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Intuitionistic fuzzy sets and interval valued fuzzy sets

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To Prof. Lotfi Zadeh

Abstract Some relationships and differences between two extensions of the fuzzy sets – intuitionistic fuzzy sets and interval valued fuzzy sets are discussed.

The advent of the concept of *"fuzzy set"* introduced by Lotfi Zadeh in 1965 is one of the most important events in the mathematics of the second half of twentieth century. It is not only an abstract mathematical object, extending J. Lukasiewicz's idea for 3- and n-valued logics, but is also during the last 30 years one of the most used mathematical concept in practice. For these reasons fuzzy sets are an object of different extensions and modifications. Two of them are the concepts of Intuitionistic Fuzzy Sets (IFSs) and Interval Valued Fuzzy Sets (IVFSs). Here we shall discuss some relationships and differences between both types of sets.

As it is noted in [3] IFSs and IVFSs are *aequipollens* concepts, because they have equal sense, but different in form. Really, if E is a fixed universe and set $A \subset E$ is given, both concepts are defined by:

$$IFS(A) = \{< x, \mu_A(x), \nu_A(x) >: x \in E\},$$

$$IVFS(A) = \{< x, M_A(x) >: x \in E\},$$

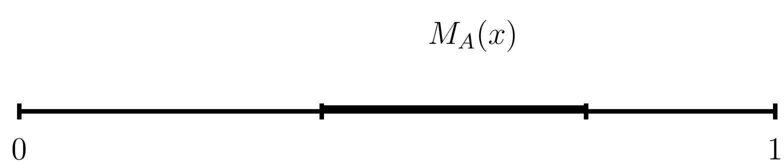
where $\mu_A, \nu_A : E \to [0,1]$ are functions determining the degrees of membership and of non-membership of element $x \in E$ to A and

$$\mu_A(x) + \nu_A(x) \leq 1; \qquad (*)$$

$M_A(x) \subset [0,1]$ is a closed interval, that contains the exact degree of

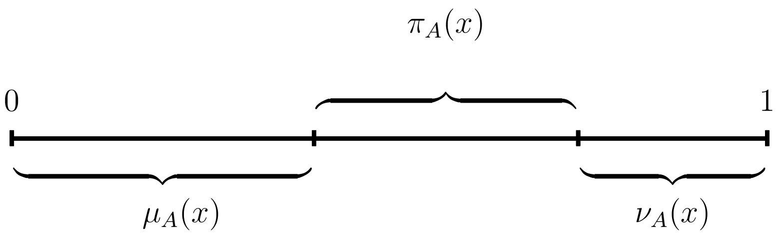
$$\mu_A(x) = \inf M_A(x) \quad \text{and} \quad \nu_A(x) = 1 - \sup M_A(x)$$

we can transform one of the sets to the contralateral. On the other hand, between both types of sets there are differences. One of the geometrical interpretations of IVFS is:



where the value of $\mu_A(x)$ is somewhere in the shown interval. The respective interpretation of the IFSs is

or



where $\pi_A(x)$ is the degree of uncertainty. The divergence in the definitions has as an effect of divergence in the definitions of the operations over both types of sets. However, as far as they are extensions of the fuzzy sets, the operations over both discussed sets are extensions of the operations over the fuzzy sets. The IFSs have different geometrical interpretations (see, e.g., [2, 5, 6, 7]). One of them is presented on Figure 1

