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The most general form of one type of intuitionistic fuzzy modal operators. Part 2

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1 Introduction

Two years ago the author published a paper under the same name, thinking that the operator, defined in it was *"the most general form of one type of intuitionistic fuzzy modal operators"*. Now, he sew that then he had not been right, because the operator introduced by him can be extended further. Now he again believes that the new operator is *"the most general form of ..."*, but he would not assert again that this is really true.

2 Basic concepts

Over Intuitionistic Fuzzy Sets (IFSs, see [2]) there have been defined not only operations and relations similar to the ordinary fuzzy set ones, but also operators that cannot be defined in the case of ordinary fuzzy sets.

In the present paper we shall discuss a new modal-like type of operator.

Let a set E be fixed. An IFS A over E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

For every two IFSs A and B a lot of relations and operations are defined (see, e.g. [2]), the most important of these are:

$$\begin{array}{lll} A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \le \mu_B(x) \& \nu_A(x) \ge \nu_B(x)); \\ A \supset B & \text{iff} & B \subset A; \end{array}$$

$$
\begin{array}{lcl}
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)); \\
\overline{A} & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}.
\end{array}
$$

We shall also define the following level operators:

$$P_{\alpha,\beta}(A) = \{\langle x, \max(\alpha, \mu_A(x)), \min(\beta, \nu_A(x))\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1;$$

$$Q_{\alpha,\beta}(A) = \{\langle x, \min(\alpha, \mu_A(x)), \max(\beta, \nu_A(x))\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1.$$

Now, we will describe couples of three types of other modal-like operators, following [1, 2, 3, 11]. We shall start with the first two – the simplest operators:

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle | x \in E\}.$$

Let $\alpha \in [0, 1]$ and let A be an IFS. Then we can define the first extension of this type of operators:

$$\boxplus_\alpha A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + 1 - \alpha\rangle | x \in E\},$$

$$\boxtimes_\alpha A = \{\langle x, \alpha.\mu_A(x) + 1 - \alpha, \alpha.\nu_A(x)\rangle | x \in E\}.$$

The second extension of operators $\boxplus$ and $\boxtimes$ is introduced in [11] by Katerina Dencheva. She extended the last two operators to the forms:

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x) + \beta, \alpha.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

The third extension of the above operators is described in [3]. They will have the forms:

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) + \gamma\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

In [13] Gökhan Cuvalcioĝlu introduced operator $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha.\mu_A(x) + 1 - \alpha), \alpha(\beta.\nu_A(x) + 1 - \beta)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$ and studied some of its properties.

A natural extension of all these operators is the operator

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and

$$\max(\alpha, \beta) + \gamma + \delta \leq 1$$

introduced in [4, 5].

3 Main results

A new extension of the above operators is the operator

$$\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A = \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma, \beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1, \tag{1}$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0. \tag{2}$$

Below we assume that in the particular cases $\alpha - \zeta > -\varepsilon, \beta = \delta = 0$ and $\delta - \zeta < \beta, \gamma = \varepsilon = 0$, the inequalities

$$\gamma \geq \varepsilon \text{ and } \beta + \delta \leq 1$$

hold. Obviously, for every IFS A:

$$\boxplus A = \boxed{\circ}_{0.5,0.5,0,0.5,0,0}A,$$

$$\boxtimes A = \boxed{\circ}_{0.5,0.5,0.5,0,0,0}A,$$

$$\boxplus_\alpha A = \boxed{\circ}_{\alpha,\alpha,0,1-\alpha,0,0}A,$$

$$\boxtimes_\alpha A = \boxed{\circ}_{\alpha,\alpha,1-\alpha,0,0,0}A,$$

$$\boxplus_{\alpha,\beta} A = \boxed{\circ}_{\alpha,\alpha,0,\beta,0,0}A,$$

$$\boxtimes_{\alpha,\beta} A = \boxed{\circ}_{\alpha,\alpha,\beta,0,0,0}A.$$

$$\boxplus_{\alpha,\beta,\gamma} A = \boxed{\circ}_{\alpha,\beta,0,\gamma,0,0}A,$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \boxed{\circ}_{\alpha,\beta,\gamma,0,0,0}A.$$

