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Evolution problem with intuitionistic fuzzy fractional derivative

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Abstract: We introduce the generalized intutionistic fuzzy derivative, this concept used in order to give a generalized intuitionistic fuzzy Caputo fractional derivative. And we descuse the intuitionistic fuzzy fractional evolution problem.
Keywords: Generalized intuitionistic fuzzy Hukuhara difference, Generalized intuitionistic fuzzy derivative, generalized intuitionistic fuzzy Caputo-derivative, intuitionistic fuzzy fractional evolution problem.
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1 Introduction

The present paper studied the existence of intuitionistic fuzzy fractional evolution problem intuitionistic fuzzy nonlinear differential equations of fractional order

$$\begin{cases} \left({}_{gH}^{\ c}D^q x(t) \right) = Ax(t) + f\left(t, x(t)\right), \ \ t \in I = [t_0, T] \\ x(t_0) = x_0 \in IF_1. \end{cases} \tag{1}$$

where $0 < q < 1$, A is an operator of $\mathbb{F}_1$ from $\mathbb{F}_1$ generated an intuitionistic fuzzy α-semigroup T_α, the operator ${}_{gH}^{\ c}D^\gamma$ denote the Caputo fractional generalized derivative of order γ, $f : I \times \mathbb{F}_1(\mathbb{R}) \longrightarrow \mathbb{F}_1(\mathbb{R})$.

The concept of intuitionistic fuzzy sets is intoduced by K. Atanassov [3]. The authors in [8] built the concept of intuitionistic fuzzy metric space and intuionistic fuzzy numbers.

In [9], S. Melliani introduce the extension of Hukuhara difference in the intuitionistic fuzzy case. T. Allahviranloo, A. Armand and Z. Gouyandeh in [1] solve the fuzzy fractional differential equations under generalized fuzzy Caputo derivative. From this end idea we introduce in this paper the concept of generalized intuitionistic fuzzy Caputo derivative, and we give an integral solution of an intuitionistic fuzzy fractional equation.

This paper is organized as follows. In Section 2 we recall some concept concerning the intuitionistic fuzzy numbers. The concept of generalized intuitionistic fuzzy derivative and generalized intuitionistic fuzzy Caputo derivative, takes place in Section 3. The integral solution has descused in Section 4. Finally, in Section 5 we illustrate by an example.

2 Preliminaries

Definition 2.1. *[8] The set of all intuitionistic fuzzy numbers is given by*

$$I\!F_1 = I\!F_1(\mathbb{R}) = \Big\{ \langle u,v\rangle : \mathbb{R} \longrightarrow [0,1]^2,\ 0 \le u+v \le 1 \Big\}$$

with the following conditions:

1. *For each $\langle u,v\rangle \in I\!F_1$ is normal, i.e., $\exists x_0,\, x_1 \in \mathbb{R}$, such that $u(x_0)=1$ and $v(x_1)=1$.*
2. *For each $\langle u,v\rangle \in I\!F_1$ is a convex intuitionistic set, i.e., u is fuzzy convex and v is fuzzy concave.*
3. *For each $\langle u,v\rangle \in I\!F_1$, u is a lower continuous and v is appear continuous.*
4. *$cl\,\{x \in \mathbb{R},\ v(x) \le \alpha\}$ is bounded.*

Definition 2.2. *[8] For $\alpha \in [0,1]$, we define the upper and lower α-cut by*

$$\Big[\langle u,v\rangle\Big]_\alpha = \{x \in \mathbb{R},\ u(x) \ge \alpha\}, \quad \Big[\langle u,v\rangle\Big]^\alpha = \{x \in \mathbb{R},\ v(x) \le 1-\alpha\}.$$

Definition 2.3. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$\widetilde{0}(x) = \begin{cases} (1,0) & x = 0 \\ (0,1) & x \ne 0 \end{cases}.$$

Proposition 2.1. *[8] We can write*

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[[\langle u,v\rangle]_l^+(\alpha), [\langle u,v\rangle]_r^+(\alpha)\Big], \quad \Big[\langle u,v\rangle\Big]^\alpha = \Big[[\langle u,v\rangle]_l^-(\alpha), [\langle u,v\rangle]_r^-(\alpha)\Big]$$