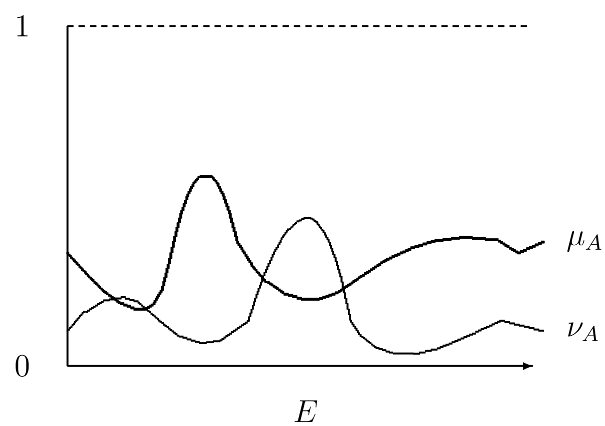


Figure 1

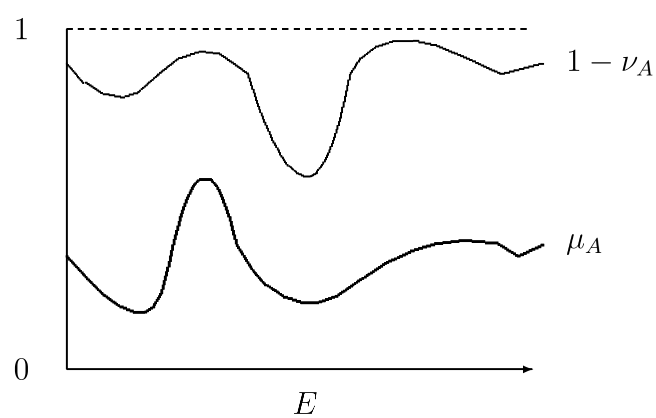


Figure 2

Its analogue is given in Figure 2.

Let a universe E be given. Consider the figure F (see Figure 3) in the Euclidean plane with a Cartesian coordinate system.

Let $A \subset E$ be a fixed set. Then we can construct a function f_A from E to F such that if $x \in E$, then $p(x) = f_A(x) \in F$. Point $p(x)$ has coordinates $\langle a, b \rangle$ for which: $0 \le a + b \le 1$ and these coordinates are such that $a = \mu_A(x), b = \nu_A(x)$.

The IFS-interpretation from Figure 3 has analogue neither in the ordinary fuzzy set theory nor in the theories of the fuzzy set extensions. The elements of a given fuzzy set are interpreted by points over the triangle hypotenuse, while the elements of a given IFS can be in each one point of the triangle.

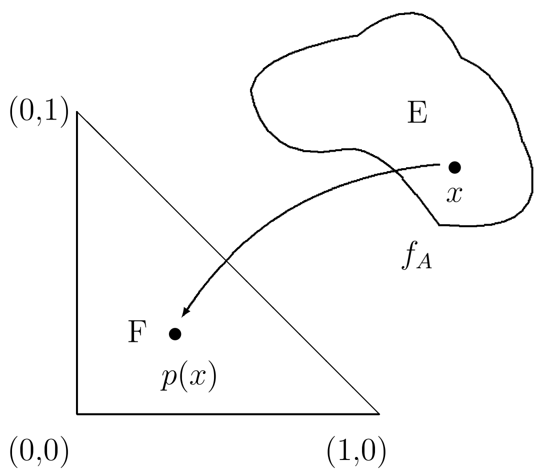


Figure 3

In general, these are the resemblances between IFSs and IVFSs. The differences between them are based mainly on the aims for defining of the concepts. The IVFSs are generated to target some problems of the interval analysis (mathematical area developed very actively in 70's). The IFSs are defined not only as extensions of the fuzzy sets, but also of the modal logic. Over the IFS (and only for them!) operators are defined, that are analogous of the modal logic ones *"necessity"* and *"possibility"*. These operators are defined by

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in E\};$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in E\}.$$

When A is a proper IFS, i.e, there is an element $x \in E$ for which $\mu_A(x) > 0$, then

$$\Box A \subset A \subset \Diamond A$$

and

$$\Box A \neq A \neq \Diamond A,$$

where

$$A \subset B \text{ if and only if } (\forall x \in E)(\mu_A(x) \le \mu_B(x) \;\&\; \nu_A(x) \ge \nu_B(x)).$$

On the other hand, for each fuzzy set, i.e., for which $\pi_A(x) = 0$ for all $x \in E$:

$$\Box A = A = \Diamond A.$$

Therefore, both operators lose their sense in ordinary fuzzy set theory. These two operators are extended to a lot of other modal-type operator defined over IFSs that have analogue neither in IVFS theory, nor in modal logic theory.

