal{A}$-measurable intuitionistic fuzzy sets on X.

Remark 1 *If $A \in \mathcal{I}_\mathcal{A}(X)$ then $A^c \in \mathcal{I}_\mathcal{A}(X)$ and $D_\alpha(A) \in \mathcal{I}_\mathcal{A}(X)$ for every $\alpha \in [0,1]$.*

In this general setting, it appears necessary to rewrite the definition of the intuitionistic entropies. In addition we will consider only the normalized intuitionistic entropies, that is $I(A) \leq 1$ for all $A \in \mathcal{I}_\mathcal{A}(X)$.

Definition 2 *A real function* $I : \mathcal{I}_{\mathcal{A}}(X) \to \mathbf{R}_+$ *is called an intuitionistic entropy on* $\mathcal{I}_{\mathcal{A}}(X)$, *if and only if* I *has the following properties:*

$(IP\,1^*)$ $I(A) = 0$ *if and only if* $\mu_A(x) + \nu_A(x) = 1$ *almost everywhere*
$(IP\,2^*)$ $I(A) = 1$ *if and only if* $\mu_A(x) = \nu_A(x) = 0$ *almost everywhere*
$(IP\,3^*)$ $I(A) = I(A^c)$ *for all* $A \in \mathcal{I}_{\mathcal{A}}(X)$
$(IP\,4^*)$ *If* $A, B \in \mathcal{I}_{\mathcal{A}}(X)$ *and* $A \preceq B$ *then* $I(A) \geq I(B)$

Remark 2 *The case of Definition* 1 *is covered by letting* X *be finite,* $\mathcal{A}$ *be the Boolean algebra of all subsets of* X *and* $m(\{x\}) = 1$ *for every* $x \in X$.

Inspired by [4] we introduce a function corresponding to the definition of the intuitionistic entropy.

Definition 3 *Let* $(X, \mathcal{A}, m)$ *be a measure space and we denote* $D = \{(\mu, \nu) \in [0,1] \times [0,1] : \mu + \nu \leq 1\}$. *An intuitionistic norm function is a* $\mathcal{A}$*-measurable function with respect to the first variable,* $\Phi : X \times D \to [0,1]$ *with the following properties for every element* $x \in X$:

$(IN\,1)$ $\Phi(x, \mu, \nu) = 0$ *if and only if* $\mu + \nu = 1$
$(IN\,2)$ $\Phi(x, \mu, \nu) = 1$ *if and only if* $\mu = \nu = 0$
$(IN\,3)$ $\Phi(x, \mu, \nu) = \Phi(x, \nu, \mu)$.
$(IN\,4)$ *If* $\mu \leq \mu'$ *and* $\nu \leq \nu'$ *then* $\Phi(x, \mu, \nu) \geq \Phi(x, \mu', \nu')$

If the function Φ *verifies* $(IN\,1), (IN\,2), (IN\,3)$ *and besides the property*

(IRN) *If* $\mu + \nu = \mu' + \nu'$ *then* $\Phi(x, \mu, \nu) = \Phi(x, \mu', \nu')$
then Φ *is called intuitionistic refined norm function.*

Now, using the abstract integral we can give the main result of this section

Theorem 4 *Let* $(X, \mathcal{A}, m)$ *denote any measure space with* $0 < m(X) < \infty$ *and let* Φ *denote an intuitionistic norm function. Then the function* $I_\Phi : \mathcal{I}_{\mathcal{A}}(X) \to [0,1]$ *defined by*

$$I_\Phi(A) = \frac{1}{m(X)} \int_X \Phi(x, \mu_A(x), \nu_A(x))\, dm$$

is an intuitionistic entropy.

Proof. Thanks to $(IN\,3)$ and $(IN\,4)$ the properties $(IP\,3^*)$ and $(IP\,4^*)$ are immediate.

We notice that $A = B$ almost everywhere, that is $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ almost everywhere, implies $\Phi\left(x, \mu_A(x), \nu_A(x)\right) =$ $= \Phi\left(x, \mu_B(x), \nu_B(x)\right)$ almost everywhere therefore $I_\Phi(A) = I_\Phi(B)$.