$$\boxed{\circ}_{\alpha,\beta,\gamma,\delta} A = \boxed{\circ}_{\alpha,\beta,\gamma,\delta,0,0}A.$$

$$E_{\alpha,\beta} A = \boxed{\circ}_{\alpha.\beta,\alpha.\beta,\beta.(1-\alpha),\alpha.(1-\beta)}A.$$

The following assertions hold for the new operator.

Theorem 1: For every IFS A and for every $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ for which (1) and (2) are valid, the equality

$$\neg\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}\neg A = \boxed{\circ}_{\beta,\alpha,\delta,\gamma,\zeta,\varepsilon}A$$

holds.

Theorem 2: For every IFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$ for which conditions that are similar to (1) and (2) are valid, the equality

$$\boxed{\circ}_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(\boxed{\circ}_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A)$$

$$= \boxed{\circ}_{\alpha_1.\alpha_2+\varepsilon_1.\zeta_2,\ \beta_1.\beta_2+\zeta_1.\varepsilon_2,\ \alpha_1.\gamma_2-\varepsilon_1.\delta_2+\gamma_1,\ \beta_1.\delta_2-\zeta_1.\gamma_2+\delta_1,\ \alpha_1.\varepsilon_2+\varepsilon_1.\beta_2,\ \beta_1.\zeta_2+\zeta_1.\alpha_2}A.$$

holds.

Proof: Let the conditions of the Theorem are valid. Then:

$$\boxed{\circ}_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(\boxed{\circ}_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A)$$

$$= \boxcircle_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(\{\langle x, \alpha_2.\mu_A(x) - \varepsilon_2.\nu_A(x) + \gamma_2, \beta_2.\nu_A(x) - \zeta_2.\mu_A(x) + \delta_2\rangle | x \in E\})$$
$$= \langle x, \alpha_1.\alpha_2.\mu_A(x) - \alpha_1.\varepsilon_2.\nu_A(x) + \alpha_1.\gamma_2 - \varepsilon_1.\beta_2.\nu_A(x) + \varepsilon_1.\zeta_2.\mu_A(x) - \varepsilon_1.\delta_2 + \gamma_1,$$
$$\beta_1.\beta_2.\nu_A(x) - \beta_1.\zeta_2.\mu_A(x) + \beta_1.\delta_2 - \zeta_1.\alpha_2.\mu_A(x) + \zeta_1.\varepsilon_2.\nu_A(x) - \zeta_1.\gamma_2 + \delta_1\rangle | x \in E\})$$
$$= \langle x, (\alpha_1.\alpha_2 + \varepsilon_1.\zeta_2).\mu_A(x) - (\alpha_1.\varepsilon_2 + \varepsilon_1.\beta_2).\nu_A(x) + (\alpha_1.\gamma_2 - \varepsilon_1.\delta_2 + \gamma_1),$$
$$(\beta_1.\beta_2 + \zeta_1.\varepsilon_2).\nu_A(x) - (\beta_1.\zeta_2 + \zeta_1.\alpha_2).\mu_A(x) + (\beta_1.\delta_2 - \zeta_1.\gamma_2 + \delta_1)\rangle | x \in E\})$$
$$= \boxcircle_{\alpha_1.\alpha_2+\varepsilon_1.\zeta_2,\ \beta_1.\beta_2+\zeta_1.\varepsilon_2,\ \alpha_1.\gamma_2-\varepsilon_1.\delta_2+\gamma_1,\ \beta_1.\delta_2-\zeta_1.\gamma_2+\delta_1,\ \alpha_1.\varepsilon_2+\varepsilon_1.\beta_2,\ \beta_1.\zeta_2+\zeta_1.\alpha_2} A.$$

Theorem 3: For every two IFSs A and B and for every $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ for which (1) and (2) are valid, the equality

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A@B) = \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A@\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B.$$

We must note that equalities

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A \cap B) = \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \cap \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B$$

and

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A \cup B) = \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \cup \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B$$

that are valid for operator $\boxdot_{\alpha,\beta,\gamma,\delta}$, now are not always valid.

