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Characterizing Regular Semigroups Using Intuitionistic Fuzzy Sets

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Abstract: In the present paper, we give some properties of intuitionistic fuzzy ideals and intuitionistic fuzzy biideal in a semigroup, and characterize a regular semigroup in terms of intuitionistic fuzzy left, right, and two-sided and biideals.

Keywords: Intuitionistic fuzzy ideal; intuitionistic fuzzy biideal; regular semigroup;

1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1983, Antanassov [10] introduces the concept of intuitionistic fuzzy sets, which constitute a extension of fuzzy sets theory: intuitionistic fuzzy sets give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must smaller than or equal to 1. The semigroup theory of fuzzy sets was deeply studied by many authors[1-6]. In [6], authors characterize the regular semigroup by fuzzy left, right, two-sided and biideals. In the present paper, we will use intuitionistic fuzzy left, right, two-sided and biideals to characterize the regular semigroup. Since intuitionistic fuzzy sets theory is a generalization of fuzzy sets theory, and fuzzy sets can be seen as a special situation of the intuitionistic fuzzy sets.

2. Preliminaries

Let S be a semigroup, a subsemigroup of S is a nonempty subset A of S such that $A^2 \subseteq A$ and a left (right) ideal of S is a nonempty subset A of S such that $SA \subseteq A$ $(AS \subseteq A)$, a two-ideal (or simply ideal) is a subset of S which is both a left and a right ideal of S. A semigroup S is called regular if for every element a of S there exists an element x in S such that $a = axa$.

Definition1[12]: An intuitionistic fuzzy set A in S is an object

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in S\}$$

where, for all $x \in S$, $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of x in S, and furthermore satisfy $\mu_A(x) + \nu_A(x) \leq 1$.

Obviously, each ordinary fuzzy set may be written as

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in S\}.$$

Definition2[12]: Let A, B be two intuitionistic fuzzy sets in S, then

$$A \subseteq B \quad \text{iff} \quad (\forall x \in S)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \leq \nu_B(x)),$$

$$A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle | x \in S\},$$

$$A \circ B = \{\langle x, \mu_{A\circ B}(x), \nu_{A\circ B}(x)\rangle | x \in S\}$$

where

$$\mu_{A\circ B}(x) = \begin{cases} \sup\limits_{x=yz}\{\min\{\mu_A(y), \mu_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise} \end{cases},$$

$$\nu_{A\circ B}(x) = \begin{cases} \inf\limits_{x=yz}\{\max\{\nu_A(y), \nu_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 1 & \text{otherwise} \end{cases}.$$

Definition3: If S be a semigroup, an intuitionistic fuzzy set A in S is called an intuitionistic fuzzy semigroup in S if

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\} \quad \text{for all } x, y \in S.$$

And is called a left (right) ideal of S if

$$\mu_A(xy) \geq \mu_A(y) \text{ and } \nu_A(x,y) \leq \nu_A(y)(\mu_A(xy) \geq \mu_A(x) \text{ and } \nu_A(x,y) \leq \nu_A(x))$$

for all $x, y \in S$. An intuitionistic fuzzy set A in S is called an intuitionistic fuzzy two-sided ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic right ideal of S.

Definition4: An intuitionistic fuzzy semigroup A in S is called an intuitionistic fuzzy biideal of S if

$$\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}, \nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(z)\}$$

for all $x, y, z \in S$.

Let A be a subset of a semigroup S, then we denote

$$\widetilde{A}=\left\{\left\langle x,\mu_{\widetilde{A}}(x),v_{\widetilde{A}}(x)\right\rangle \middle| x\in S\right\}$$

where

$$\mu_{\widetilde{A}}(x)=\begin{cases}1 & x\in A\\ 0 & \text{otherwise}\end{cases},\quad v_{\widetilde{A}}(x)=\begin{cases}0 & x\in A\\ 1 & \text{otherwise}\end{cases}.$$

Obviously $\widetilde{A}$ is an intuitionistic fuzzy set in S. Obviously, semigroup S also can be seen as an intuitionistic fuzzy set $\widetilde{S}=\left\{\left\langle x,1,0\right\rangle \middle| x\in S\right\}$. In the present paper, we will use S represent S and $\widetilde{S}$, it is easy to see their means from context.