Now, if $E = \{\langle x, \mu_E(x), \nu_E(x)\rangle : x \in X\} \in \mathcal{I}_\mathcal{A}(X)$ verifies $\mu_E(x) + \nu_E(x) = 1, \forall x \in X$ then

$$I_\Phi(E) = \frac{1}{m(X)} \int_X 0 dm = 0$$

Also, if E verifies $\mu_E(x) = \nu_E(x) = 0, \forall x \in X$ then

$$I_\Phi(E) = \frac{1}{m(X)} \int_X 1 dm = 1$$

thus the inverse implication in $(IP\,1^*)$ and $(IP\,2^*)$ are satisfied.

On the other hand, if $A \in \mathcal{I}_\mathcal{A}(X)$ is any intuitionistic fuzzy set with $I_\Phi(A) = 0$ then $\Phi\left(x, \mu_A(x), \nu_A(x)\right) = 0$ almost everywhere $x \in X$ and $(IN\,1)$ implies $\mu_A(x) + \nu_A(x) = 1$ almost everywhere $x \in X$. Also, if $I_\Phi(A) = 1$ then $\int_X \Phi\left(x, \mu_A(x), \nu_A(x)\right) dm = \int_X 1 dm$ that is $\Phi\left(x, \mu_A(x), \nu_A(x)\right) = 1$ almost everywhere $x \in X$. Thanks to $(IN\,4)$ we obtain $\mu_A(x) = \nu_A(x) = 0$ almost everywhere $x \in X$ ■

Some simple properties of the intuitionistic entropies introduced by the previous theorem are given by the next theorems.

Theorem 5 *Let $(X, \mathcal{A}, m)$ be a measure space and Φ be an intuitionistic refined norm function. The intuitionistic entropies defined as above enjoys the properties:*
(a) If f is a continuous triangular norm or a continuous triangular conorm which satisfies $f(x,y) + f^c(x,y) = x + y \; \forall x, y \in [0,1]$ then

$$I_\Phi(A) = I_\Phi(A \tilde{f} A^c) \; \forall A \in \mathcal{I}_\mathcal{A}(X)$$

(b) If we represent a crisp set C (subset of X) as an intuitionistic fuzzy set through $A = \{\langle x, \chi_C(x), 1 - \chi_C(x)\rangle \,|x \in X\}$, where χ_C is the indicator function of C, then

$$I_\Phi(\alpha X) = I_\Phi(\alpha C) \; \forall \alpha \in [0,1], \forall C \in \mathcal{A}.$$

Proof. (*a*) Because

$$\Phi\left(x, f\left(\mu_A(x), \nu_A(x)\right), f^c\left(\nu_A(x), \mu_A(x)\right)\right) = \Phi\left(x, \mu_A(x), \nu_A(x)\right)\ \forall x \in X$$

the conclusion is immediate.

(*b*) $\alpha C = \{\langle x, \alpha\chi_C(x), \alpha - \alpha\chi_C(x)\rangle : x \in X\}$ and $\alpha X = \{\langle x, \alpha, 0\rangle : x \in X\}$ implies

$$\Phi\left(x, \mu_{\alpha C}(x), \nu_{\alpha C}(x)\right) = \Phi\left(x, \mu_{\alpha X}(x), \nu_{\alpha X}(x)\right)\ \forall x \in X \ \blacksquare$$

Theorem 6 *Let $(X, \mathcal{A}, m)$ be a measure space and Φ be an intuitionistic norm function. If Φ is a continuous function with respect to the second and third variables then the intuitionistic entropy $I_\Phi : \mathcal{I}_\mathcal{A}(X) \to [0,1]$ defined by*

$$I_\Phi(A) = \frac{1}{m(X)} \int_X \Phi\left(x, \mu_A(x), \nu_A(x)\right) dm$$

is a continuous function relative to the metric $d : \mathcal{I}_\mathcal{A}(X) \times \mathcal{I}_\mathcal{A}(X) \to [0,1]$ defined by

$$d(A, B) = \max\left(\sup_{x\in X} \left|\mu_A(x) - \mu_B(x)\right|, \sup_{x\in X} \left|\nu_A(x) - \nu_B(x)\right|\right)$$

where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}, B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \,|x \in X\}$.

