Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy goal geometric programming problem

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Abstract: This paper deals with goal geometric programming problem which is discussed on intuitionistic fuzzy environment. Also a more general concept of intuitionistic fuzzy set is proposed and it is applied on goal geometric programming problem. Some basic properties on intuitionistic fuzzy optimization are described in this paper. Numerical examples are also provided for illustration. A design of Industrial Wastewater Treatment Plant, operating on pulp and paper manufacturing wastes is taken as an application. Decision Maker sets some objectives and its targets in purifying wastewater such as removal of maximum five day biochemical oxygen demand (BOD_5) at the minimum cost.

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1 Introduction

The concept of intuitionistic fuzzy set is introduced by Atanassov [1] in 1986 adding an additional degree of non-membership. Further some basic concepts and theorems on intuitionistic fuzzy set have been developed by Grzegorzewski, Mrowka [15], Deschrijver, Kerre [7], Cattaneo, Ciucci [4] etc. Goal programming has been used as a useful tool for multi-objective optimization problems. But intuitionistic fuzzy goal programming is rare in this context. Intuitionistic fuzzy linear programming is defined by Nagoorgani, Ponnalago [17] and Dubey, Mehera [8]. They both have used triangular intuitionistic fuzzy number in solving linear programming problem. Nachammai, Thangaraj [16] have also discussed intuitionistic fuzzy linear programming problem but they have used similarity measures. Parvathi, Malathi

[18, 19] have developed simplex method and decisive set method on intuitionistic fuzzy linear programming problem. Apart from linear programming on intuitionistic fuzzy environment, Chakrabortty et. al. [5] and N. K. Mahapatra [13] have described on intuitionistic fuzzy non-linear programming problem.

In this paper, we have worked on intuitionistic fuzzy goal programming problem where equations are non-linear. In a more generalized manner, we have taken generalized intuitionistic fuzzy set in the model of intuitionistic fuzzy goal programming problem. Generalized intuitionistic fuzzy set is proposed by Mondal, Samanta [14]. They have discussed definition and some properties on generalized intuitionistic fuzzy set. We have applied the same concept of generalized intuitionistic fuzzy set in intuitionistic fuzzy goal programming and described a basic property on generalized intuitionistic fuzzy goal programming. Since, in this paper, we have taken non-linear equations, so we have used one of the useful techniques, geometric programming to solve intuitionistic fuzzy goal programming problem and generalized intuitionistic fuzzy goal programming problem. Geometric programming gives better result than nonlinear programming (K-K-T conditions) which is already described in Ghosh, Roy [11, 12]. Cao [3] has introduced a method of solving geometric programming and applied it in fuzzy environment. In this paper we have followed Cao's method of solving goal geometric programming problem in intuitionistic fuzzy environment. Sakawa et. al. [20] has described Pareto optimality, M-Pareto optimality for fuzzy multi-objective stochastic programming and multi-objective fuzzy random programming etc. Following him we have constructed M-N Pareto optimality for intuitionistic fuzzy multi-objective nonlinear programming. Real life applications on material removal economics case study [6] and industrial wastewater treatment design are illustrated here in both models: intuitionistic fuzzy goal geometric programming problem (IFG^2P^2), generalized intuitionistic fuzzy goal geometric programming problem ($GIFG^2P^2$). Shih, Krishnan [21], Evenson [10], Ecker, McNamara [9] are the pioneer of industrial wastewater treatment design. Later Beightler, Philips [2] have discussed the design using geometric programming and Cao [3] also has discussed it using his method of geometric programming on it. In this paper, Cao's method of geometric programming is used on intuitionistic fuzzy environment to illustrate industrial wastewater treatment design.

Definition 1.1 (Intuitionistic fuzzy set (IFS)) Let X is a non null set. An intuitionistic fuzzy set in X is given by $\tilde{A}^{\mathrm{I}}= \{\{x,\mu_A(x),\vartheta_A(x)\}: x\in X\}$ where $\mu_{\tilde{A}^{\mathrm{I}}}(x)$: $X\to[0,1]$ and $\vartheta_{\tilde{A}^{\mathrm{I}}}(x)$: $X\to[0,1]$ satisfy the condition $0\le \mu_{\tilde{A}^{\mathrm{I}}}(x)+\vartheta_{\tilde{A}^{\mathrm{I}}}(x)\le 1$, for every $x\in X$.

2 Intuitionistic fuzzy goal programming model

We consider an intuitionistic fuzzy goal programming problem of minimization type objective function. Mathematically, the problem can be stated as:

$$\left.\begin{array}{l}\widetilde{\text{Minimize}}^{I} f_i(X) \text{ with target value } a_i \text{ and acceptance tolerance } t_i, \\ \qquad\qquad \text{rejection tolerance } t_i^0,\ i = 1, 2, ..., m \\ \text{subject to } g_j(X) \le bj,\ j = 1, 2, ..., n; \\ X = (x_1, x_2, ..., x_q)^T > 0\end{array}\right\} \quad (2.1)$$

In addition with degree of acceptance (membership), when degree of non-acceptance (non-membership) is taken into consideration without complementing each other, then an intuitionistic fuzzy set can be used as a general tool for decision making under uncertainty. Let $\mu_{f_i}(x_1, x_2 \dots x_q)$ be the degree of acceptance (membership) and $\vartheta_{f_i}(x_1, x_2 \dots x_q)$ be the degree of non-acceptance (non-membership) of X to i-th intuitionistic fuzzy set. Then in optimization problem membership function should be maximized and non-membership function should be minimized. Hence, the above model (2.1) can be written in crisp programming as:

$$\left.\begin{array}{l} \text{Maximize } \mu_{f_i}(x_1, x_2, \dots, x_q), \text{ Minimize } \vartheta_{f_i}(x_1, \dots, x_q),\ i = 1, 2, \dots, m \\ \text{subject to } g_j(X) \le b_j,\ j = 1, 2, \dots, n;\ X = (x_1, x_2, \dots, x_q)^T > 0, \\ 0 \le \mu_{f_i}(x_1, x_2, \dots, x_q) + \vartheta_{f_i}(x_1, x_2, \dots, x_q) \le 1 \\ 0 \le \mu_{f_i}(x_1, x_2, \dots, x_q) \le 1, 0 \le \vartheta_{f_i}(x_1, x_2, \dots, x_q) \le 1. \end{array}\right\} \quad (2.2)$$