Remark 2.1. *We can write $[\langle u,v\rangle]_\alpha = [u]^\alpha$ and $[\langle u,v\rangle]^\alpha = [1-v]^\alpha$, in the fuzzy case.*

Proposition 2.2. *[8] For all $\langle u,v\rangle, \langle u',v'\rangle \in I\!F_1$, we have*

$$\langle u,v\rangle = \langle u',v'\rangle \Longleftrightarrow [\langle u,v\rangle]_\alpha \text{ and } [\langle u,v\rangle]^\alpha,\ \forall t \in [0,1].$$

We define two operations on $\mathbb{F}_1$ by

$$\langle u,v\rangle \oplus \langle u',v'\rangle = \langle u \vee v, u' \wedge v'\rangle \quad \forall \langle u,v\rangle, \langle u',v'\rangle \in \mathbb{F}_1$$
$$\lambda\langle u,v\rangle = \langle \lambda u, \lambda v\rangle \quad \forall \lambda \in \mathbb{R},\ \forall \langle u,v\rangle \in \mathbb{F}_1.$$

According to Zadeh extension, we have

$$\Big[\langle u,v\rangle \oplus \langle u',v'\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha + \Big[\langle u',v'\rangle\Big]_\alpha, \quad \Big[\langle u,v\rangle \oplus \langle u',v'\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha + \Big[\langle u',v'\rangle\Big]^\alpha$$
$$\Big[\lambda\langle u,v\rangle\Big]_\alpha = \lambda\Big[\langle u,v\rangle\Big]_\alpha, \quad \Big[\lambda\langle u,v\rangle\Big]^\alpha = \lambda\Big[\langle u,v\rangle\Big]^\alpha.$$

Theorem 2.3. *[8] Let $\mathcal{M} = \{M_\alpha, M^\alpha, \alpha \in [0,1]\}$ be a family of subsets in $\mathbb{R}$ stisfying the following conditions*

1. *$\alpha \le s \Longrightarrow M_s \subset M_\alpha$ and $M^s \subset M^\alpha$, for each $\alpha, s \in [0,1]$.*
2. *M_α and M_s are nonempty compact convex sets in $\mathbb{R}$ for each $\alpha \in [0,1]$.*
3. *for any nondecreasing sequence $\alpha_i \longrightarrow \alpha$ on $[0,1]$, we have $M_\alpha \in [0,1] = \bigcap_i M_{\alpha_i}$ and $M^\alpha = \bigcap_i M^{\alpha_i}$.*

We define u and v by

$$u(x) = \begin{cases} 0 & x \notin M_0 \\ \sup\limits_{\alpha\in[0,1]} M_\alpha & x \in M_0 \end{cases}$$
$$v(x) = \begin{cases} 1 & x \notin M^0 \\ 1 - \sup\limits_{\alpha\in[0,1]} M_\alpha & x \in M^0 \end{cases}$$

Then $\langle u,v\rangle \in \mathbb{F}_1$ with $M_\alpha = \Big[\langle u,v\rangle\Big]_\alpha$ and $M^\alpha = \Big[\langle u,v\rangle\Big]^\alpha$.

Remark 2.2. *[8]*

1. *The family $\{[\langle u,v\rangle]_\alpha, [\langle u,v\rangle]^\alpha,\ \alpha \in [0,1]\}$ satisfying conditions 1–3 of the previous theorem.*
2. *For all $\alpha \in [0,1]$, $\quad [\langle u,v\rangle]_\alpha \subset [\langle u,v\rangle]^\alpha$.*

Theorem 2.4. *[8] On $\mathbb{F}_1$ we can define the metric*

$$d_\infty\Big((u,v),(z,w)\Big) = \frac{1}{4}\Bigg(\sup_{0<\alpha\le1}\bigg|\Big[(u,v)\Big]_r^+(\alpha) - \Big[(z,w)\Big]_r^+(\alpha)\bigg| + \sup_{0<\alpha\le1}\bigg|\Big[(u,v)\Big]_l^+(\alpha) - \Big[(z,w)\Big]_l^+(\alpha)\bigg| + \sup_{0<\alpha\le1}\bigg|\Big[(u,v)\Big]_r^-(\alpha) - \Big[(z,w)\Big]_r^-(\alpha)\bigg| + \sup_{0<\alpha\le1}\bigg|\Big[(u,v)\Big]_l^-(\alpha) - \Big[(z,w)\Big]_l^-(\alpha)\bigg|\Bigg)$$