The first group of extended modal operators are the following:

$$\begin{aligned}
&D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\},\\
&F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1,\\
&G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},\\
&H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},\\
&H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},\\
&J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},\\
&J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\},
\end{aligned}$$

where $\alpha, \beta \in [0,1]$ are fixed numbers. These operators are extended to the operators

$$\begin{aligned}
F_B(A) &= \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},\\
G_B(A) &= \{\langle x, \mu_B(x).\mu_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},\\
H_B(A) &= \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},\\
H^*_B(A) &= \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).(1 - \mu_B(x).\mu_A(x)\\
&\qquad -\nu_A(x))\rangle \mid x \in E\},\\
J_B(A) &= \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},\\
J^*_B(A) &= \{\langle x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_B(x).\nu_A(x)),\\
&\qquad \nu_B(x).\nu_A(x)\rangle \mid x \in E\},
\end{aligned}$$

where B is a given IFS; and modified to operators

$$\begin{aligned}
d_\alpha(A) &= \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \mu_A(x) + (1-\alpha).\pi_A(x)\rangle \mid x \in E\},\\
f_{\alpha,\beta}(A) &= \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\},\\
&\qquad \text{where } \alpha + \beta \leq 1,\\
g_{\alpha,\beta}(A) &= \{\langle x, \alpha.\nu_A(x), \beta.\mu_A(x)\rangle \mid x \in E\},\\
h_{\alpha,\beta}(A) &= \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\},\\
h^*_{\alpha,\beta}(A) &= \{\langle x, \alpha.\nu_A(x), \mu_A(x) + \beta.(1 - \alpha.\nu_A(x) - \mu_A(x))\rangle \mid x \in E\},\\
j_{\alpha,\beta}(A) &= \{\langle x, \nu_A(x) + \alpha.\pi_A(x), \beta.\mu_A(x)\rangle \mid x \in E\},\\
j^*_{\alpha,\beta}(A) &= \{\langle x, \nu_A(x) + \alpha.(1 - \nu_A(x) - \beta.\mu_A(x)), \beta.\mu_A(x)\rangle \mid x \in E\}.
\end{aligned}$$

A series of new extensions of the modal operators were introduced in the last two years.

Other operators defined over IFSs are the topological operators that are analogous of operators *"closure"* and *"interior"* from topology (see [2]):

$$C(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\},$$

$$I(A) = \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}.$$

Weight operator W is also defined over IFSs. These operators have no analogues in fuzzy set theory and in the theories of the other fuzzy set extensions.

Now, following [4] we shall discuss another relationship between IFSs and IVFSs. Let us have the set of intervals $\{[a_i, b_i] : 1 \leq i \leq s\}$, where for every i $(1 \leq i \leq s) : a_i \leq b_i$. Let

$$A \leq \min_{1 \leq i \leq s} a_i < \max_{1 \leq i \leq s} b_i \leq B.$$

If we replace "$<$" with "$=$", then all intervals will be transformed to real numbers, because $a_1 = ... = a_s = b_1 = ... = b_s$. Therefore, we can define numbers for each i $(1 \leq i \leq s)$:

$$\mu_i = \frac{a_i - A}{B - A},$$

$$\nu_i = \frac{B - b_i}{B - A}$$

which are represented by points of the triangle from Figure 3. Obviously, it is more convenient to work with points than to work with intervals. If the above points have the geometrical interpretation from Figure 4, then by topological operators C and I we can determine points U and V and the region in which all above constructed points lie (see Figure 5). If it is necessary, by operator W we can find the point of the triangle that is the mass centre of this region.

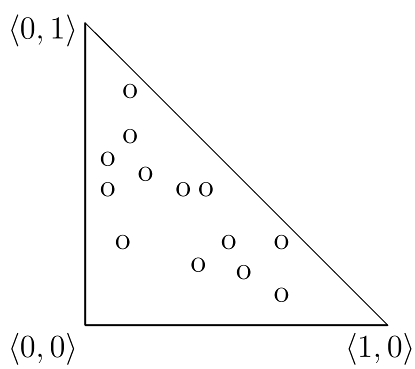


Figure 4.

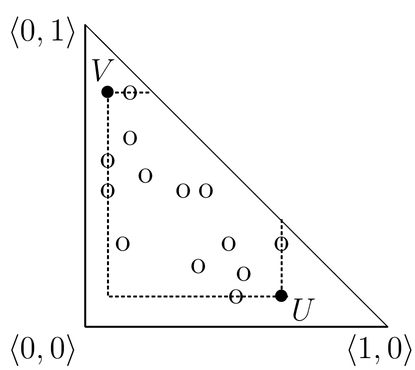


Figure 5.

Similarly to the modal operators, the topological operators also are extended so, that the new operators determine not only the two boundary points U and V, but also the boundaries themselves.