Lemma1: Let A be a nonempty subset of a semigroup S, then

(1) A is a subsemigroup of S if and only if $\widetilde{A}$ is an intuitionistic fuzzy semigroup of S.

(2) A is a left (right, two-sided) ideal of S if and only if $\widetilde{A}$ is an intuitionistic fuzzy left (right, two-sided) of S.

Proof: (1) For any element x,y of A, since A is a subsemigroup of S, then $xy\in S$,

$$\mu_{\widetilde{A}}(xy)=1=\mu_{\widetilde{A}}(x)=\mu_{\widetilde{A}}(y)=\min\{\mu_{\widetilde{A}}(x),\mu_{\widetilde{A}}(y)\},$$
$$v_{\widetilde{A}}(xy)=0=v_{\widetilde{A}}(x)=v_{\widetilde{A}}(y)=\max\{v_{\widetilde{A}}(x),v_{\widetilde{A}}(y)\}.$$

So $\widetilde{A}$ is an intuitionistic fuzzy semigroup of S.

Conversely, for any element x,y of A, since $\widetilde{A}$ is an intuitionistic fuzzy semigroup of S, then

$$\mu_{\widetilde{A}}(xy)\geq\min\{\mu_{\widetilde{A}}(x),\mu_{\widetilde{A}}(y)\}=1, v_{\widetilde{A}}(xy)\leq\max\{v_{\widetilde{A}}(x),v_{\widetilde{A}}(y)\}=0.$$

So $xy\in A$, thus $A^2\subseteq A$.

(2) A is a left ideal of S, then for arbitrary $x,y\in S$,

In the case when $y\in A$, then

$$\mu_{\widetilde{A}}(xy)=1\geq\mu_{\widetilde{A}}(y), v_{\widetilde{A}}(xy)=0\leq v_{\widetilde{A}}(y).$$

In the case when $y \notin A$, then

$$\mu_{\tilde{A}}(xy) \geq \mu_{\tilde{A}}(y) = 0, \nu_{\tilde{A}}(xy) \leq \nu_{\tilde{A}}(y) = 1.$$

Lemma2: A nonempty subset A of a semigroup S is a biideal of S if and only if $\tilde{A}$ is an intuitionistic fuzzy biideal of S.

Proof: First assume that A is a biideal of a semigroup S, then for any element x, y, z of S.

In the case when $x \in A$ and $z \in A$, then

$$\mu_{\tilde{A}}(xyz) = 1 \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\}, \nu_{\tilde{A}}(xyz) = 0 \leq \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{A}}(z)\}.$$

In the case when $x \notin A$ and $z \in A$,

$$\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0, \nu_{\tilde{A}}(xyz) \leq \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{A}}(z)\} = 1.$$

In the case when $x \in A$ and $z \notin A$,

$$\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0, \nu_{\tilde{A}}(xyz) \leq \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{A}}(z)\} = 1.$$

In the case when $x \notin A$ and $z \notin A$,

$$\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0, \nu_{\tilde{A}}(xyz) \leq \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{A}}(z)\} = 1.$$

Thus $\tilde{A}$ is an intuitionistic fuzzy biideal of S.

Conversely, assume $\tilde{A}$ is an intuitionistic fuzzy biideal of S, then for any element $x, z \in A, y \in S$,

$$\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 1, \nu_{\tilde{A}}(xyz) \leq \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{A}}(z)\} = 0.$$

Thus $xyz \in A$, and $ASA \subseteq A$. This completes the proof.

Lemma3[2]: A semigroup S is regular if and only if $B = BSB$ for all biideals B of S.

Lemma4: An intuitionistic fuzzy set A in a semigroup S is an intuitionistic fuzzy semigroup of S if and only if $A \circ A \subseteq A$.