Proof. Because $D = \{(\mu, \nu) \in [0,1] \times [0,1] : \mu + \nu \leq 1\}$ is compact, $\Phi(x, \cdot, \cdot)$ is a uniform continuous function. Let $\varepsilon > 0$. There is $\delta > 0$ so that $x \in X : |\mu_A(x) - \mu_B(x)| < \delta$ and $|\nu_A(x) - \nu_B(x)| < \delta$ implies

$$\left|\Phi\left(x, \mu_A(x), \nu_A(x)\right) - \Phi\left(x, \mu_B(x), \nu_B(x)\right)\right| < \varepsilon.$$

But $d(A, B) < \delta$ implies that $|\mu_A(x) - \mu_B(x)| < \delta$ and $|\nu_A(x) - \nu_B(x)| < \delta$ $\forall x \in X$ therefore

$$\left|\Phi\left(x, \mu_A(x), \nu_A(x)\right) - \Phi\left(x, \mu_B(x), \nu_B(x)\right)\right| < \varepsilon\ \forall x \in X.$$

We obtain the existence of a $\delta > 0$ such that $d(A, B) < \delta$ implies

$$\left|\frac{1}{m(X)} \int_X \Phi\left(x, \mu_A(x), \nu_A(x)\right) dm - \frac{1}{m(X)} \int_X \Phi\left(x, \mu_B(x), \nu_B(x)\right) dm\right| \leq \varepsilon$$

therefore $|I_\Phi(A) - I_\Phi(B)| \leq \varepsilon$ ■

Between all the possible intuitionistic entropies characterized previously, the one obtained by setting $\Phi(x,\mu,\nu) = 1-\mu-\nu$ that is

$$\overline{I}(A) = \frac{1}{m(X)} \int_X (1-\mu_A(x)-\nu_A(x))\, dm \quad \text{for all } A \in \mathcal{I}_{\mathcal{A}}(X)$$

is very simple. The next properties of this entropy are true (the same with those from the last proposition by [4]).

Theorem 7 *Let $(X,\mathcal{A},m)$ be a measure space. Then*

(i) $\quad \overline{I}(A) = 2 \cdot \delta_{IFS(X)}(A, D_\alpha(A))$
(ii) $\quad \overline{I}(A) = \delta_{IFS(X)}(D_0(A), D_1(A))$
(iii) $\quad \delta_{IFS(X)}(D_\alpha(A), D_\beta(A)) = (\alpha-\beta)\cdot \overline{I}(A)$ *with* $\alpha,\beta \in [0,1], \beta \leq \alpha$

where

$$\delta_{IFS(X)}(A,B) = \frac{1}{m(X)} \int_X \frac{|\mu_A(x)-\mu_B(x)| + |\nu_A(x)-\nu_B(x)|}{2} dm$$

is the Hamming distance between A, B belonging to $\mathcal{I}_{\mathcal{A}}(X)$,
$A = \{\langle x,\mu_A(x),\nu_A(x)\rangle : x \in X\}, B = \{\langle x,\mu_B(x),\nu_B(x)\rangle : x \in X\}.$

Proof. See [4] ■

3 Intuitionistic entropy and intuitionistic (fuzzy) measure

Definition 8 *([2]) Let f be a triangular norm or a triangular conorm. A subfamily $\mathcal{I} \subseteq IFS(X)$ which satisfies:*

(i) $\quad \emptyset = \{\langle x,0,1\rangle \,|x \in X\} \in \mathcal{I}$
(ii) $\quad A \in \mathcal{I}$ *implies* $A^c \in \mathcal{I}$
(iii) $\quad (A_n)_{n\in\mathbf{N}} \subseteq \mathcal{I}$ *implies* $\underset{n\in\mathbf{N}}{\widetilde{f}} A_n \in \mathcal{I}$

will be called an intuitionistic f-algebra on X.