The IFSs are not only extensions of the fuzzy sets. They also are object of extensions. One of them are the *intuitionistic L-fuzzy sets,* where the values of functions μ_A,and ν_A are element of some fixed lattice L. Another extension are *IFSs of type 2,* for which (*) is changed to

$$\mu_A(x)^2 + \nu_A(x)^2 \leq 1.$$

It is clear that the latter inequality is a natural extension of the ordinary fuzzy set condition $\mu_A(x) \in [0, 1]$. Of course, we can continue in the direction of increasing the powers. Therefore, for natural number $n \geq 2$ we can define *IFSs of type n,* for which (*) is changed to

$$\mu_A(x)^n + \nu_A(x)^n \leq 1.$$

We can easily see that for every natural number $n \geq 2$, if a given set is an IFS of type n, then it is an IFS of type $n + 1$, but the opposite is not always valid.

The author's oppinion is that one of the most useful extensions of the IFS are the so called *"temporal IFS"* (see [1]). All operations, relations and operators over IFS can be transferred to them as well. They have the form (see Fig. 6)

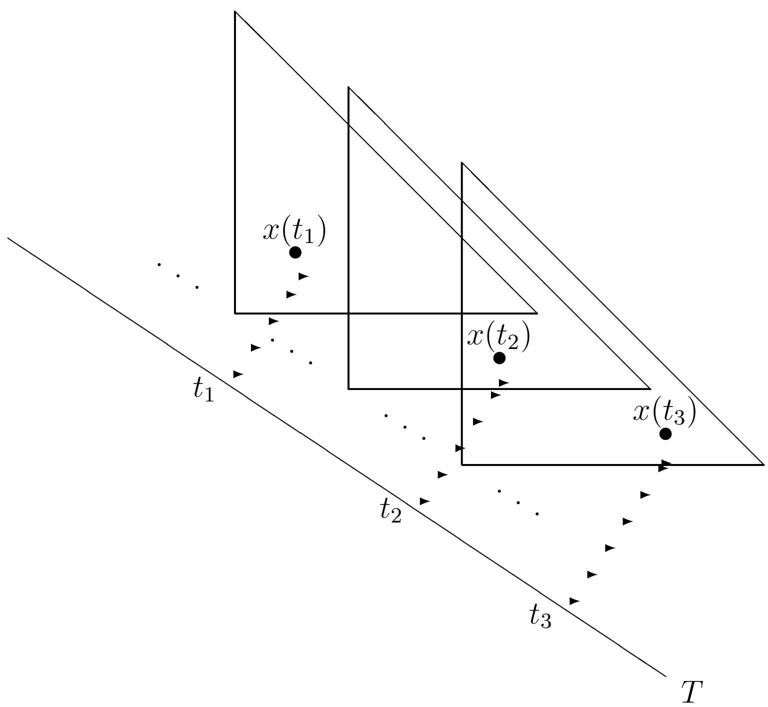


Figure 6.

$$A(T) = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle \;:\; \langle x, t\rangle \in E \times T\},$$

where E is a universe, T is a non-empty set and

(a) $A \subset E$ is a fixed set,

(b) $\mu_A(x,t) + \nu_A(x,t) \leq 1$ for every $\langle x, t\rangle \in E \times T$,

(c) $\mu_A(x, t)$ and $\nu_A(x, t)$ are the degrees of membership and non-membership, respectively, of the element $x \in E$ at the time-moment $t \in T$.

As generalization of the IFS and IVFS another fuzzy set extension was introduced in [3, 2] - Interval Valued IFSs (IVIFS):

$$IVIFS(A) = \{< x, M_A(x), N_A(x) > : \ x \in E\},$$

where for each $x \in E : M_A(x)$ and $N_A(x)$ are two intervals for which $M_A(x), N_A(x) \subset [0, 1]$ and $\sup M_A(x) + \sup N_A(x) \leq 1$. Obviously, every IFS and every IVFS can be represented as an IVIFS.

All above mentioned or defined and a lot of other operations, relations and operators are defined over IVIFSs. The IVIFSs also can be extended to IVIFSs from n-th type, to IVIL-FS, to Temporal IVIFSs and others.

As it is seen from the above short remarks, the IFSs, in comparison of IVFSs, are more flexible and more suitable for further extensions.

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