Proof: If $A \circ A \subseteq A$, then for $\forall x, y \in S$

$$\mu_A(xy) \geq \mu_{A \circ A}(xy) \geq \min\{\mu_A(x), \mu_A(y)\},$$

$$\nu_A(xy) \leq \nu_{A \circ A}(xy) \leq \max\{\nu_A(x), \nu_A(y)\}.$$

So A is an intuitionistic fuzzy semigroup of S.

Conversely, for $\forall x \in S$, if there are $y, z \in S$ such that $x = yz$, then

$$\mu_{A \circ A}(x) \leq \min\{\mu_A(y), \mu_A(z)\} \leq \mu_A(yz) = \mu_A(x),$$

$$\nu_{A \circ A}(x) \geq \max\{\nu_A(y), \nu_A(z)\} \geq \nu_A(yz) = \nu_A(x);$$

otherwise

$$\mu_{A \circ A}(x) = 0 \leq \mu_A(x), \nu_{A \circ A}(x) = 1 \geq \nu_A(x).$$

Thus $A \circ A \subseteq A$, this completes the proof.

Lemma5: For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy left ideal of S.

(2) $S \circ A \subseteq A$.

Proof: First assume that (1) holds. Let a be any element of S. In the case when there exist element $x, y \in S$ such that $a = xy$. Then, since A is an intuitionistic fuzzy left ideal of S, we have

$$\mu_{S \circ A}(a) = \sup_{a=xy}\{\min\{\mu_S(x), \mu_A(y)\}\} \leq \sup_{a=xy}\{\min\{1, \mu_A(xy)\}\} = \sup_{a=xy}\{\min\{1, \mu_A(a)\}\} = \mu_A(a),$$

$$\nu_{S \circ A}(a) = \inf_{a=xy}\{\max\{\nu_S(x), \nu_A(y)\}\} \geq \inf_{a=xy}\{\max\{0, \nu_A(xy)\}\} = \inf_{a=xy}\{\max\{0, \nu_A(a)\}\} = \nu_A(a).$$

Otherwise

$$\mu_{S \circ A}(a) = 0 \leq \mu_A(a), \nu_{S \circ A}(a) = 1 \geq \nu_A(a).$$

So we have $S \circ A \subseteq A$.

Conversely, assume (2) holds. Let x, y be any element of S, put $a = xy$. Since $S \circ A \subseteq A$, we have

$$\mu_A(xy) \geq \mu_{S \circ A}(a) = \sup_{a=bc}\{\min\{\mu_S(b), \mu_A(c)\}\} \geq \min\{\mu_S(x), \mu_A(y)\} = \min\{1, \mu_A(y)\} = \mu_A(y),$$

$$\nu_A(xy) \leq \nu_{S \circ A}(a) = \inf_{a=bc}\{\max\{\nu_S(b), \nu_A(c)\}\} \leq \max\{\nu_S(x), \nu_A(y)\} = \max\{0, \mu_A(y)\} = \nu_A(y).$$

This means that A is an intuitionistic fuzzy left ideal of S, thus (2) implies (1). This completes the proof.

The left-right dual of lemma5 states as follows:

Lemma6: For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy right ideal of S.

(2) $A \circ S \subseteq A$.

Combining the above two lemmas, we can obtain the following result:

Lemma7: For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy two-sided ideal of S.

(2) $S \circ A \subseteq A$ and $A \circ S \subseteq A$.

For an intuitionistic fuzzy biideal of a semigroup we have the following:

Lemma8: For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy biideal of S.

(2) $A \circ A \subseteq A$ and $A \circ S \circ A \subseteq A$.

Proof: First assume that (1) holds. Since A is an intuitionistic fuzzy semigroup of S, by Lemma4 that $A \circ A \subseteq A$. Let a be any element of S, in the case when $\mu_{A\circ S\circ A}(a) = 0, v_{A\circ S\circ A}(a) = 1$, it is obvious that $A \circ S \circ A \subseteq A$.