The pair $(X,\mathcal{I})$ will be called an intuitionistic f-measurable space.

Definition 9 *([2]) A function $\widetilde{m} : \mathcal{I} \to \mathbf{R}_+, \mathcal{I}$ being an intuitionistic f-algebra on X, which satisfies the following conditions:*

$$\begin{array}{ll} (i) & \widetilde{m}(\emptyset) = 0 \\ (ii) & A, B \in \mathcal{I} \text{ implies } \widetilde{m}\left(A\tilde{f}B\right) + \widetilde{m}\left(A\widetilde{f^c}B\right) = \widetilde{m}(A) + \widetilde{m}(B) \\ (iii) & (A_n)_{n\in\mathbf{N}} \subseteq \mathcal{I},\ A_n \leq A_{n+1} \text{ for every } n \in \mathbf{N} \text{ and } \lim\limits_{n\to\infty} A_n \in \mathcal{I} \text{ implies} \\ & \qquad \lim\limits_{n\to\infty} \widetilde{m}(A_n) = \widetilde{m}\left(\lim\limits_{n\to\infty} A_n\right) \end{array}$$

will be called an intuitionistic f-measure on $\mathcal{I}$.

After the idea from [4] we give the next definition.

Definition 10 *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\varphi : [0,1] \to [0,1]$ a continuous function such that if $\alpha + \beta \leq 1$,then $\varphi(\alpha) + \varphi(\beta) \leq 1$, with $\alpha, \beta \in [0,1]$. The function $I_\varphi : \mathcal{I}_{\mathcal{A}}(X) \to [0,1]$ defined by*

$$I_\varphi(A) = \frac{1}{m(X)} \int_X (1 - \varphi(\mu_A(x)) - \varphi(\nu_A(x)))\, dm$$

we will call I_φ-function.

Analogous with [4] we obtain a characterization theorem of intuitionistic fuzzy entropies using $I_\varphi-$functions

Theorem 11 *Let $I : \mathcal{I}_{\mathcal{A}}(X) \to [0,1]$ and $\varphi : [0,1] \to [0,1]$ a continuous function. The function I is an intuitionistic entropy and an I_φ-function if and only if*

$$I(A) = \frac{1}{m(X)} \int_X (1 - \varphi(\mu_A(x)) - \varphi(\nu_A(x)))\, dm$$

where φ satisfies the conditions:

$$\begin{array}{ll} (i) & \varphi \text{ is increasing} \\ (ii) & \varphi(\alpha) = 0 \text{ if and only if } \alpha = 0 \\ (iii) & \varphi(\alpha) + \varphi(\beta) = 1 \text{ if and only if } \alpha + \beta = 1. \end{array}$$

Proof. ($\Leftarrow$) Let us consider $\Phi : X \times D \to [0,1]$, $\Phi(x,\mu,\nu) = 1 - \varphi(\mu) - \varphi(\nu)$, where $D = \{(\mu,\nu) \in [0,1] \times [0,1] : \mu + \nu \le 1\}$. The function Φ is constant with respect to x therefore it is $\mathcal{A}$-measurable with respect to the first variable.

We verify $(IN\,1) - (IN\,4)$ by Definition 3:

$(IN\,1)$ $\Phi(x,\mu,\nu) = 1 - (\varphi(\mu) + \varphi(\nu)) = 0$ if and only if $\varphi(\mu) + \varphi(\nu) = 1$ if and only if $\mu + \nu = 1$.

$(IN\,2)$ $\Phi(x,\mu,\nu) = 1$ if and only if $\varphi(\mu) + \varphi(\nu) = 0$ if and only if $\varphi(\mu) = \varphi(\nu) = 0$ if and only if $\mu = \nu = 0$.

$(IN\,3)$ $\Phi(x,\mu,\nu) = 1 - \varphi(\mu) - \varphi(\nu) = \Phi(x,\nu,\mu)$.