Theorem 4: For every IFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$ for which the conditions that are similar to (1) and (2) are valid, the inclusion

$$\boxcircle_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(A) \subset \boxcircle_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A$$

is valid if and only if $\alpha_1 \leq \alpha_2, \varepsilon_1 \geq \varepsilon_2, \gamma_1 \leq \gamma_2, \beta_1 \geq \beta_2, \zeta_1 \leq \zeta_2, \delta_1 \geq \delta_2$.

In [12] an implication $\rightarrow_*$ is introduced on the base of another, already defined implication $\rightarrow$ by

$$x \rightarrow_* y = \Box x \rightarrow \Diamond y.$$

Here, extending this construction and having in mind the already defined more than 100 implications (see [6, 7, 8, 9, 10]) we introduce for each one of the existing implications (let us mark it by $\rightarrow_i$) a new implication

$$A \rightarrow_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2} B = \boxcircle_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(A) \rightarrow_i \boxcircle_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A.$$

Below we shall illustrate the new implication using as a base the ordinary IFS implication ($\rightarrow_1$) that for two IFSs A and B has the form:

$$A \rightarrow_1 B = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \rightarrow \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}$$
$$= \langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle | x \in E\}.$$

Therefore, for $\rightarrow_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}$ we obtain:

$$A \rightarrow_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2} B$$
$$= \{\langle x, \max(\beta_1.\nu_A(x) - \zeta_1.\mu_A(x) + \delta_1, \alpha_2.\mu_B(x) - \varepsilon_2.\nu_B(x) + \gamma_2),$$
$$\min(\alpha_1.\mu_A(x) - \varepsilon_1.\nu_A(x) + \gamma_1, \beta_2.\nu_B(x) - \zeta_2.\mu_B(x) + \delta_2)\rangle | x \in E\}.$$

As it is well known, having some implication, we can construct an operation negation by formula

$$\neg\langle a, b\rangle = \langle a, b\rangle \rightarrow \langle 0, 1\rangle.$$

For example, implication $\rightarrow_1$ generates negation

$$\neg_1\langle a, b\rangle = \langle a, b\rangle \rightarrow_1 \langle 0, 1\rangle = \langle b, a\rangle,$$

i.e., the standard negation $\neg$ mentioned above.

Therefore, now, we can construct a negation related to the above defined implication $\rightarrow_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}$ in the form:

$$\begin{aligned} &\neg_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2} A \\ &= \{\langle x, \max(\beta_1.\nu_A(x) - \zeta_1.\mu_A(x) + \delta_1, \gamma_2 - \varepsilon_2), \\ &\min(\alpha_1.\mu_A(x) - \varepsilon_1.\nu_A(x) + \gamma_1, \beta_2 + \delta_2\rangle | x \in E\}. \end{aligned}$$

Theorem 5: For every IFS A and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$ for which conditions that are similar to (1) and (2) are valid, the equalities

$$\neg_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2} A = P_{\gamma_2-\varepsilon_2,\beta_2+\delta_2}(\boxed{\circ}_{\beta_1,\alpha_1,\delta_1,\gamma_1,\zeta_1,\varepsilon_1}\neg_1 A)$$

and

$$\neg_{i;\ \alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1;\ \alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2} A = \neg_1 Q_{\beta_2+\delta_2,\gamma_2-\varepsilon_2}(\boxed{\circ}_{\beta_1,\alpha_1,\delta_1,\gamma_1,\zeta_1,\varepsilon_1} A).$$

hold.

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