Otherwise, there exist elements $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Since A is an intuitionistic fuzzy biideal of S, we have

$$\mu_A(pqy) \geq \min\{\mu_A(p), \mu_A(y)\}, v_A(pqy) \leq \max\{v_A(p), v_A(y)\}.$$

Then we have

$$\begin{aligned}
\mu_{A\circ S\circ A}(a) &= \sup_{a=xy}\{\min\{\mu_{A\circ S}(x), \mu_A(y)\}\} \\
&= \sup_{a=xy}\left\{\min\left\{\sup_{x=pq}\{\min\{\mu_A(p), \mu_S(q)\}\}, \mu_A(y)\right\}\right\} \\
&= \sup_{a=xy}\left\{\min\left\{\sup_{x=pq}\{\min\{\mu_A(p), 1\}\}, \mu_A(y)\right\}\right\} \\
&= \sup_{a=xy}\{\min\{\mu_A(p), \mu_A(y)\}\} \\
&\leq \sup_{a=xy}\{\mu_A(pqy)\} \\
&= \sup_{a=xy}\{\mu_A(xy)\} \\
&= \mu_A(a),
\end{aligned}$$

$$\begin{aligned}
v_{A\circ S\circ A}(a) &= \inf_{a=xy}\{\max\{v_{A\circ S}(x), v_A(y)\}\} \\
&= \inf_{a=xy}\left\{\max\left\{\inf_{x=pq}\{\max\{v_A(p), v_S(q)\}\}, v_A(y)\right\}\right\} \\
&= \inf_{a=xy}\left\{\max\left\{\inf_{x=pq}\{\max\{v_A(p), 0\}\}, v_A(y)\right\}\right\} \\
&= \inf_{a=xy}\{\max\{v_A(p), v_A(y)\}\} \\
&\geq \inf_{a=xy}\{v_A(pqy)\} \\
&= \inf_{a=xy}\{v_A(xy)\} \\
&= v_A(a),
\end{aligned}$$

and so we have $A \circ S \circ A \subseteq A$.

Conversely, assume that (2) holds. Since $A \circ A \subseteq A$,by lemma4 that A is an intuitionistic fuzzy semigroup of S. Let x, y, z be any element of S. Put $a = xyz$, since $A \circ S \circ A \subseteq A$, we have

$$\begin{aligned}
\mu_A(xyz) &= \mu_A(a) \\
&\geq \mu_{A \circ S \circ A}(a) \\
&= \sup_{a=bc}\{\min\{\mu_{A \circ S}(b), \mu_A(c)\}\} \\
&\geq \min\{\mu_{A \circ S}(xy), \mu_A(z)\} \\
&= \min\left\{\sup_{xy=pq}\{\min\{\mu_A(p), \mu_S(q)\}\}, \mu_A(z)\right\} \\
&\geq \min\{\min\{\mu_A(x), \mu_S(y)\}, \mu_A(z)\} \\
&= \min\{\min\{\mu_A(x), 1\}, \mu_A(z)\} \\
&= \min\{\mu_A(x), \mu_A(z)\}
\end{aligned}$$

$$\begin{aligned}
v_A(xyz) &= v_A(a) \\
&\leq v_{A \circ S \circ A}(a) \\
&= \inf_{a=bc}\{\max\{v_{A \circ S}(b), v_A(c)\}\} \\
&\leq \max\{v_{A \circ S}(xy), v_A(z)\} \\
&= \max\left\{\inf_{xy=pq}\{\max\{v_A(p), v_S(q)\}\}, v_A(z)\right\} \\
&\leq \max\{\max\{v_A(x), v_S(y)\}, v_A(z)\} \\
&= \max\{\max\{v_A(x), 0\}, v_A(z)\} \\
&= \max\{\mu_A(x), \mu_A(z)\}
\end{aligned}$$

This means that A is an intuitionistic fuzzy biideal of S. This completes the proof.

Lemma9: Let A be any intuitionistic fuzzy set in a semigroup S, and B be any intuitionistic fuzzy biideal of S. Then $A \circ B$ and $B \circ A$ are both intuitionistic fuzzy biideals of S.