and

$$d_p\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)=\bigg(\frac{1}{4}\int_0^1\Big|[\langle u,v\rangle]_l^+(\alpha)-[\langle u',v'\rangle]_l^+(\alpha)\Big|^p dt$$
$$+\frac{1}{4}\int_0^1\Big|[\langle u,v\rangle]_r^+(\alpha)-[\langle u',v'\rangle]_r^+(\alpha)\Big|^p dt+\frac{1}{4}\int_0^1\Big|[\langle u,v\rangle]_l^-(\alpha)-[\langle u',v'\rangle]_l^-(\alpha)\Big|^p dt$$
$$+\frac{1}{4}\int_0^1\Big|[\langle u,v\rangle]_r^-(\alpha)-[\langle u',v'\rangle]_r^-(\alpha)\Big|^p dt\bigg)^{\frac{1}{p}}$$

For $p\in[1,\infty)$, we have $\Big(\mathbb{F}_1,d_p\Big)$ is a complete metric space.

3 The generalized Hukuhara derivative of an intuitionistic fuzzy-valued function

The concept of intuitionistic fuzzy Hukuhara difference is introduced by the authors in [9], in this paper we will give the definition of generalized Hukuhara difference betwen two intuitionistic fuzzy number.

Definition 3.1. *The generalized Hukuhara difference of two fuzzy number $\langle u,v\rangle$, $\langle u',v'\rangle\in\mathbb{F}_1$ is defined as follows*

$$\langle u,v\rangle-_{gH}\langle u',v'\rangle=\langle z,w\rangle\Longleftrightarrow\langle u,v\rangle=\langle u',v'\rangle\oplus\langle z,w\rangle.$$

Note that the α-level representation of fuzzy-valued function $f:[0,T]\longrightarrow\mathbb{F}_1$ expressed by $[f]_\alpha=[f_{\alpha,l},f_{\alpha,r}]$ and $[f]^\alpha=\big[f^{\alpha,l},f^{\alpha,r}\big]$.

Definition 3.2. *The generalized Hukuhara derivative of a intuitionistic fuzzy-valued function $f:[0,T]\longrightarrow\mathbb{F}_1$ at t_0 is defined as*

$$f'_{gH}(t_0)=\lim_{t\to t_0}\frac{f(t)-_{gH}f(t_0)}{t-t_0}$$

if $f'_{gH}(t_0)\in\mathbb{F}_1$, we say that f is generalized Hukuhara differentiable at t_0.

Also we say that f is $[(i)-gH]$-differentiable at t_0 if

$$\begin{cases}\Big(f'_{gH}\Big)_\alpha=\Big[(f_{\alpha,l})',(f_{\alpha,r})'\Big]\\ \Big(f'_{gH}\Big)^\alpha=\Big[(f^{\alpha,l})',(f^{\alpha,r})'\Big]\end{cases}$$

and that f is $[(ii)-gH]$-differentiable at t_0 if

$$\begin{cases}\Big(f'_{gH}\Big)_\alpha=\Big[(f_{\alpha,r})',(f_{\alpha,l})'\Big]\\ \Big(f'_{gH}\Big)^\alpha=\Big[(f^{\alpha,r})',(f^{\alpha,l})'\Big]\end{cases}.$$

Remark 3.1. *We can define the generalized derivative of higher order by*

$$\begin{cases} f^{(0)} = f \\ f_{gH}^{(n)} = \left(f^{(n-1)}\right)'_{gH}, \quad \forall n \in \mathbb{N} \end{cases}. \tag{2}$$

Definition 3.3. *Let $f : (0,T) \longrightarrow I\!F_1$. We say that f is of class $\mathcal{C}^m$, $m \in \mathbb{N}$, if $f_{gh}^{(m)}$ exists and continues, by respect to metric d_∞.*