$(IN\,4)$ If $\mu \le \mu'$ and $\nu \le \nu'$ then $\varphi(\mu) + \varphi(\mu') \le \varphi(\nu) + \varphi(\nu')$ that is $\Phi(x,\mu,\nu) \ge \Phi(x,\mu',\nu')$.

Φ being an intuitionistic norm function, Theorem 4 implies that I is an intuitionistic entropy.

If $\alpha + \beta \le 1$ then

$$I(\{\langle x,\alpha,\beta\rangle : x \in X\}) = \frac{1}{m(X)} \int_X (1 - \varphi(\alpha) - \varphi(\beta))\, dm = 1 - \varphi(\alpha) - \varphi(\beta) \ge 0$$

implies $\varphi(\alpha) + \varphi(\beta) \le 1$,therefore I is an I_φ-function.

($\Rightarrow$) We assume that I is an intuitionistic entropy and an I_φ-function.

Being I_φ-function it has the form

$$I(A) = \frac{1}{m(X)} \int_X (1 - \varphi(\mu_A(x)) - \varphi(\nu_A(x)))\, dm$$

We prove that φ satisfies the conditions $(i) - (iii)$.

(i) If $\alpha \le \alpha'$ and $\alpha, \alpha' \in [0,1]$, we construct the following intuitionistic fuzzy sets: $A = \{\langle x,\alpha,0\rangle : x \in X\}$, $B = \{\langle x,\alpha',0\rangle : x \in X\}$. Because $A \preceq B, I(A) \ge I(B)$ that is

$$\int_X (1 - \varphi(\alpha) - \varphi(0))\, dm \ge \int_X (1 - \varphi(\alpha') - \varphi(0))\, dm$$

and

$$1 - \varphi(\alpha) - \varphi(0) \ge 1 - \varphi(\alpha') - \varphi(0).$$

We obtain $\varphi(\alpha) \le \varphi(\alpha')$.

(ii) $I(\{\langle x,0,0\rangle : x \in X\}) = 1$ by $(IP\,2^*)$, Definition 2. But

$$I(\{\langle x,0,0\rangle : x \in X\}) = \frac{1}{m(X)} \int_X (1 - \varphi(0) - \varphi(0))\, dm = 1 - 2\varphi(0),$$

therefore $1 = 1 - 2\varphi(0)$ and $\varphi(0) = 0$.

Reciprocally, if $\varphi(\alpha) = 0$, then

$$I(\{\langle x, \alpha, 0\rangle : x \in X\}) = \frac{1}{m(X)} \int_X (1 - \varphi(\alpha) - \varphi(0))\, dm = 1 - \varphi(\alpha) = 1$$

and $(IP\,2^*)$ implies $\alpha = 0$.

(iii) If $\alpha + \beta = 1$, we take $A = \{\langle x, \alpha, \beta\rangle : x \in X\}$. We know by $(IP\,1^*)$, Definition 2 that $I(A) = 0$. But

$$I(A) = \frac{1}{m(X)} \int_X (1 - \varphi(\alpha) - \varphi(\beta))\, dm = 1 - \varphi(\alpha) - \varphi(\beta).$$

We obtain $\varphi(\alpha) + \varphi(\beta) = 1$.

Reciprocally, let $\varphi(\alpha) + \varphi(\beta) = 1$. If we assume that $\alpha + \beta \neq 1$, then two things may occur:

(a) $\alpha + \beta < 1$

Then

$$I(\{\langle x, \alpha, \beta\rangle : x \in X\}) = \frac{1}{m(X)} \int_X (1 - \varphi(\alpha) - \varphi(\beta))\, dm = 1 - \varphi(\alpha) - \varphi(\beta) = 0,$$

therefore $\alpha + \beta = 1$ (see $(IP\,1^*)$), contradicting the hypothesis.

(b) $\alpha + \beta > 1$

Because $\alpha + (1 - \alpha) = 1$ and $\beta + (1 - \beta) = 1$,

$$\varphi(\alpha) + \varphi(1 - \alpha) = 1$$

and

$$\varphi(\beta) + \varphi(1 - \beta) = 1.$$

For all α, β we can write

$$\varphi(\alpha) + \varphi(\beta) + \varphi(1 - \alpha) + \varphi(1 - \beta) = 2,$$

therefore $\varphi(1 - \alpha) + \varphi(1 - \beta) = 1$.