Proof: Since B be an intuitionistic fuzzy biideal of S, it follows from lemma8 that $(A \circ B) \circ (A \circ B) = A \circ (B \circ (A \circ B)) \subseteq A \circ (B \circ (S \circ B)) \subseteq A \circ B$. Then it follows from lemma1 that $A \circ B$ is an intuitionistic fuzzy semigroup of S. And we have $(A \circ B) \circ S \circ (A \circ B) = A \circ (B \circ (S \circ A) \circ B) \subseteq A \circ (B \circ S \circ B) \subseteq A \circ B$. Then it follows from lemma4 that $A \circ B$ is an intuitionistic fuzzy biideal of S. It can be seen in a similar way that $B \circ A$ is an intuitionistic fuzzy biideal of S.

Lemma10: Every intuitionistic left (right) ideal of a semigroup is also an intuitionistic fuzzy biideal.

Proof: The proof is obvious, we omit it.

Lemma11[13]: For a semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $B \cap C = BCB$ holds for every biideal B and every two-sided ideal C of S.

Lemma12[1]: For a semigroup S, the following conditions are equivalent:

(1) S is regular;
(2) $BC = B \bigcap C$ for every right ideal B and every left ideal C of S.

3. Characterizing regular semigroup

Theorem1: For a semigroup S, the following conditions are equivalent:
(1) S is regular;
(2) $A = A \circ S \circ A$ for all intuitionistic fuzzy biideals of S.

Proof: First assume that (1) holds. Let A be any intuitionistic fuzzy biideal of S, and a any element of S. Since S is regular, there exists an element x in S such that $a = axa$. Then we have

$$\begin{aligned} \mu_{A\circ S\circ A}(a) &= \sup_{a=yz}\{\min\{\mu_{A\circ S}(y), \mu_A(z)\}\} \\ &\geq \min\{\mu_{A\circ S}(ax), \mu_A(a)\} \\ &= \min\left\{\sup_{ax=pq}\{\min\{\mu_A(p), \mu_S(q)\}, \mu_A(a)\}\right\} \\ &\geq \min\{\min\{\mu_A(a), \mu_S(x)\}, \mu_A(a)\} \\ &= \min\{\min\{\mu_A(a), 1\}, \mu_A(a)\} \\ &= \mu_A(a) \end{aligned}$$

$$\begin{aligned} \nu_{A\circ S\circ A}(a) &= \inf_{a=yz}\{\max\{\nu_{A\circ S}(y), \nu_A(z)\}\} \\ &\leq \max\{\nu_{A\circ S}(ax), \nu_A(a)\} \\ &= \max\left\{\inf_{ax=pq}\{\max\{\nu_A(p), \nu_S(q)\}, \nu_A(a)\}\right\} \\ &\leq \max\{\max\{\nu_A(a), \nu_S(x)\}, \nu_A(a)\} \\ &= \max\{\max\{\nu_A(a), 0\}, \nu_A(a)\} \\ &= \nu_A(a) \end{aligned}$$

so we have $A \subseteq A \circ S \circ A$. Since A is an intuitionistic fuzzy biideal of S, it follows from lemma8 that $A \circ S \circ A \subseteq A$. Thus we obtain $A \circ S \circ A = A$.

Conversely, assume (2) holds, let B be any biideal of S, and a any element of B, then $\widetilde{B}$ is a intuitionistic fuzzy biideal of S by lemma 2, so we have

$$\sup_{a=yz}\{\min\{\mu_{\widetilde{B}\circ S}(y), \mu_{\widetilde{B}}(z)\}\} = \mu_{\widetilde{B}\circ S\circ \widetilde{B}}(a) = \mu_{\widetilde{B}}(a) = 1,$$
$$\inf_{a=yz}\{\max\{\mu_{\widetilde{B}\circ S}(y), \mu_{\widetilde{B}}(z)\}\} = \nu_{\widetilde{B}\circ S\circ \widetilde{B}}(a) = \nu_{\widetilde{B}}(a) = 0.$$

This implies that there exist $b, c \in S, a = bc$ such that

$$\mu_{\widetilde{B}\circ S}(b) = 1, \mu_{\widetilde{B}}(c) = 1, \nu_{\widetilde{B}\circ S}(b) = 0, \nu_{\widetilde{B}}(c) = 0.$$

