Eleventh Int. Conf. on IFSs, Sofia, 28-30 April 2007
NIFS 13 (2007), 2, 23-29

On the probability theory on IF-events

Beloslav Riečan
Faculty of Natural Sciences, Matej Bel University
Department of Mathematics
Tajovského 40
974 01 Banská Bystrica, Slovakia
riecan@fpv.umb.sk
and
Mathematical Institute of Slovak Acad. of Sciences
Štefánikova 49
SK-81473 Bratislava

April 18, 2007

Abstract

From a general point of view of t-norms and t-conorms there are considered some possibilities for the probability theory on IF-events.
Keywords: IF-events, t-norms, probability.

1 Introduction

In the paper we discuss main notions of probability theory (probability, observable) from the point of view of two modern concepts of fuzzy sets theory with valuable applications: IF-sets theory [1] and t-norms theory [3].

Let $(\Omega, \mathcal{S})$ be a measurable space. By an IF-event we mean any pair

$$A = (\mu_A, \nu_A)$$

of $\mathcal{S}$-measurable functions, such that $\mu_A \geq 0, \nu_A \geq 0$, and

$$\mu_A + \nu_A \leq 1.$$

An important notion is the ordering

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Denote by $\mathcal{F}$ the set of all IF-events with respect to $(\Omega, \mathcal{S})$.

Recall that a t-norm is a commutative and associative binary operation T on $[0,1]$ non-decreasing in any variable and such that $T(u,1) = u$ for any $u \in [0,1]$. Similarly a t-conorm is a commutative and associative binary operation S on $[0,1]$ non-decreasing in any variable and such that $S(u,0) = u$ for any $u \in [0,1]$

We shall consider pairs (S,T) of dual t-conorms and t-norms, i.e.

$$S(u,v) = 1 - T(1-u, 1-v)$$

for any $u, v \in [0,1]$. Moreover we shall assume that $S, T : [0,1] \times [0,1] \to [0,1]$ are Borel measurable functions. We shall use also the notation $uSu = S(u,v), uTv = T(u,v)$.

In the probability theory on IF-events still the most frequent pairs (S,T) were the following dual pairs $(S_L, T_L), (S_M, T_M)$

$$S_L(u,v) = \min(u+v, 1), T_L(u,v) = \max(u+v-1, 0),$$

$$S_M(u,v) = \max(u,v), T_M(u,v) = \min(u,v).$$

Of course, for any dual pair (S,T) we can define the correspoding operations on $\mathcal{F}$. Let $a, b \in \mathcal{F}, a = (\mu_a, \nu_a), b = (\mu_b, \nu_b)$. Then

$$S(a,b) = (\mu_a S \mu_b, \nu_a T \nu_b),$$

$$T(a,b) = (\mu_a T \mu_b, \nu_a S \nu_b).$$

It is easy to see that $S(a,b) \in \mathcal{F}$ because of inequalities $\nu_a \leq 1 - \mu_a, \nu_b \leq 1 - \mu_b$:

$$S(\mu_a, \mu_b) + T(\nu_a, \nu_b) =$$

$$= 1 - T(1-\mu_a, 1-\mu_b) + T(\nu_a, \nu_b) \leq$$

$$\leq 1 - T(1-\mu_a, 1-\mu_b) + T(1-\mu_a, 1-\mu_b) = 1.$$

Therefore it is possible to define a probability and an observable with respect to the pair (S,T). The first problem is realized in Section 2, the second in Section 3. In Section 4 we consider the probability distribution and in Section 5 we propose some directions for further research.

2 Probability

1. Definition. Denote by $\mathcal{F}$ the set of all IF-events, by $\mathcal{J}$ the family of all compact intervals on R. A probability is a mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following properties:

(I) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1], \mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$;

(II) $aTb = (0_\Omega, 1_\Omega) \Longrightarrow \mathcal{P}(a) + \mathcal{P}(b) = \mathcal{P}(aSb)$ for all $a, b \in \mathcal{F}$;

(III) $a_n \nearrow a \Longrightarrow \mathcal{P}(a_n) \nearrow \mathcal{P}(a)$.

Here $a_n \nearrow a$ means that $\mu_{a_n} \nearrow \mu_a, \nu_{a_n} \searrow \nu_a$. Of course, $\mathcal{P}(a_n) = [\mathcal{P}^\flat(a_n), \mathcal{P}^\sharp(a_n)] \nearrow \mathcal{P}(a) = [\mathcal{P}^\flat(a), \mathcal{P}^\sharp(a)]$ means that $\mathcal{P}^\flat(a_n) \nearrow \mathcal{P}^\flat(a), \mathcal{P}^\sharp(a_n) \nearrow \mathcal{P}^\sharp(a)$.

2. Definition. A mapping $m : \mathcal{F} \to [0,1]$ is called a state, if the following properties are satisfied:

(i) $m((1_\Omega, 0_\Omega)) = 1, m((0_\Omega, 1_\Omega)) = 0$;

(ii) $aTb = (0_\Omega, 1_\Omega) \Longrightarrow m(a) + m(b) = m(aSb)$ for all $a, b \in \mathcal{F}$

(iii)$a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

3. Proposition. If $\mathcal{P} = [\mathcal{P}^\flat, \mathcal{P}^\sharp] : \mathcal{F} \to \mathcal{J}$ is a probability, then $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ are states.