The linear membership and non-membership functions for Intuitionistic fuzzy objective goals are

$$\mu_{f_i}(x_1, x_2 \dots x_q) = \begin{cases} 1, & f_i(x_1, x_2 \dots x_q) \le a_i \\ 1 - \frac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i}, & a_i \le f_i(x_1, x_2 \dots x_q) \le a_i + t_i \\ 0, & f_i(x_1, x_2 \dots x_q) \ge a_i + t_i \end{cases}$$

and

$$\vartheta_{f_i}(x_1, x_2 \dots x_q) = \begin{cases} 0 & f_i(x_1, x_2 \dots x_q) \le a_i \\ \dfrac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i^0}, & a_i \le f_i(x_1, x_2 \dots x_q) \le a_i + t_i^0 \\ 1, & f_i(x_1, x_2 \dots x_q) \ge a_i + t_i^0 \end{cases}$$

with $t_i \le t_i^0$, for $i = 1, 2, \dots, m$.

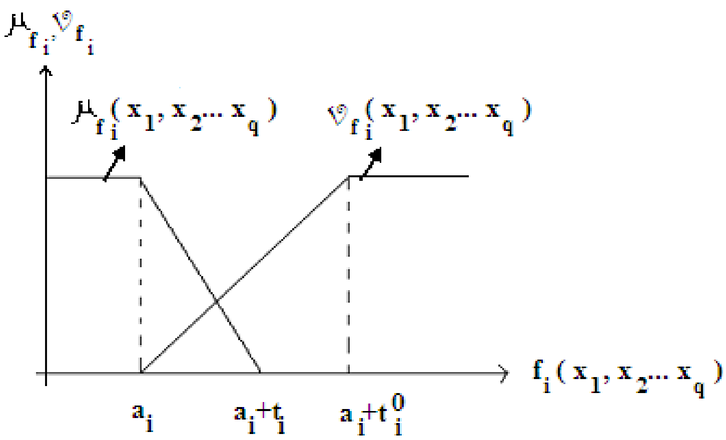

Figure 1. Linear membership and non-membership functions

Definition 2.1 (M-N Pareto optimal solution) x^* is said to be a M-N Pareto optimal solution if and only if there does not exist another $x \in X$ such that $\mu_{f_i}(x) \geq \mu_{f_i}(x^*)$ and $\vartheta_{f_i}(x) \leq \vartheta_{f_i}(x^*)$ for all *i* and strict inequality holds for at least one *i*.

Theorem 2.1 $x^* \in X$ is M-N Pareto optimal solution of (2.1) if and only if x^* is Pareto optimal solution of

$$\left.\begin{array}{l} \text{Minimize } f_i(X),\ i = 1, 2 \dots m \\ \text{subject to } g_j(X) \leq b_j,\ j = 1, 2 \dots n;\ X = (x_1, x_2 \dots x_q)^{\mathrm{T}} > 0. \end{array}\right\} \tag{2.3}$$

Proof: Let $x^* \in X$ is M-N Pareto optimal solution of (2.1), then there does not exist any $x \in X$ such that $\mu_{f_i}(x_1, x_2 \dots x_q) \geq \mu_{f_i}(x_1, x_2 \dots x_q)^*$ and $\vartheta_{f_i}(x_1, x_2 \dots x_q) \leq \vartheta_{f_i}(x_1, x_2 \dots x_q)^*$ for all *i* and strictly inequality holds for at least one *i*.

So from the expression of membership function we have

$$1 - \frac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i} \geq 1 - \frac{f_i(x_1, x_2 \dots x_q)^* - a_i}{t_i} \text{ i.e. } f_i(x_1, x_2 \dots x_q) \leq f_i(x_1, x_2 \dots x_q)^*$$

Also for non-membership function

$$\frac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i^0} \leq \frac{f_i(x_1, x_2 \dots x_q)^* - a_i}{t_i^0}$$

gives $f_i(x_1, x_2 \dots x_q) \leq f_i(x_1, x_2 \dots x_q)^*$ with strict inequality holding for at least one *i*. So x^* is Pareto optimal solution of (2.3).

On the other hand, let x^* be a Pareto optimal solution of (2.3). Then there does not exist $x \in X$ such that $f_i(x_1, x_2 \dots x_q) \leq f_i(x_1, x_2 \dots x_q)^*$ with strict inequality holding for at least one *i*.

So $f_i(x_1, x_2 \dots x_q) - a_i \leq f_i(x_1, x_2 \dots x_q)^* - a_i$, i.e.

$$\frac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i^0} \leq \frac{f_i(x_1, x_2 \dots x_q)^* - a_i}{t_i^0}$$

tells that $\vartheta_{f_i}(x_1, x_2 \dots x_q) \leq \vartheta_{f_i}(x_1, x_2 \dots x_q)^*$ and

$$1 - \frac{f_i(x_1, x_2 \dots x_q) - a_i}{t_i} \geq 1 - \frac{f_i(x_1, x_2 \dots x_q)^* - a_i}{t_i}$$

tells that $\mu_{f_i}(x_1, x_2 \dots x_q) \geq \mu_{f_i}(x_1, x_2 \dots x_q)^*$, with strict inequality holding for at least one *i*.

Hence, x^* is M-N Pareto optimal solution of (2.1).