Now if the α-levels of $f : (0,T] \longrightarrow \mathbb{F}_1$, are given by $[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}]$ and $[f]^\alpha = \left[f^{\alpha,l}, f^{\alpha,r}\right]$ and $f_{\alpha,l}$, $f_{\alpha,r}$, $f^{\alpha,l}$, $f^{\alpha,r}$ are Riemann integrable on $[0,T]$. Since the family

$$\left\{ [f_{\alpha,l}, f_{\alpha,r}], \ \left[f^{\alpha,l}, f^{\alpha,r}\right] \right\}$$

builds an intuitionistic element and the integral preserves the monotony, then by Theorem 2.3 the family

$$\left\{ \left[\int_{[0,T]} f_{\alpha,l}, \int_{[0,T]} f_{\alpha,r} \right], \left[\int_{[0,T]} f^{\alpha,l}, \int_{[0,T]} f^{\alpha,r} \right] \right\}$$

defines an intuitionistic fuzzy element, which is the integral of f on $[0,T]$, we denote by $\int_0^T f$.

Definition 3.4. *Let $f : [0,T] \longrightarrow I\!F_1$ be a intuitionistic fuzzy-valued function. We say that f is integrable on $[0,T]$ if $f_{\alpha,l}$, $f_{\alpha,r}$, $f^{\alpha,l}$, $f^{\alpha,r}$ defined in the previous are integrable on $[0,T]$.*

4 Intuitionistic fuzzy generalized Caputo-derivative

Let $f : [0,T] \longrightarrow \mathbb{F}_1$ be a intuitionistic fuzzy-valued integrable function on $[0,T]$, and $\delta \in (m-1, m]$ and $m \in \mathbb{N}^*$, its α-levels are defined by $[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}]$ and $[f]^\alpha = \left[f^{\alpha,l}, f^{\alpha,r}\right]$ where $f_{\alpha,l}$, $f_{\alpha,r}$, $f^{\alpha,l}$, $f^{\alpha,r} \in \mathcal{C}^m([0,T])$.
So

$$M_\alpha = \left[\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \Big(f_{\alpha,l}\Big)^{(m)}(s), \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \Big(f_{\alpha,r}\Big)^{(m)}(s) \right]$$

and

$$M^\alpha = \left[\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \Big(f^{\alpha,l}\Big)^{(m)}(s), \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \Big(f^{\alpha,r}\Big)^{(m)}(s) \right].$$

Proposition 4.1. *The the family $\left\{M_\beta, M^\beta, \ \beta \in [0,1]\right\}$ defines an intuitionistic fuzzy element.*

Proof. Just use Theorem 2.3. □

Definition 4.1. *The intuitionistic fuzzy preceding item is called the generalized Caputo derivative of f, we denote $D^\alpha f$. We say that f is ${}^{cf}[(i)-gH]$-differentiable at t_0 if*

$$\left[{}_{gH}D^\delta f \right]_\alpha = \left[D^\delta f_{\alpha,l}, D^\delta f_{\alpha,r}\right]$$
$$\left[{}_{gH}D^\delta f \right]^\alpha = \left[D^\delta f^{\alpha,l}, D^\delta f^{\alpha,r}\right]$$

and that f is ${}^{cf}[(ii)-gH]$-differentiable at t_0 if

$$\left[{}_{gH}D^{\delta}f\right]_{\alpha}=\left[D^{\delta}f_{\alpha,r},D^{\delta}f_{\alpha,l}\right]$$
$$\left[{}_{gH}D^{\delta}f\right]^{\alpha}=\left[D^{\delta}f^{\alpha,r},D^{\delta}f^{\alpha,l}\right].$$

As in the previuos definition we will give the difinition of intuitionistic fuzzy fractional Riemann–Liouville integral. If the α-levels of $f:(0,T]\longrightarrow \mathbb{F}_1$, are given by $[f]_{\alpha}=[f_{\alpha,l},f_{\alpha,r}]$ and $[f]^{\alpha}=\left[f^{\alpha,l},f^{\alpha,r}\right]$ and $f_{\alpha,l},f_{\alpha,r},f^{\alpha,l},f^{\alpha,r}$ are Riemann integrable on $(0,T]$. Since the family

$$\left\{[f_{\alpha,l},f_{\alpha,r}],\ \left[f^{\alpha,l},f^{\alpha,r}\right]\right\}$$

builds an intuitionistic element and the integral preserves the monotony, then by Theorem 2.3 the family

$$\{\mathcal{A}_{\alpha},\mathcal{A}^{\alpha},\alpha\in[0,1]\},$$

where

$$\mathcal{A}_{\alpha}=\left[\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f_{\alpha,l}(s),\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f_{\alpha,r}(s)\right]$$

and

$$\mathcal{A}^{\alpha}=\left[\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f^{\alpha,l}(s),\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f^{\alpha,r}(s)\right],$$

defines an intuitionistic fuzzy element, which is the Riemann–Liouville fractional integral of f on $(0,T)$, we denote $\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f(s)ds$.

