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Some new equalities connected with intuitionistic fuzzy sets

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Abstract: In this paper, some new equalities connected with intuitionistic fuzzy sets based on operations (denoted by $\cup, \cap, \cdot, +, *, @, \$, \#$) are proved.

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1 Introduction

The notion of intuitionistic fuzzy sets was introduced by Atanassov [1, 2], as a generalization of the concept of fuzzy sets [3]. Intuitionistic fuzzy sets are characterized by two functions expressing the degree of membership and the degree of non-membership respectively. In this paper, we prove some new equalities connected with IFSs based on operations (denoted by $\cup, \cap, \cdot, +, *, @, \$, \#$).

The paper is organized as follows: In Section 2, some basic definitions related to intuitionistic fuzzy set theory are presented. In Section 3, new equalities connected with IFSs are proved.

2 Preliminaries

Definition 1 (*Intuitionistic Fuzzy Set*): An intuitionistic fuzzy set A [1] defined on a universe of discourse X is mathematically represented as

$$A = \left\{ \left\langle x,\ \mu_A(x), \nu_A(x) \right\rangle \middle| x \in X \right\}, \tag{1}$$

where functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, and for every $x \in X$, such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1. \qquad (2)$$

Furthermore, we call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),\ x \in X$, the intuitionistic index or hesitancy degree of x in A. It is obvious that $0 \le \pi_A(x) \le 1$, for each $x \in X$.

Definition 2 (*Set operations on IFSs*): Let *IFS*(*X*) denote the family of all IFSs in the universe *X*, assume *A*, *B* ∈ *IFS*(*X*) given as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$
$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\},$$

then some operations defined as follows:

(i) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$;

(ii) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$;

(iii) $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cdot B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)\rangle \mid x \in X\}$;

(v) $A @ B = \left\{\left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\right\rangle \mid x \in X\right\}$;

(vi) $A \$ B = \left\{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)}\right\rangle \mid x \in X\right\}$;

(vii) $A \# B = \left\{\left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)}\right\rangle \mid x \in X\right\}$ for which we shall accept that if $\mu_A(x) = \mu_B(x) = 0$, then $\frac{\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)} = 0$ and if $\nu_A(x) = \nu_B(x) = 0$, then $\frac{\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} = 0$;

(viii) $A * B = \left\{\left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2(\mu_A(x)\mu_B(x) + 1)}, \frac{\nu_A(x) + \nu_B(x)}{2(\nu_A(x)\nu_B(x) + 1)}\right\rangle \mid x \in X\right\}$.

In the next section, we formulate and prove some new equalities connected with operations on IFSs.

3 New equalities connected with IFSs

Theorem 1: For A, $B \in IFS(X)$, it holds that

$$(A\# B)\$(A\# B) = (A\# B).$$

Proof:

$$(A\# B)\$(A\# B)$$

$$= \left\{\left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x \in X\right\} \$ \left\{\left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x \in X\right\}$$

$$= \left\{\left\langle x, \sqrt{\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}}, \sqrt{\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}}\right\rangle \mid x \in X\right\}$$

$$= \left\{\left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x \in X\right\}$$

$$= (A\# B).$$

This proves the result. □

Theorem 2: For A, $B \in IFS(X)$, it holds that

$$(A+B)\$(A+B) = (A+B).$$

Proof:

$$(A+B)\$(A+B)$$

$$= \{\langle x, \sqrt{(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x))(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x))}\rangle \mid x \in X\}$$

$$\$ \{\langle x, \sqrt{(\nu_A(x)\nu_B(x))(\nu_A(x)\nu_B(x))}\rangle \mid x \in X\}$$

$$= \{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x \in X\}$$

$$= (A+B).$$

This proves the result. □

Theorem 3: For A, $B \in IFS(X)$, it holds that

$$(A\cdot B)\$(A\cdot B) = (A\cdot B).$$

Proof:

$$(A\cdot B)\$(A\cdot B)$$

$$= \{\langle x, \sqrt{(\mu_A(x)\mu_B(x))(\mu_A(x)\mu_B(x))}\rangle \mid x \in X\}$$

$$\$ \{\langle x, \sqrt{(\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x))(\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x))}\rangle \mid x \in X\}$$

$$= \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)\rangle \mid x \in X\}$$

$$= (A+B).$$

This proves the result. □

Theorem 4: For A, $B \in IFS(X)$, it holds that,

$$(A@B)\$(A@B) = (A@B).$$

Proof:

$$(A@B)\$(A@B)$$

$$= \left\{\left\langle x, \sqrt{\left(\frac{\mu_A(x)+\mu_B(x)}{2}\right)\left(\frac{\mu_A(x)+\mu_B(x)}{2}\right)}\right\rangle \mid x \in X\right\}$$

$$\$ \left\{\left\langle x, \sqrt{\left(\frac{\nu_A(x)+\nu_B(x)}{2}\right)\left(\frac{\nu_A(x)+\nu_B(x)}{2}\right)}\right\rangle \mid x \in X\right\}$$

$$= \left\{\left\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\right\rangle \mid x \in X\right\}$$

$$= (A@B).$$

This proves the result. □

Theorem 5: For A, $B \in IFS(X)$, it holds that

$$((A\#B)\$(A\#B))\$((A@B)\$(A@B)) = (A\$B).$$

Proof:

$$((A\#B)\$(A\#B)) = \left\{\left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x \in X\right\}, \quad (3)$$

and

$$((A@B)\$(A@B)) = \left\{\left\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\right\rangle \mid x \in X\right\}. \quad (4)$$