2.1 Goal geometric programming model

Model (2.2) is a standard crisp programming model, which is derived from the intuitionistic fuzzy goal programming problem (2.1). The crisp programming model (2.2) can be formulated as:

$$\left.\begin{aligned}&\text{Maximize } \alpha, \text{ Minimize } \beta\\&\text{subject to } 1-\frac{f_i(x_1,x_2 \dots x_q)-a_i}{t_i} \geq \alpha, \frac{f_i(x_1,x_2 \dots x_q)-a_i}{t_i^0} \leq \beta, i=1, 2 \dots m\\&\qquad g_j(x_1, x_2 \dots x_q) \leq b_j, j= 1, 2 \dots n\\&\qquad X=(x_1, x_2 \dots x_q)^{\mathrm{T}}>0, \alpha+\beta \leq 1,\ 0 \leq \alpha, \beta \leq 1\end{aligned}\right\} \quad (2.4)$$

It is easily seen that Minimize β is equivalent to Maximize $(1-\beta)$ as $0 \leq \beta \leq 1$. Therefore using arithmetic mean method [20] the above model (2.4) can be written as

$$\left.\begin{aligned}&\text{Maximize } \frac{\alpha+1-\beta}{2}\\&\text{subject to } \frac{f_i(x_1,x_2 \dots x_q)}{2}\left(\frac{1}{t_i}+\frac{1}{t_i^0}\right) \leq 1+\frac{a_i}{2}\left(\frac{1}{t_i}+\frac{1}{t_i^0}\right)-\frac{\alpha+1-\beta}{2}, i= 1, 2 \dots m,\\&\qquad g_j(x_1, x_2 \dots x_q) \leq b_j, j= 1, 2 \dots n,\\&\qquad X=(x_1, x_2 \dots x_q)^{\mathrm{T}}>0, \alpha+\beta \leq 1, 0 \leq \alpha, \beta \leq 1.\end{aligned}\right\} \quad (2.5)$$

Let us consider $u = \frac{\alpha+1-\beta}{2}$, then the above model (2.5) reduces to following primal geometric programming form as

$$\left.\begin{aligned}&\text{Minimize } (u)^{-1}\\&\text{subject to } \frac{f_i(x_1,x_2 \dots x_q)}{2\left(1+\frac{a_i}{2}\left(\frac{1}{t_i}+\frac{1}{t_i^0}\right)-u\right)}\left(\frac{1}{t_i}+\frac{1}{t_i^0}\right) \leq 1, i=1,2 \dots m,\\&\qquad \frac{g_j(x_1,x_2 \dots x_q)}{b_j} \leq 1, j=1, 2 \dots n,\\&\qquad X=(x_1, x_2 \dots x_q)^{\mathrm{T}}>0, u>0.\end{aligned}\right\} \quad (2.6)$$

which can be solved by geometric programming technique with u (> 0) as parameter.

2.2 Generalized Intuitionistic fuzzy set (GIFS)

Let X be a non-null set. A generalized intuitionistic fuzzy set $\widetilde{A^I}$ on X is defined as $\widetilde{A^I}= \{\{x,\mu_{\tilde{A}}(x),\vartheta_{\tilde{A}}(x)\}:x\in X\}$ where $\mu_{\widetilde{A^I}}(x)$: $X\to[0,w]$ and $\vartheta_{\widetilde{A^I}}(x)$: $X\to[0,w^0]$ if it satisfies the condition $0\leq \mu_{\widetilde{A^I}}(x)+\vartheta_{\widetilde{A^I}}(x)\leq w+w^0$ $(0 \leq w+w^0 \leq 1)$, for every $x \in X$ where w $(0 \leq w \leq 1)$ and w^0 $(0 \leq w^0 \leq 1)$ are the gradations of the membership and the non-membership function, respectively.

Theorem 2.2. For a generalized intuitionistic fuzzy goal geometric programming model

$\widetilde{Minimize}^I$ $f_i(X)$ with target value a_i and acceptance tolerance p_i rejection tolerance p_i^0

subject to $\quad g_j(X) \leq b_j, j = 1, 2 \dots n$

$\qquad X=(x_1, x_2 \dots x_q)^{\mathrm{T}}>0, i = 1, 2 \dots m$

where the sum of membership and non-membership functions will lie between 0 and $w+w^0$.

Proof: Membership function and non-membership functions are defined as follows (Fig. 2)

$$\mu^{w}(f_i(x_1, x_2 \dots x_q)) = \begin{cases} w, & f_i(x_1, x_2 \dots x_q) \le a_i \\ w\left(1 - \frac{f_i(x_1, x_2 \dots x_q) - a_i}{p_i}\right), & a_i \le f_i(x_1, x_2 \dots x_q) \le a_i + p_i \\ 0, & f_i(x_1, x_2 \dots x_q) \ge a_i + p_i \end{cases}$$

and

$$\vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = \begin{cases} 0, & f_i(x_1, x_2 \dots x_q) \le a_i \\ w^0\left(\frac{f_i(x_1, x_2 \dots x_q) - a_i}{p_i^0}\right), & a_i \le f_i(x_1, x_2 \dots x_q) \le a_i + p_i^0 \\ w^0, & f_i(x_1, x_2 \dots x_q) \ge a_i + p_i^0 \end{cases}$$

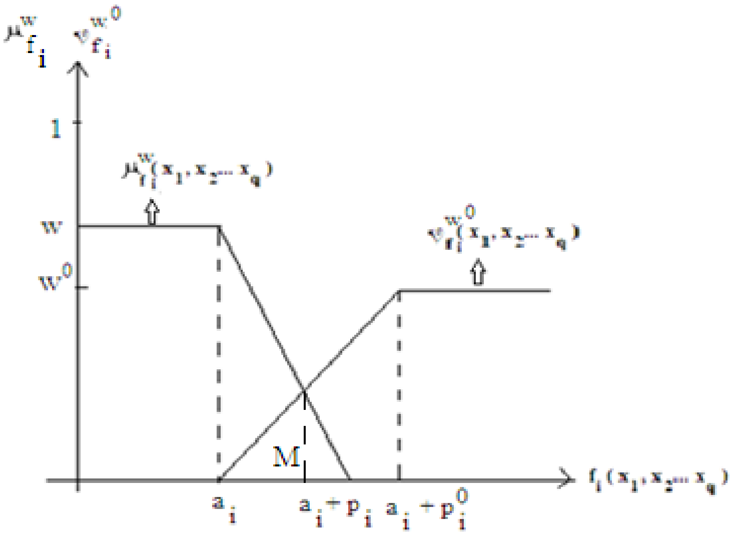

Figure 2. Membership and non-membership functions of $f_i(X)$

From Fig. 2 and the above definition, we see that $0 \le \mu^{w}(f_i(x_1, x_2 \dots x_q)) \le w$ and $0 \le \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \le w^0$. Further, for $f_i(x_1, x_2 \dots x_q) \le a_i$, $\mu^{w}(f_i(x_1, x_2 \dots x_q)) = w$, $\vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = 0$. Therefore $\mu^{w}(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w \le w + w^0$ [Since $w^0 \ge 0$]. Again $w \ge 0$ gives that $\mu^{w}(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \ge 0$.