Since any probability on $\mathcal{F}$ can be decomposed into two states, in the paper we shall study only states $m : \mathcal{F} \to [0,1]$. Moreover, since the properties (i) and (iii) does not depend on the pair (S,T), we shall be concentrated to the additivity (ii) only.

4. Example. Consider the Lukasiewicz connectives (S_L, T_L). The review of the corresponding theory will be published in [7]. The additivity is defined by the following way:

$$(w)\ a \odot b = (0_\Omega, 1_\Omega) \Longrightarrow m(a \oplus b) = m(a) + m(b).$$

As a consequence of the representation theorem [6] one can prove the equivalence (w) and

$$(s)\ m(a) + m(b) = m(a \odot b) + m(a \oplus b), a, b \in \mathcal{F}.$$

Indeed, by[6] to any state satisfying (w) there exists a probability measure $P : \Omega \to [0,1]$ and $\alpha \in [0,1]$ such that

$$m(a) = (1-\alpha)\int_\Omega \mu_a dP + \alpha(1 - \int_\Omega \nu_a dP).$$

From this expression (s) can be derived immediately.

Recall that the family $\mathcal{F}$ in the Lukasiewicz case can be imbedded to the well developed MV-algebras probability theory ([9],[10]).

5. Example. M. Krachounov [4] used the pair (S_M, T_M). His additivity condition is defined by the following way:

$$m(a) + m(b) = m(a \vee b) + m(a \wedge b), a, b \in \mathcal{F}.$$

Evidently this property implies the condition (ii).

3 Observable

6. Definition. An observable is a mapping $x : \mathcal{B}(R) \to \mathcal{F}(\mathcal{B}(R)$ being the σ-algebra of subsets of R) satisfying the following conditions:

$$1. \quad x(R) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega).$$

$$2. \quad A \cap B = \emptyset \Longrightarrow x(A)Tx(B) = (0_\Omega, 1_\Omega),$$

$$x(A \cup B) = x(A)Sx(B).$$

$$3. \quad A_n \searrow \emptyset \Longrightarrow x(A_n) \searrow (0_\Omega, 1_\Omega).$$

7. Example. Consider again the pair (S_L, T_L). In [7] the additivity is defined by the following way:

$$A \cap B = \emptyset \Longrightarrow x(A) \odot x(B) = (0_\Omega, 1_\Omega)$$

$$x(A \cup B) = x(A) \oplus x(B).$$

Evidently it is a special case on the point 2 of Definition 6.

8. Example. In [7] the pair (S_M, T_M) has been considered and the additivity defined by the following way: We shall use the following connectives for $a, b \in R$:

$$x(A \cup B) = x(A) \vee x(B), x(A \cap B) = x(A) \wedge x(B).$$

We shall prove that this definition implies the point 2 of definition 6. Indeed, if $A \cap B = \emptyset$, then $x(A)Tx(B) = x(A) \wedge x(B) = x(A \cap B) = x(\emptyset) = (0_\Omega, 1_\Omega)$, and

$$x(A)Sx(B) = x(A) \vee x(B) = x(A \cup b).$$

It seems that a valuable mean in the max - min probability theory should be the existence of the joint observable in this case. It is the main result of [7].

4 Probability distribution

9. Theorem. Let $(\Omega, \mathcal{S})$be a measurable space, $\mathcal{F}$ the family of all IF-events. Let $m : \mathcal{F} \to [0,1]$ be a state, $x : \mathcal{B}(R) \to \mathcal{F}$ be an observable. Then the composite map $p = m \circ x : \mathcal{B}(R) \to [0,1]$ is a probability measure.

Proof. Evidently

$$p(R) = m(x(R)) = m((1_\Omega, 0_\Omega)) = 1.$$

If $A \cap B = \emptyset$, then $x(A \cup B) = x(A)Sx(B)$. Moreover

$$x(A)Tx(B) = (0_\Omega, 1_\Omega).$$

Therefore by (ii) of definition 2

$$p(A \cup B) = m(x(A \cup B)) =$$

$$= m(x(A)Sx(B)) =$$

$$= m(x(A)) + m(x(B)) = p(A) + p(B).$$

Finally, let $A_n \searrow \emptyset$. Then $x(A_n) \searrow (0_\Omega, 1_\Omega)$, hence

$$p(A_n) = m(x(A_n)) \searrow m((0_\Omega, 1_\Omega)) = 0.$$

5 Conclusion

1. Since the special case (S_L, T_L) can be embedded to a suitable MV-algebra, it is reasonable to give an attention to the MV-algebra probability theory. In [9] there is a review of the theory. It could be developed further at least in two directions: a) conditional probability, b) entropy theory.

2. Although MV-algebra probability theory contains many valuable results and Lukasiewicz IF-probability theory can be considered as its special case, it is important to translate the general results to a simple language of IF sets for to be these results applied in some practical circumstances.

3. The max-min probability theory on IF sets has in this moment two useful means: a) the existence of the joint observable of any two observables ([7]); b) a local representation theorem of sequences of observables by sequences of random variables in the framework of the Kolmogorov probability

theory ([8]). Therefore a very effective way has been discovered for achieving new results in the max-min theory.

4. For any dual pair (S, T) the methods of the theory of independency ([9], Chapter 2) can be applied immediately. Possibly some interesting results could be obtained by this way.

5. It would be reasonable from our point of view study some connections with the Lendelová concept of probability theory on so-called L-posets [4].

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