Now with \$ of (3) and (4), we have

$$((A\#B)\$(A\#B))\$((A@B)\$(A@B))$$

$$= \left\{\left\langle x, \sqrt{2\frac{\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}\frac{\mu_A(x)+\mu_B(x)}{2}}, \sqrt{2\frac{\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\frac{\nu_A(x)+\nu_B(x)}{2}}\right\rangle \mid x \in X\right\}$$

$$= \left\{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)}\right\rangle \mid x \in X\right\}$$

$$= (A\$B).$$

This proves the result. □

Theorem 6: For A, $B \in IFS(X)$, it holds that

$$((A+B)\$(A+B))@((A\cdot B)\$(A\cdot B)) = (A@B).$$

Proof:

$$((A+B)\$(A+B)) = \{\langle x, (\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)), \nu_A(x)\nu_B(x)\rangle \mid x \in X\} \quad (5)$$

and

$$((A\cdot B)\$(A\cdot B))=\{\langle x,\mu_A(x)\mu_B(x),\nu_A(x)+\nu_B(x)\nu_A(x)\nu_B(x)\rangle \mid x\in X\} \tag{6}$$

Now with @ of (5) and (6),

$$((A+B)\$(A+B))@((A\cdot B)\$(A\cdot B))$$

$$=\left\{\left\langle x,\frac{\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)+\mu_A(x)\mu_B(x)}{2},\frac{\nu_A(x)\nu_B(x)+\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)}{2}\right\rangle \mid x\in X\right\}$$

$$=\left\{\left\langle x,\frac{\mu_A(x)+\mu_B(x)}{2},\frac{\nu_A(x)+\nu_B(x)}{2}\right\rangle \mid x\in X\right\}$$

$$=(A@B).$$

This proves the result. □

Theorem 7: For *A*, *B* ∈ *IFS*(*X*), it holds that

$$(A\cup B)\#(A\cap B)=(A\# B)\$(A\# B).$$

Proof:

$$(A\cup B)\#(A\cap B)$$

$$=\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\nu_A(x),\nu_B(x))\rangle \mid x\in X\}$$

$$\#\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle \mid x\in X\}$$

$$=\left\{\left\langle x,\frac{2\max(\mu_A(x),\mu_B(x))\min(\mu_A(x),\mu_B(x))}{\max(\mu_A(x),\mu_B(x))+\min(\mu_A(x),\mu_B(x))},\frac{2\min(\nu_A(x),\nu_B(x))\max(\nu_A(x),\nu_B(x))}{\min(\nu_A(x),\nu_B(x))+\max(\nu_A(x),\nu_B(x))}\right\rangle \mid x\in X\right\} \tag{7}$$

$$=\left\{\left\langle x,\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)},\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x\in X\right\}$$

and

$$(A\# B)\$(A\# B)$$

$$=\left\{\left\langle x,\sqrt{\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)}},\sqrt{\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}}\right\rangle \mid x\in X\right\} \tag{8}$$

$$=\left\{\left\langle x,\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)},\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)}\right\rangle \mid x\in X\right\}.$$

From (7) and (8) we get the result. □

Theorem 8: For *A*, *B* ∈ *IFS*(*X*), it holds that

$$(A\cup B)\#(A\cap B)=(A\# B)@(A\# B).$$

Proof:

$$(A\cup B)\#(A\cap B)$$

$$= \left(\begin{array}{l} \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in X\} \\ \qquad\qquad \# \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in X\} \end{array} \right)$$

$$= \left\{ \left\langle x, \frac{2\max(\mu_A(x), \mu_B(x))\min(\mu_A(x), \mu_B(x))}{\max(\mu_A(x), \mu_B(x)) + \min(\mu_A(x), \mu_B(x))}, \frac{2\min(\nu_A(x), \nu_B(x))\max(\nu_A(x), \nu_B(x))}{\min(\nu_A(x), \nu_B(x)) + \max(\nu_A(x), \nu_B(x))} \right\rangle \mid x \in X \right\} \quad (9)$$

$$= \left\{ \left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} \right\rangle \mid x \in X \right\}$$

and

$$(A \# B) @ (A \# B)$$

$$= \left\{ \left\langle x, \frac{\frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)} + \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}}{2}, \frac{\frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} + \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)}}{2} \right\rangle \mid x \in X \right\} \quad (10)$$

$$= \left\{ \left\langle x, \frac{2\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \frac{2\nu_A(x)\nu_B(x)}{\nu_A(x) + \nu_B(x)} \right\rangle \mid x \in X \right\}.$$

From (9) and (10) we get the result. □

Corollary 1: For A, $B \in IFS(X)$, it holds that

$$(A \cup B) \# (A \cap B) = (A \# B) \$ (A \# B) = (A \# B) @ (A \# B) = (A \# B).$$

Proof: It obvious follows Theorems 7 and 8. □

References

[1] Atanassov, K. Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, Vol. 20, 1986, No. 1, 87–96.

[2] Atanassov, K. *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer Physica-Verlag, Heidelberg, 1999.

[3] Zadeh, L. Fuzzy sets, *Information and Control*, Vol. 8, 1965, No. 3, 338–353.