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Measures of Contradiction for Intuitionistic Fuzzy Sets and for Fuzzy Classifications

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Abstract: Let us denote by $u = (u_1, \ldots, u_n)$ a vector of membership degrees (compatibility degrees) of a subject or an element x from the finite set X to one of n classes. Let it hold $0 \leq u_i \leq 1$ for all i, another restrictions such as $\sum_i u_i = 1$ are not given. In this case the vector x contains inherently not only the uncertainty about the possible final crisp classification of the element x into one of n classes but also, to some extent, a contradiction among its components. Both categories - uncertainty and contradiction - should be treated separately. In the article properties of the measure of contradiction are studied and formulas for the evaluation of the contradiction for a given vector u are proposed. The case $n = 2$ is closely related to intuitionistic fuzzy sets [1] and to the intuitionistic fuzzy logic [2].
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Introduction

As the result of the fuzzy partition procedure [3] we get for each element x of a finite universum X a vector $u = (u_1, \ldots, u_n)$ where u_i denotes the membership degree (compatibility degree) of x to a class i, $1 \leq i \leq n$. It is supposed that $0 \leq u_i \leq 1$ for all i and $\sum_i u_i = 1$. By normalizing an arbitrary vector u', the latter condition $\sum_i u_i = 1$ can be always attained. Doing this, we are eliminating a possible contradiction which may exist in the original n-tuple $(u'_1, \ldots, u'_n)$ before the normalizing operation. Hence the condition $\sum_i u_i = 1$ will not be considered in the sequel.

Measures of Contradiction

The measure of contradiction of a vector u will be formalized as a function denoted by C which fulfils some properties. First let us introduce some notations. $u = (u_1, \ldots, u_n)$ denotes a vector represented by elements from $[0,1]^n$. Let us denote for short $\sum_i^n u_i = S(u_1, \ldots, u_n) = S(u)$ and s denotes the value of S for some u.

C is defined on $[0,1]^n$ with values in $\mathcal{R}^+ = \{y \in \mathcal{R} : y \geq 0\}$. C fulfils the following properties:

P1. C is continuous on $[0,1]^n$,

P2. C is symmetric in its arguments,

P3. $C^{n+1}(u_1, \ldots, u_n, 0) = C^n(u_1, \ldots, u_n)$,

P4. $C(u_1, \ldots, u_n) \leq C(1, \ldots, 1)$,

P5. $C(u_1, \ldots, u_n) \geq C(1, 0, \ldots, 0)$,

P6. $S(u^2) = S(u^1) \Rightarrow C(u^2) = C(u^1)$,

P7. $S(u^2) > S(u^1) \geq 1 \Rightarrow C(u^2) > C(u^1)$,

P8. $1 \geq S(u^2) \geq S(u^1) \Rightarrow C(u^2) \geq C(u^1)$.

It is necessary to distinguish between P7 and P8 for the values of $S(u)$ smaller or greater than 1 because writing simply $S(u^2) > S(u^1) \Rightarrow C(u^2) > C(u^1)$ is in contradiction with P5 as soon as $u^2 = (1, 0, \ldots, 0)$ and $S(u^1) < 1$. P8 together with P5 leads to the
Corollary $S(u) \leq 1 \Rightarrow C(u) = C(1, 0, \ldots, 0)$. It means $C(u)$ is constant for all u such that $S(u) \leq 1$. P4 is a direct consequence of P7 and P8.
Theorem 1: C has the following form

$$C(u) = \begin{cases} f(S(u)) & \ldots \quad S(u) \geq 1 \\ f(1) & \ldots \quad S(u) \leq 1 \end{cases} \tag{1}$$

where f is a continuous, strictly monotone increasing function on $[1, n]$, $\quad f : [1, n] \to \mathcal{R}^+$.
Proof (a) Let C be given as in (1). Then P1-P8 can be easily verified.

(b) Let C has properties P1-P8. f can be constructed as follows. For $s \in [1, 2]$ we put $f(s) = C(1, s-1, 0, \ldots, 0)$. By P6 for $s = S(u)$ we can write $C(x) = C(u_1, \ldots, u_n) = C(1, s-1, 0, \ldots, 0) = f(s) = f(S(u))$, hence the desired form of (1). f is by P7 strictly monotone increasing on $[1, 2]$. f is by P1 also continuous on $[1, 2]$. Similarly for $s \in [2, 3], \ldots, s \in [n-1, n]$. For $s \in [0, 1]$ according to Corollary 1 we have $C(u) = C(1, 0, \ldots, 0)$ and $C(1, 0, \ldots, 0) = f(1)$ as already defined for the case $s \in [1, 2]$. Hence we have $C(u) = f(1)$ and this is the form (1). QED.

Hence choosing f as identity mapping the simlest form of C is

$$C(u) = \begin{cases} S(u) = \sum u_i & \ldots \quad S(u) \geq 1 \\ 1 & \ldots \quad S(u) \leq 1 \end{cases} \tag{2}$$

or

$$C(u) = \begin{cases} \sum u_i - 1 & \ldots \quad S(u) \geq 1 \\ 0 & \ldots \quad S(u) \leq 1 \end{cases} \tag{3}$$

where (3) seems to be more natural because of its value 0 for $C(1, 0, \ldots, 0)$.

The above properties P1-P8 are related to such a kind of crisp classification, for which it holds that the classified element does not need to belong to any of the n classes ,i.e., $u = (0, \ldots, 0)$ does not mean any contradiction.