From Fig. 2, it is seen that $\mu^{w}(f_i(x_1, x_2 \dots x_q))$ and $\vartheta^{w^0}(f_i(x_1, x_2 \dots x_q))$ are intersecting in the interval $[a_i, a_i + p_i]$. M is the intersecting point on horizontal axis whose co-ordinate is

$$(a_i + \frac{w}{(\frac{w}{p_i} + \frac{w^0}{p_i^0})}, 0).$$

For $f_i(x_1, x_2 \dots x_q) \in (a_i, a_i + p_i]$,

$$\mu^{w}(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w\left(1 - \frac{f_i(x_1, x_2 \dots x_q) - a_i}{p_i}\right) + w^0\left(\frac{f_i(x_1, x_2 \dots x_q) - a_i}{p_i^0}\right)$$

When $f_i(x_1, x_2 \dots x_q) \le a_i + \frac{w}{(\frac{w}{p_i}+\frac{w^0}{p_i^0})}$,

$$\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \le \frac{(w+w^0)^2}{w+w^0} - \frac{w^2+{w^0}^2}{w+w^0} \le w + w^0,$$

When $f_i(x_1, x_2 \dots x_q) \le a_i + p_i$, $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \le w^0 \frac{p_i}{p_i^0} < w^0$

$$\le w + w^0 \quad \text{[Since } \frac{p_i}{p_i^0} < 1 \text{, according to Fig. 2]}$$

Again when $f_i(x_1, x_2 \dots x_q) > a_i$, $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) > w \ge 0$.

For $a_i + p_i < f_i(x_1, x_2 \dots x_q) \le a_i + p_i^0$, $\mu^w(f_i(x_1, x_2 \dots x_q)) = 0$,

$$\vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w^0 \left(\frac{f_i(x_1, x_2 \dots x_q) - a_i}{p_i^0}\right).$$

When $f_i(x_1, x_2 \dots x_q) \le a_i + p_i^0$, $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \le w^0 \le w + w^0$ and $f_i(x_1, x_2 \dots x_q) > a_i + p_i$, $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) > w^0 \frac{p_i}{p_i^0} > w^0 \ge 0$ [Since $w^0 \ge 0$ and $\frac{p_i}{p_i^0} < 1$, according to Fig. 2].

For $f_i(x_1, x_2 \dots x_q) > a_i + p_i^0$, $\mu^w(f_i(x_1, x_2 \dots x_q)) = 0$, $\vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w^0$. Hence $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w^0 \le w + w^0$. Also since $w^0 \ge 0$, therefore in that region of $f_i(x_1, x_2 \dots x_q)$, $\mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) = w^0 \ge 0$.

Hence in all cases $0 \le \mu^w(f_i(x_1, x_2 \dots x_q)) + \vartheta^{w^0}(f_i(x_1, x_2 \dots x_q)) \le w + w^0$.

3 Illustrative numerical example

Every manufacturing unit wants to minimize the expenditure like loading unloading cost, cutting cost, tool cost and tool changing cost, while machining a 150 mm long and 25 mm in diameter cylindrical bar with cutting speed v m/min and feed rate f mm/rev. Decision maker of the manufacturing unit sets some flexible target of 1.8 $ of total expenditure. The maximum feed to be used to control the surface finish is 0.115 mm/rev with some flexibility. Uncertainty of total expenditure mainly depends upon tool life T which is related to the cutting condition via Taylor's equation

$$T = Cv^{-\frac{1}{n}} f^{-\frac{1}{m}}.$$

In spite of some fixed cutting conditions, tool life T may change due to non-homogeneity of the machined and cutting material. The required data for material removal economics case study are:

- R_0 = operator rate ($/min) = 0.60 $/min
- R_m= machine rate ($/min) = 0.40 $/min

- C_t= tool cost (\$/cutting edge) = 2.00 \$/edge
- t_l = machine loading & unloading time (min) = 1.5 min
- t_{ch}= tool changing time (min) = 0.8 min
- 1/m = feed rate exponent = 1.25 (m = 0.80)
- 1/n = cutting speed exponent = 4.00 (n = 0.25)
- C= Taylor's Modified Tool Life Constant (min) = 2.46×10^8 min
- Q= fraction of cutting path that tool is cutting material = 1.0 (for turning)
- B= cutting path surface factor of tool= 11.77286 (mm-m).
- Loading unloading cost k_{00} = $(R_0+R_m)\ t_l$ = 1.50,
- Cutting cost= $k_{01}f^{-1}v^{-1}$ = $(R_0+R_m)Bf^{-1}v^{-1}$ = $1.78f^{-1}v^{-1}$,
- Tool cost and tool changing cost= $k_{02}f^{(1/m-1)}v^{(1/n-1)}$

$$= [(R_0+R_m)\ t_{ch}+C_t]QBC^{-1}f^{(1/m-1)}v^{(1/n-1)}$$
$$= 1.34\times 10^{-7}f^{0.25}v^3$$

Hence the total expenditure C_u= $1.50+11.78f^{-1}v^{-1}+1.34\times 10^{-7}f^{0.25}v^3$ along with the feed rate f is to be minimized with some flexible targets. The intuitionistic fuzzy goal programming problem is:

- $\widetilde{Minimize}^I C_u = 1.50+11.78\,f^{-1}v^{-1}+1.34\times 10^{-7}f^{0.25}v^3$ with target value 1.8 and acceptance tolerance 0.3, rejection tolerance 0.5
- $\widetilde{Minimize}^I f_{ed} = f$ with target value 0.115 and acceptance tolerance 0.24, rejection tolerance 0.26 subject to f, v >0.