Then we have

$$\sup_{b=pq}\{\min\{\mu_{\tilde{B}}(p),\mu_S(q)\}\}=\mu_{\tilde{B}\circ S}(b)=1,$$
$$\inf_{b=pq}\{\max\{\nu_{\tilde{B}}(p),\nu_S(q)\}\}=\nu_{\tilde{B}\circ S}(b)=0.$$

This implies that there exist $d,e\in S,b=de$ such that

$$\mu_{\tilde{B}}(d)=1,\nu_{\tilde{B}}(d)=0,\mu_S(e)=1,\nu_S(e)=0.$$

So we have $d,c\in B,e\in S$, and $a=bc=(de)c\in BSB$, thus we have $B\subseteq BSB$. On the other hand, since B is a biideal of S, $BSB\subseteq B$ holds. So we obtain that $B=BSB$. This means that S is regular semigroup. This completes the proof.

Theorem2: Let A be an intuitionistic fuzzy set in a regular semigroup S, then the following conditions are equivalent:
(1) A is an intuitionistic fuzzy biideal of S;
(2) A may be represented in the form $A=B\circ C$, where B is an intuitionistic fuzzy right ideal and C an intuitionistic fuzzy left ideal of S.

Proof: First assume that (1) holds, since S is regular, it follows from theorem1 that

$$\begin{aligned}A&=A\circ S\circ A\\&=A\circ S\circ(A\circ S\circ A)\\&=(A\circ(S\circ A))\circ(S\circ A)\\&\subseteq(A\circ S)\circ(S\circ A)\\&=A\circ(S\circ S)\circ A\\&\subseteq A\circ S\circ A\\&=A\end{aligned}$$

and so we have $A=(A\circ S)\circ(S\circ A)$. On the other hand, we have $(A\circ S)\circ S=A\circ(S\circ S)\subseteq A\circ S$, then it follows form lemma8 that $A\circ S$ is an intuitionistic fuzzy ideal of S. Similarly, we can prove that $S\circ A$ is an intuitionistic fuzzy left ideal of S.

Conversely, assume (2) holds, then it follows from lemma9 and lemma10 that A is an intuitionistic fuzzy biideal of S. This completes the proof.

Theorem3: For a semigroup S, the following conditions are equivalent:
(1) S is regular;
(2) $B\cap C=B\circ C\circ B$ holds for every intuitionistic fuzzy biideal B and every intuitionistic fuzzy two-sided ideal C of S.

Proof: First assume (1) holds, let B be any intuitionistic fuzzy biideal, and C any intuitionistic fuzzy two-sided ideal of S. Then by lemma8 we obtain

$$B\circ C\circ B\subseteq B\circ S\circ B\subseteq B.$$

By lemma7 we have $B\circ C\circ B\subseteq S\circ C\circ S\subseteq S\circ C\subseteq C$. Thus $B\circ C\circ B\subseteq B\cap C$.

Next we will prove the converse inclusion holds, let a be any element of S, since S is a regular semigroup, there exists an element x in S such that $a = axa$, then

$$\mu_C(xax) \geq \mu_C(ax) \geq \mu_C(a), v_C(xax) \leq v_C(ax) \leq v_C(a).$$

Thus

$$\begin{aligned}
\mu_{B \circ C \circ B}(a) &= \sup_{a=yz}\{\min\{\mu_B(y), \mu_{C \circ B}(z)\}\} \\
&\geq \min\{\mu_B(a), \mu_{C \circ B}(xaxa)\} \\
&\geq \min\left\{\mu_B(a), \sup_{xaxa=pq}\{\min\{\mu_C(xax), \mu_B(a)\}\}\right\} \\
&\geq \min\{\mu_B(a), \min\{\mu_C(a), \mu_B(a)\}\} \\
&= \min\{\mu_B(a), \mu_C(a)\} \\
&= \mu_{B \cap C}(a),
\end{aligned}$$

$$\begin{aligned}
v_{B \circ C \circ B}(a) &= \inf_{a=yz}\{\max\{v_B(y), v_{C \circ B}(z)\}\} \\
&\leq \max\{v_B(a), v_{C \circ B}(xaxa)\} \\
&\leq \max\left\{v_B(a), \inf_{xaxa=pq}\{\max\{v_C(xax), v_B(a)\}\}\right\} \\
&\leq \max\{v_B(a), \max\{v_C(a), v_B(a)\}\} \\
&= \max\{v_B(a), v_C(a)\} \\
&= v_{B \cap C}(a).
\end{aligned}$$