Another possibility is the crisp classification, for which it holds that the classified element must necessarily belong to one of the n classes. If not then we have a contradiction. In this case $u = (0, \ldots, 0)$ means the same contradiction as $u = (1, \ldots, 1)$. A generalization for the fuzzy case is as follows: the vector u such that $S(u) = 1$ does not mean any contradiction, any deviation from this state leads to a contradiction.

In the case of intuitionistic fuzzy sets we have $n = 2$ properties which are mutually complementary. Hence it means that any deviation from the state $S(u) = 1$ leads to a contradiction.

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For the second type of the classification some of the Properties P1-P8 must be changed.

P1. C is continuous on $[0,1]^n$.

P2. C is symmetric in its arguments.

The extension of $u = (u_1, \ldots, u_n)$ by a component $u_{n+1} = 0$ did not mean by the former P3 any further contradiction because the classified element did not need to belong to any of the classes. This does not hold any more for the second type of the classification. Instead of P3 we write a new property

P3' $C(0,0,\ldots,0) = C(1,1,\ldots,1)$.

P4. $C(u_1,\ldots,u_n) \leq C(1,\ldots,1)$.

P5. $C(u_1,\ldots,u_n) \geq C(1,0,\ldots,0)$.

P6. $S(u^2) = S(u^1) \Rightarrow C(u^2) = C(u^1)$.

P7. $S(u^2) > S(u^1) \geq 1 \Rightarrow C(u^2) > C(u^1)$.

P8'. $1 \geq S(u^2) > S(u^1) \Rightarrow C(u^2) < C(u^1)$.

Theorem 2: C has the following form

$$C(u) = \begin{cases} f(S(u)) & \ldots \quad S(u) \geq 1 \\ g(S(u)) & \ldots \quad S(u) \leq 1 \end{cases} \tag{4}$$

where f is a continuous, strictly monotone increasing function on $[1,n]$, g is a continuous, strictly monotone decreasing function on $[0,1]$, $f(1) = g(1)$ and $f(n) = g(0)$.

The proof is analog to Theorem 1.

A simple form for C is

$$C(u) = \begin{cases} S(u) & \ldots \quad 1 \leq S(u) \leq n \\ -(n-1)S(u) + n & \ldots \quad 0 \leq S(u) \leq 1 \end{cases} \tag{5}$$

or

$$C(u) = \begin{cases} S(u) - 1 & \ldots \quad 1 \leq S(u) \leq n \\ -(n-1)S(u) + (n-1) & \ldots \quad 0 \leq S(u) \leq 1 \end{cases} \tag{6}$$

In the case of $n = 2$ (4) and (5) are

$$C(u) = \begin{cases} u_1 + u_2 & \ldots \quad 1 \leq u_1 + u_2 \leq 2 \\ -u_1 - u_2 + 2 & \ldots \quad 0 \leq u_1 + u_2 \leq 1 \end{cases} \tag{7}$$

or

$$C(u) = \begin{cases} u_1 + u_2 - 1 & \ldots \quad 1 \leq u_1 + u_2 \leq 2 \\ 1 - u_1 - u_2 & \ldots \quad 0 \leq u_1 + u_2 \leq 1 \end{cases} \tag{8}$$

In the case of an intuitionistic fuzzy set A u_1 and u_2 are denoted as $u_1 = \mu_A(x)$, $u_2 = \nu_A(x)$, $x \in X$, cf. [4] and it holds $\mu_A(x) + \nu_A(x) \leq 1$. According to (8) we can write $C(u^x) = 1 - \mu_A(x) - \nu_A(x)$. We see that $C(u^x)$ is identical with the intuitionistic index of the element

x. Hence we can understand the intuitionistic index actually as a measure of contradiction. For $\mu_A(x) = \nu_A(x) = 0$ we get the maximal contradiction.

$C(u^x)$ is a measure of contradiction for an element $x \in X$, we can also speak about a local measure of contradiction. If the contradiction for the whole set X is considered, we need a global measure of contradiction. In analogy with local and global measures of uncertainty (entropy) c.f. [5,6], global measures can be constructed from local measures e.g. by adding the values $C(u^x)$ over all $x \in X$.

Contradiction Measures and Intuitionistic Fuzzy Logic

A valuation function V assigns to each propositional form A a pair $< \mu(A), \gamma(A) >$, $0 \leq \mu(A), \gamma(A) \leq 1$, $\mu(A) + \gamma(A) \leq 1$. The valuations of the operations $\neg A, A \wedge B, A \vee B$ are given by valuations of their constituents cf. [2]. Using the results of Sec.2 we can define the contradiction in A by $C(A) = 1 - \mu(A) - \gamma(A)$. Using the valuations for $\neg A, A \wedge B, A \vee B$ as in [2], we can derive

$$\begin{array}{lll} C(A \vee T) = 0, & C(A \vee F) = C(A), & C(A) = C(\neg A), \\ C(A \wedge T) = C(A), & C(A \wedge F) = 0, & \end{array} \tag{9}$$

For a set of propositional forms $\{A_1, \ldots, A_m\}$ let A_L be the A_i with the lowest value of γ and A_H be the A_i with the highest value of μ. Then we have

$$C(A_L) \geq C(A_1 \vee \ldots \vee A_m) \geq C(A_H). \tag{10}$$

Let A'_L be the A_i with the lowest value of μ and A'_H be the A_i with the highest value of γ. Then we have

$$C(A'_L) \geq C(A_1 \wedge \ldots \wedge A_m) \geq C(A'_H). \tag{11}$$

From (10) and (11) we have

$$\max_i C(A_i) \geq C(A_1 \vee \ldots \vee A_m), C(A_1 \wedge \ldots \wedge A_m) \geq \min_i C(A_i). \tag{12}$$

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