Membership and non-membership functions for intuitionistic fuzzy objective functions are:

$$\mu_{c_u}(f,v)=\begin{cases} 1 & C_u(f,v) \le 1.8 \\ 1-\frac{C_u(f,v)\ -1.8}{0.3}, & 1.8 \le C_u(f,v) \le 2.1 \\ 0 & C_u(f,v) \ge 2.1 \end{cases},$$

$$\vartheta_{c_u}(f,v) = \begin{cases} 0 & C_u(f,v) \le 1.8 \\ \frac{C_u(f,v)\ -1.8}{0.5}, & 1.8 \le C_u(f,v) \le 2.3 \\ 1, & C_u(f,v) \ge 2.3 \end{cases},$$

$$\mu_{f_{ed}}(f,v)=\begin{cases} 1 & f_{ed}(f,v) \le 0.115 \\ 1-\frac{f_{ed}(f,v)-0.115}{0.24}, & 0.115 \le f_{ed}(f,v) \le 0.355 \\ 0 & f_{ed}(f,v) \ge 0.355 \end{cases} \quad \text{and}$$

$$\vartheta_{f_{ed}}(f,v)=\begin{cases} 0 & f_{ed}(f,v) \le 0.115 \\ \frac{f_{ed}(f,v)-0.115}{0.26}, & 0.115 \le f_{ed}(f,v) \le 0.375 \\ 1 & f_{ed}(f,v) \ge 0.375 \end{cases}$$

Now the crisp model is

$$\left.\begin{array}{l}\text{Maximize } \mu_{c_u}(f, v), \text{ Minimize } \vartheta_{c_u}(f, v)\\ \text{Maximize } \mu_{f_{ed}}(f, v), \text{ Minimize } \vartheta_{f_{ed}}(f, v)\\ \text{subject to } 0\leq \mu_{c_u}(f, v), \mu_{f_{ed}}(f, v), \vartheta_{c_u}(f, v), \vartheta_{f_{ed}}(f, v) \leq 1,\\ \mu_{c_u}(f, v) + \vartheta_{c_u}(f, v) \leq 1, \mu_{f_{ed}}(f, v) + \vartheta_{f_{ed}}(f, v) \leq 1.\end{array}\right\} \quad (3.1)$$

Now we have got the below model of goal geometric programming problem by following expressions (2.4) and (2.5) as

$$\left.\begin{array}{l}\text{Minimize } (\frac{\alpha+1-\beta}{2})^{-1}\\ \text{subject to } \frac{\frac{5.333\times 11.78}{2}f^{-1}v^{-1}}{1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}} + \frac{\frac{5.333\times 1.34}{2}\times 10^{-7}f^{0.25}v^{3}}{1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}} \leq 1\\ \qquad \frac{\frac{9.109731}{2}f}{1+\frac{9.109731\times 0.115}{2}-\frac{\alpha+1-\beta}{2}} \leq 1\\ \qquad f, v>0, \alpha+\beta \leq 1,\ 0 \leq \alpha, \beta \leq 1.\end{array}\right\} \quad (3.2)$$

We solve the above geometric programming model where degree of difficulty is 3 – (2+1) = 0. Dual of the above model is

$$d= (\frac{2}{(\alpha+1-\beta)\delta_{01}})^{\delta_{01}} \frac{\frac{5.333\times 11.78}{2}}{\left(1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}\right)\delta_{11}})^{\delta_{11}} (\frac{\frac{5.333\times 1.34}{2}\times 10^{-7}}{\left(1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}\right)\delta_{12}})^{\delta_{12}}$$

$$\left(\frac{\frac{9.109731}{2}}{\left(1+\frac{9.109731\times 0.115}{2}-\frac{\alpha+1-\beta}{2}\right)\delta_{21}}\right)^{\delta_{21}} (\delta_{11}+\delta_{12})^{(\delta_{11}+\delta_{12})}{\delta_{21}}^{\delta_{21}}$$

such that $\delta_{01} = \delta_{11} + \delta_{12} = 1, - \delta_{11} + 0.25\delta_{12} + \delta_{21} = 0, - \delta_{11} + 3\delta_{12}=0$.

Solving them we have $\delta_{01} = 1, \delta_{11}= \frac{3}{4}, \delta_{12} = \frac{1}{4}, \delta_{21} = \frac{2.75}{4}$.

Hence from primal dual relation

$$(\frac{\alpha+1-\beta}{2})^{-1}$$

$$=(\frac{\alpha+1-\beta}{2})^{-1} \frac{\frac{5.333\times 11.78}{2}\times 4}{\left(1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}\right)\times 3})^{\frac{3}{4}} (\frac{\frac{5.333\times 1.34}{2}\times 10^{-7}\times 4}{\left(1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}\right)})^{\frac{1}{4}} \left(\frac{\frac{9.109731}{2}}{\left(1+\frac{9.109731\times 0.115}{2}-\frac{\alpha+1-\beta}{2}\right)}\right)^{\frac{2.75}{4}} \quad (3.3)$$

$$\frac{\frac{5.333\times 11.78}{2}f^{-1}v^{-1}}{1+\frac{5.333\times 0.3}{2}-\frac{\alpha+1-\beta}{2}} = \frac{\delta_{11}}{\delta_{11}+\delta_{12}} \quad (3.4)$$

$$\frac{\frac{5.333\times1.34}{2}\times10^{-7}f^{0.25}v^{3}}{1+\frac{5.333\times0.3}{2}-\frac{\alpha+1-\beta}{2}}=\frac{\delta_{12}}{\delta_{11}+\delta_{12}} \tag{3.5}$$

Considering $\frac{\alpha+1-\beta}{2}$= u in primal dual relation, the optimal values of decision variables and objective functions are obtained solving equation number (3.3), (3.4) and (3.5). The equations are solved using Lindo-Lingo software and the lists of values are given in Table 1.