Knowing that

$$1 - \alpha + 1 - \beta = 2 - (\alpha + \beta) \leq 1$$

we have

$$I(\{\langle x, 1 - \alpha, 1 - \beta\rangle : x \in X\}) = \frac{1}{m(X)} \int_X (1 - \varphi(1 - \alpha) - \varphi(1 - \beta))\, dm = 0.$$

Then $(IP1^*)$, Definition 2, implies $1-\alpha+1-\beta=1$ therefore $\alpha+\beta=1$, contradiction with the hypothesis.

We obtain the only possibility $\alpha+\beta=1$ ■

Now, we can estabilish the next result.

Theorem 12 *Let $(X,\mathcal{A},m)$ be a measure space. If $I:\mathcal{I}_{\mathcal{A}}(X)\rightarrow[0,1]$ is an intuitionistic entropy and an $I_{\varphi}-$function then I is an intuitionistic $\wedge$-measure on $\mathcal{I}_{\mathcal{A}}(X)$. Moreover, if φ is additive then I is an intuitionistic $f-$measure on $\mathcal{I}_{\mathcal{A}}(X)$ for every triangular norm which verifies $f(x,y)+f^{c}(x,y)=x+y$ for all $x,y\in[0,1]$.*

Proof. By the previous theorem we obtain

$$I(\{\langle x,0,1\rangle:x\in X\})=\frac{1}{m(X)}\int_X(1-(\varphi(0)+\varphi(1)))\,dm=0$$

Let $(A_n)_{n\in\mathbf{N}}\subseteq\mathcal{I}_{\mathcal{A}}(X)$ be an increasing sequence. Then

$$\begin{aligned}\lim_{n\to\infty}I(A_n)&=\lim_{n\to\infty}\frac{1}{m(X)}\int_X(1-(\varphi(\mu_{A_n}(x))+\varphi(\nu_{A_n}(x))))\,dm=\\&=\frac{1}{m(X)}\int_X\left(1-\left(\varphi\left(\lim_{n\to\infty}\mu_{A_n}(x)\right)+\varphi\left(\lim_{n\to\infty}\nu_{A_n}(x)\right)\right)\right)dm=\\&=I\left(\langle x,\lim_{n\to\infty}\mu_{A_n}(x),\lim_{n\to\infty}\nu_{A_n}(x)\rangle:x\in X\right)=I\left(\lim_{n\to\infty}A_n\right)\end{aligned}$$

because φ is continuous(see and [2]).

If φ is additive and $f(x,y)+f^{c}(x,y)=x+y$ for all $x,y\in[0,1]$ then we have

$$\begin{aligned}&I\left(A\widetilde{f}B\right)+I\left(A\widetilde{f^{c}}B\right)\\=&\frac{1}{m(X)}\int_X(1-(\varphi(f(\mu_A(x),\mu_B(x)))+\varphi(f^{c}(\nu_A(x),\nu_B(x)))))\,dm+\\&+\frac{1}{m(X)}\int_X(1-(\varphi(f^{c}(\mu_A(x),\mu_B(x)))+\varphi(f(\nu_A(x),\nu_B(x)))))\,dm\\=&\frac{1}{m(X)}\int_X(2-(\varphi(\mu_A(x))+\varphi(\mu_B(x))+\varphi(\nu_A(x))+\varphi(\nu_B(x))))\,dm=\\=&\frac{1}{m(X)}\int_X(1-(\varphi(\mu_A(x))+\varphi(\nu_A(x))))\,dm+\\&+\frac{1}{m(X)}\int_X(1-(\varphi(\mu_B(x))+\varphi(\nu_B(x))))\,dm\\=&I(A)+I(B).\end{aligned}$$

For $f \equiv \wedge$ and $f^c \equiv \vee$ the proof is similar and it is true even if φ is not additive(see [4]) ■

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