So we obtain $B \circ C \subseteq B \circ C \circ B$, therefore $B \circ C = B \circ C \circ B$.

Conversely, since S itself is an intuitionistic fuzzy two-sided ideal of S, we have $B = B \cap S = B \circ S \circ B$, thus S is regular. This completes the proof.

Theorem4: For a semigroup S, the following conditions are equivalent:

(1) S is regular;

(2) $B \circ C = B \cap C$ for every intuitionistic fuzzy right ideal B and every intuitionistic fuzzy left ideal C of S.

Proof: First assume (1) holds, let B be any intuitionistic fuzzy right ideal and C any intuitionistci fuzzy left ideal of S. Then by lemma6 and lemma5 we obtain

$B \circ C \subseteq B \circ S \subseteq B, B \circ C \subseteq S \circ C \subseteq C$, and we have $B \circ C \subseteq B \cap C$.

Next we will prove the converse inclusion holds, let a be any element of S, since S is regular, there exists an element $x \in S$ such that $a = axa$, then we have

$$\begin{aligned}
\mu_{B \circ C}(a) &= \sup_{a=yz}\{\min\{\mu_B(y), \mu_C(z)\}\} \\
&\geq \min\{\mu_B(ax), \mu_C(a)\} \\
&\geq \min\{\mu_B(a), \mu_C(a)\} \\
&= \mu_{B \circ C}(a),
\end{aligned}$$

$$
\begin{aligned}
\nu_{B\circ C}(a) &= \inf_{a=yz}\{\max\{\nu_B(y),\nu_C(z)\}\} \\
&\le \max\{\nu_B(ax),\nu_C(a)\} \\
&\le \max\{\nu_B(a),\nu_C(a)\} \\
&= \nu_{B\circ C}(a).
\end{aligned}
$$

Then we obtain $B\cap C\subseteq B\circ C$, thus $B\cap C=B\circ C$.

Conversely, let P be any right ideal and Q be any left ideal of S, a be any element of $P\cap Q$. Then by lemma1 , $\widetilde{P},\widetilde{Q}$ are an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively, so we have $\widetilde{P}\circ\widetilde{Q}=\widetilde{P}\cap\widetilde{Q}$ and

$$
\sup_{a=yz}\{\min\{\mu_{\widetilde{P}}(y),\mu_{\widetilde{Q}}(z)\}\}=\mu_{\widetilde{P}\circ\widetilde{Q}}(a)=\mu_{\widetilde{P}\cap\widetilde{Q}}(a)=\min\{\mu_{\widetilde{P}}(a),\mu_{\widetilde{Q}}(a)\}=1,
$$
$$
\inf_{a=yz}\{\max\{\nu_{\widetilde{P}}(y),\nu_{\widetilde{Q}}(z)\}\}=\nu_{\widetilde{P}\circ\widetilde{Q}}(a)=\nu_{\widetilde{P}\cap\widetilde{Q}}(a)=\max\{\nu_{\widetilde{P}}(a),\nu_{\widetilde{Q}}(a)\}=0.
$$

This means that there exist element $b,c\in S, a=bc$ such that

$$
\mu_{\widetilde{P}}(b)=1,\mu_{\widetilde{Q}}(c)=1,\nu_{\widetilde{P}}(b)=0,\nu_{\widetilde{Q}}(c)=0.
$$

Thus $a=bc\in PQ$, so we have $P\cap Q\subseteq PQ$, since the converse inclusion always holds, we obtain that $PQ=P\cap Q$, so S is regular. This completes the proof.

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