Again the same example is illustrated in generalized intuitionistic fuzzy G^2P^2. Hence membership and non-membership functions for generalized intuitionistic fuzzy objective functions are

$$\mu_{C_u}{}^{w}(f,v)=\begin{cases} w & C_u \le 1.8 \\ w\left(1-\frac{C_u-1.8}{0.3}\right), & 1.8 \le C_u \le 2.1 \\ 0 & C_u \ge 2.1 \end{cases}$$

$$\vartheta_{C_u}{}^{w^0}(f,v)=\begin{cases} 0 & C_u \le 1.8 \\ w^0\left(\frac{C_u-1.8}{0.5}\right), & 1.8 \le C_u \le 2.3 \\ w^0, & C_u \ge 2.3 \end{cases}$$

$$\mu_{f_{ed}}{}^{w}(f,v)=\begin{cases} w & f_{ed} \le 0.115 \\ w\left(1-\frac{f_{ed}-0.115}{0.24}\right), & 0.115 \le f_{ed} \le 0.355 \\ 0 & f_{ed} \ge 0.355 \end{cases}$$

and

$$\vartheta_{f_{ed}}^{w^0}(f,v)=\begin{cases} 0 & f_{ed} \le 0.115 \\ w^0\left(\frac{f_{ed}-0.115}{0.26}\right), & 0.115 \le f_{ed} \le 0.375 \\ w^0 & f_{ed} \ge 0.375 \end{cases}$$

Following models (3.1), (2.4) and (2.5), the crisp goal geometric programming problem can be written as

Minimize $\left(\frac{\alpha+1-\beta}{2}\right)^{-1}$

subject to

$$\frac{1}{2}\left(\frac{w}{0.3}+\frac{w^0}{0.5}\right)\times 11.78f^{-1}v^{-1} + \frac{1}{2}\left(\frac{w}{0.3}+\frac{w^0}{0.5}\right)\times 1.34\times10^{-7}f^{0.25}v^{3} \le \frac{w+1}{2}+\frac{0.3}{2}\left(\frac{w}{0.3}+\frac{w^0}{0.5}\right)-\frac{\alpha+1-\beta}{2}$$

$$\frac{1}{2}\left(\frac{w}{0.24}+\frac{w^0}{0.26}\right)\times f \le \frac{w+1}{2}+\frac{0.115}{2}\left(\frac{w}{0.24}+\frac{w^0}{0.26}\right)-\frac{\alpha+1-\beta}{2}$$

$f, v>0, \alpha+\beta\le 1,\ 0\le \alpha,\beta\le 1.$

Considering $\frac{\alpha+1-\beta}{2}$ = u and following the solution procedure of model (3.2) we get the optimal decision variables and objective functions given in Table 1.

Table 1. Optimal values of decision variables and objective functions

Method	w, w^0	Primal Variables	Optimal objective functions	α^*, β^*	Membership and Non-membership	Sum of Membership and Non-membership
IFG^2P^2		f^*= 0.257297 mm/rev v^*= 112.4525 m/min	$C_u^*(f,v)$= 2.04285$ $f_{ed}^*(f,v)$= 0.25729 mm/rev	$\alpha^* \in$ [0,0.64801] $\beta^* \in$ [0,0.35198]	$\mu_{c_u}(f,v)$ =0.1904979 $\vartheta_{c_u}(f,v)$=0.4857013 $\mu_{f_{ed}}(f,v)$=0.407095 $\vartheta_{f_{ed}}(f,v)$=0.547296	μ_{c_u}(f, v) + ϑ_{c_u}(f, v) = 0.6761992 $\mu_{f_{ed}}(f,v)$+ $\vartheta_{f_{ed}}(f,v)$ = 0.954392
GIF G^2P^2	w = 0.7 w^0= 0.3	f^*= 0.27372 mm/rev v^*= 110.299 m/min	$C_u^*(f,v)$= 2.02025$ $f_{ed}^*(f,v)$= 0.27372 mm/rev	$\alpha^* \in$ [0, 0.526971] $\beta^* \in$ [0, 0.473029]	$\mu_{C_u}^{w}(f,v)$=0.1860902 $\vartheta_{C_u}^{w^0}(f,v)$=0.1321482 $\mu_{f_{ed}}^{w}(f,v)$=0.2370764 $\vartheta_{f_{ed}}^{w^0}(f,v)$=0.1831346	${\mu_{c_u}}^{w}(f,v)$+ ${\vartheta_{c_u}}^{w^0}(f,v)$ =0.3182385 ${\mu_{f_{ed}}}^{w}(f,v)$+ $\vartheta_{f_{ed}}^{w^0}(f,v)$ =0.420211

The table shows that all objective functions attain their goals as well as restrictions of membership and non-membership function in both IFG^2P^2 and $GIFG^2P^2$. Although the cutting conditions are different, Taylor's tool life remains unchanged (8.3949 min). But to get more minimized expenditure, $GIFG^2P^2$ is the appropriate method. Also it is noticeable that, in $GIFG^2P^2$ the sum of membership and non-membership function for each objective function is less or equal to sum of gradations ($w + w^0$). Hence all the criteria of generalized intuitionistic fuzzy set are satisfied here.

4 Industrial wastewater treatment model

A treatment plant is required to design to purify the wastewater of paper and pulp industry. The treatment units indicate the removal of suspended solid and BOD_5 from the wastewater. The process of paper and pulp wastewater treatment involves many steps (Cao in [15], Beightler and Philips in [19]).

In our study, we have taken the 1st design which has four consecutive disposal processes.

Primary Clarifier ⟶ Trickling Filter ⟶ Activated Sludge ⟶ Carbon Adsorption

After each disposal process, suspended solid and BOD_5 is removed from the wastewater. According to national standard, removal of BOD_5 from wastewater should be 97.1%. Let x_i be the percentage of remaining BOD_5 after each step. Then after four processes the remaining percentage of BOD_5 will be $x_1 x_2 x_3 x_4$. Decision Maker's (DM) aim is to minimize the remaining percentage of BOD_5 with minimum annual cost as much as possible. There are different annual costs for each disposal process given in Table 2.

Table 2. Annual Charges for different disposal process

Process	**Content**	**Annual Charge (Unit $ thousands)**
1	Primary Clarifier	$19.4x_1^{-1.47}$
2	Trickling Filter	$16.8x_2^{-1.66}$
3	Activated Sludge	$91.5x_3^{-0.3}$
4	Carbon Adsorption	$120x_4^{-0.33}$

Now, let us look into the series of disposal process with minimum annual cost and minimum percentage of remaining BOD_5 in wastewater. DM has set some targets on total annual cost and remaining percentage of BOD_5 and also gave flexibility on his targets. Hence the intuitionistic fuzzy goal programming model is

$\widetilde{Minimize}^I$ *Cost* $(x_1, x_2, x_3, x_4) = 19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5x_3^{-0.3} + 120x_4^{-0.33}$ with target 300, acceptance tolerance 200, rejection tolerance 300,

$\widetilde{Minimize}^I$ *BOD* $(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$ with target 0.012, acceptance tolerance 0.2, rejection tolerance 0.4,

subject to $x_1, x_2, x_3, x_4 > 0$.

Intuitionistic fuzzy goal geometric programming:

Here are the membership and non-membership functions of objective functions.

$$\mu_{cost}(x_1, x_2, x_3, x_4) = \begin{Bmatrix} 1, & Cost(x_1, x_2, x_3, x_4) \leq 300 \\ 1 - \frac{Cost(x_1, x_2, x_3, x_4) - 300}{200}, & 300 \leq Cost(x_1, x_2, x_3, x_4) \leq 500 \\ 0, & Cost(x_1, x_2, x_3, x_4) \geq 500 \end{Bmatrix},$$

$$\vartheta_{cost}(x_1, x_2, x_3, x_4) = \begin{Bmatrix} 0, & Cost(x_1, x_2, x_3, x_4) \leq 300 \\ \frac{Cost(x_1, x_2, x_3, x_4) - 300}{300}, & 300 \leq Cost(x_1, x_2, x_3, x_4) \leq 600 \\ 1, & Cost(x_1, x_2, x_3, x_4) \geq 600 \end{Bmatrix},$$

$$\mu_{BOD}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} 1, & BOD(x_1, x_2, x_3, x_4) \leq 0.012 \\ 1-\frac{BOD(x_1,x_2,x_3,x_4)-0.012}{0.2}, & 0.012 \leq BOD(x_1, x_2, x_3, x_4) \leq 0.212 \\ 0, & BOD(x_1, x_2, x_3, x_4) \geq 0.212 \end{array}\right\},$$

and

$$\vartheta_{BOD}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} 0, & BOD(x_1, x_2, x_3, x_4) \leq 0.012 \\ \frac{BOD(x_1,x_2,x_3,x_4)-0.012}{0.4}, & 0.012 \leq BOD(x_1, x_2, x_3, x_4) \leq 0.412 \\ 1, & BOD(x_1, x_2, x_3, x_4) \geq 0.412 \end{array}\right\}.$$

The intuitionistic fuzzy goal programming model can be transformed into geometric programming model following models (2.2), (2.4) and (2.5) as

$$\left.\begin{array}{l} \text{Minimize } \left(\frac{\alpha+1-\beta}{2}\right)^{-1} \\ \text{subject to } \frac{1}{\left(1+\frac{300\times5}{2\times600}-\frac{\alpha+1-\beta}{2}\right)}\left(\frac{19.4\times5}{2\times600}x_1^{-1.47}+\frac{16.8\times5}{2\times600}x_2^{-1.66}+\frac{91.5\times5}{2\times600}x_3^{-0.3}+\frac{120\times5}{2\times600}x_4^{-0.33}\right)\leq 1, \\ \frac{1}{\left(1+\frac{0.012\times0.6}{2\times0.08}-\frac{\alpha+1-\beta}{2}\right)}\left(\frac{0.6}{2\times0.08}x_1x_2x_3x_4\right) \leq 1 \\ x_1, x_2, x_3, x_4 > 0, \alpha+\beta \leq 1,\ 0 \leq \alpha, \beta \leq 1. \end{array}\right\} \quad (4.1)$$

Applying geometric programming technique taking $\frac{\alpha+1-\beta}{2}$ = u (>0) as parameter, the optimal values of decision variables and objective functions are given in Table-3.

Generalized intuitionistic fuzzy goal geometric programming:

$$\mu_{cost}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} w, & Cost(x_1, x_2, x_3, x_4) \leq 300 \\ w\left(1-\frac{Cost(x_1,x_2,x_3,x_4)-300}{200}\right), & 300 \leq Cost(x_1, x_2, x_3, x_4) \leq 500 \\ 0, & Cost(x_1, x_2, x_3, x_4) \geq 500 \end{array}\right\},$$

$$\vartheta_{cost}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} 0, & Cost(x_1, x_2, x_3, x_4) \leq 300 \\ w^0\left(\frac{Cost(x_1,x_2,x_3,x_4)-300}{300}\right), & 300 \leq Cost(x_1, x_2, x_3, x_4) \leq 600 \\ w^0, & Cost(x_1, x_2, x_3, x_4) \geq 600 \end{array}\right\},$$

$$\mu_{BOD}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} w, & BOD(x_1, x_2, x_3, x_4) \leq 0.012 \\ w\left(1-\frac{BOD(x_1,x_2,x_3,x_4)-0.012}{0.2}\right), & 0.012 \leq BOD(x_1, x_2, x_3, x_4) \leq 0.212 \\ 0, & BOD(x_1, x_2, x_3, x_4) \geq 0.212 \end{array}\right\},$$

and

$$\vartheta_{BOD}(x_1, x_2, x_3, x_4)=\left\{\begin{array}{cc} 0, & BOD(x_1, x_2, x_3, x_4) \leq 0.012 \\ w^0\left(\frac{BOD(x_1,x_2,x_3,x_4)-0.012}{0.4}\right), & 0.012 \leq BOD(x_1, x_2, x_3, x_4) \leq 0.412 \\ w^0, & BOD(x_1, x_2, x_3, x_4) \geq 0.412 \end{array}\right\}.$$

The generalized intuitionistic fuzzy goal programming model can be transformed into a geometric programming model following (2.2), (2.4) and (2.5) as

$$\left.\begin{aligned}
&\text{Minimize } \left(\frac{\alpha+1-\beta}{2}\right)^{-1}\\
&\text{subject to } \frac{1}{\left(1+\frac{300}{2}\left(\frac{w}{200}+\frac{w^0}{300}\right)-\frac{\alpha+1-\beta}{2}\right)}\Big[\frac{19.4}{2}\left(\frac{w}{200}+\frac{w^0}{300}\right)x_1^{-1.47}+\frac{16.8}{2}\left(\frac{w}{200}+\frac{w^0}{300}\right)x_2^{-1.66}+\\
&\quad\frac{91.5}{2}\left(\frac{w}{200}+\frac{w^0}{300}\right)x_3^{-0.3}+\frac{120}{2}\left(\frac{w}{200}+\frac{w^0}{300}\right)x_4^{-0.33}\Big]\le 1,\\
&\quad\frac{1}{\left(1+\frac{0.012}{2}\left(\frac{w}{0.2}+\frac{w^0}{0.4}\right)-\frac{\alpha+1-\beta}{2}\right)}\left(\frac{1}{2}\left(\frac{w}{0.2}+\frac{w^0}{0.4}\right)x_1x_2x_3x_4\right)\le 1\\
&\quad x_1,x_2,x_3,x_4>0,\ \alpha+\beta\le 1,\ 0\le\alpha,\beta\le 1.
\end{aligned}\right\} \quad (4.2)$$

Applying geometric programming technique taking $\frac{\alpha+1-\beta}{2}$ = u (> 0) as parameter, following table shows optimal values of decision variables and objective functions.

Table 3. Optimal values of decision variables,
total annual cost and remaining BOD_5 in wastewater

Met hods	$\boldsymbol{w}$, $\boldsymbol{w^0}$	**Primal Variables**	**Optimal objective functions**	$\boldsymbol{\alpha^*}$, $\boldsymbol{\beta^*}$	**Membership and Non-membership**	**Sum of Membership and Non-membership**
IF G^2P^2		x_1*= 0.727312 x_2*= 0.744224 x_3*= 0.185002 x_4*= 0.654756	*Cost** (x_1,x_2,x_3,x_4) =346.4679 (thousand $) *BOD** (x_1,x_2,x_3,x_4) =0.06556666	$\boldsymbol{\alpha^*}\in$ [0,0.7991250] $\boldsymbol{\beta^*}\in$ [0,0.200875]	$\mu_{cost}(x_1,x_2,x_3,x_4)$ =0.7676605 $\vartheta_{cost}(x_1,x_2,x_3,x_4)$ =0.1548930 $\mu_{BOD}(x_1,x_2,x_3,x_4)$ =0.7321667 $\vartheta_{BOD}(x_1,x_2,x_3,x_4)$ =0.1339166	$\mu_{cost}(x_1,x_2,x_3,x_4)$ + $\vartheta_{cost}(x_1,x_2,x_3,x_4)$ =0.9225535 $\mu_{BOD}(x_1,x_2,x_3,x_4)$ + $\vartheta_{BOD}(x_1,x_2,x_3,x_4)$ =0.8660834
GIF G^2P^2	$\boldsymbol{w}$= 0.98 $\boldsymbol{w^0}$ = 0.012	x_1*= 0.644288 x_2*= 0.668479 x_3*= 0.102149 x_4*= 0.381581	*Cost** (x_1,x_2,x_3,x_4) =416.1249 (thousand $) *BOD** (x_1,x_2,x_3,x_4) =0.01678772	$\boldsymbol{\alpha^*}\in$ [0,0.7131579] $\boldsymbol{\beta^*}\in$ [0,0.2868421]	$\mu_{cost}(x_1,x_2,x_3,x_4)$ =0.4193755 $\vartheta_{cost}(x_1,x_2,x_3,x_4)$ =0.3870830 $\mu_{BOD}(x_1,x_2,x_3,x_4)$ =0.9760614 $\vartheta_{BOD}(x_1,x_2,x_3,x_4)$ =0.01196930	$\mu_{cost}(x_1,x_2,x_3,x_4)$ + $\vartheta_{cost}(x_1,x_2,x_3,x_4)$ =0.8064585 $\mu_{BOD}(x_1,x_2,x_3,x_4)$ + $\vartheta_{BOD}(x_1,x_2,x_3,x_4)$ =0.9880307

Table 3 shows that membership and non-membership functions satisfy all the restrictions as in model (2.2) and Theorem 2.2. The percentage of BOD_5 removed from the wastewater in IFG^2P^2 is (100 – 0.06556666×100) = 93.443334% which doesn't attain the set quota by the national standard and the annual total cost is 346.4679 (thousands $), which is very near to the set target of DM. So, if DM gives emphasize on minimum annual cost then this method can be used. Also the remaining percentage of BOD_5 in wastewater in $GIFG^2P^2$ is 0.01678772×100 = 1.678772% and the annual total cost is 416.1249 (thousands $). Comparing with IFG^2P^2 and Cao's [15] fuzzy geometric programming process, $GIFG^2P^2$ gives better result as we can remove more BOD_5 from wastewater. In this process, we are able to remove 98.321228% BOD_5, which attains the set quota by the national standard.

5 Conclusions

Here we have considered a non-linear goal programming problem and solved it using Cao's method of geometric programming. The main objective of this work is to describe goal geometric programming in intuitionistic fuzzy environment. Also in a more general case, generalized intuitionistic fuzzy set is used in this paper to describe goal geometric programming. A real life problem on industrial wastewater treatment is illustrated here as an application. We have used here linear membership and non-membership functions. Also non-linear, exponential, parabolic membership and non-membership functions can be used in intuitionistic fuzzy goal geometric programming problem (IFG^2P^2).

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