Definition 4.2. *The Riemann–Liouville fractional integral of f on $(0,T)$, defined as*

$$I^{\delta}f(t)=\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f(s)ds$$

where $\delta\in(m-1,m)$.

5 Embedding theorem and intuitionistic fuzzy α-semigroup

Since the element of $\mathbb{F}_1$ are closed (Hausdorff topology) and convex, so we can apply the result of [11].

Theorem 5.1. *We can extend $\mathbb{F}_1$ in a normed space.*

Proof. Consider the following relation on $\mathbb{F}_1\times\mathbb{F}_1$ defined by

$$(\langle u,v\rangle,\langle z,t\rangle)\sim(\langle u',v'\rangle,\langle z',t'\rangle)\iff\langle u,v\rangle+\langle z',t'\rangle=\langle u',v'\rangle+\langle z,t\rangle.$$

It is clear that such relation a equivalence relation.
We denote that $\mathbb{F}^{*}=\mathbb{F}_1\times\mathbb{F}_1/_{\sim}$ is a vector space (see [11]).
Now consider the map

$$j:\begin{cases}\mathbb{F}_1\longrightarrow\mathbb{F}_1^{*}\\ \langle u,v\rangle\longrightarrow\overline{\left(\langle u,v\rangle,\widetilde{0}\right)}\end{cases}\tag{3}$$

is an injective mapping, indded:

$$j(\langle u,v\rangle)=j(\langle u',v'\rangle)\Longrightarrow\overline{\left(\langle u,v\rangle,\widetilde{0}\right)}=\overline{\left(\langle u',v'\rangle,\widetilde{0}\right)}$$

$$\Longrightarrow(\langle u,v\rangle,\widetilde{0})\sim(\langle u',v'\rangle,\widetilde{0})\Longrightarrow\langle u,v\rangle=\langle u',v'\rangle.$$

Further we can define the norm on $\mathbb{F}_1^*$ as

$$\|\ \overline{(\langle u,v\rangle,\langle u',v'\rangle)}\ \|=d_1\left(\langle u,v\rangle,\langle u',v'\rangle\right),$$

which proves that $(\mathbb{F}_1^*,\|.\|)$ is a normed vector space. □

Theorem 5.2. *There exists a Banach space X such that $\mathbb{F}_1$ can be embedded as a convex cone C with vertex 0 in X. Furthermore, the following conditions hold true:*

1. *The embedding j is isometric,*
2. *The addition in X induces the addition in $\mathbb{F}_1$,*
3. *The multiplication by a nonnegative real number in X induces the corresponding operation in $\mathbb{F}_1$,*
4. *$\mathcal{C}-\mathcal{C}=\{a-b,\ a,b\in\mathcal{C}\}$ is dense in X,*
5. *$\mathcal{C}$ is closed*

Proof. By Theorem 5.1, $\mathbb{F}_1$ can be embedded as a convex cone $\mathcal{C}$ in a normed linear space Y such that $\mathcal{C}$ spans Y and the conditions $1-3$ hold true.

If X is a completion of Y, then also 4. is satisfied. Since $(\mathbb{F}_1,d_1)$ is complete, which follows by combining results in [5] and [10], and the embedding j is isometric we have 5. □

Definition 5.1. *A continuous one-parameter intuitionistic fuzzy α-semigroup $\{T_\alpha(t),\ t\geq 0\}$ of operators on $\mathbb{F}_1$ is defined by the following conditions:*

1. *For any fixed $t\geq 0$, $T_\alpha(t)$ is a continuous operator defined on $\mathbb{F}_1$ into $\mathbb{F}_1$.*
2. *For any $\langle u,v\rangle\in\mathbb{F}_1$, $T_\alpha(t)\langle u,v\rangle$ is strongly continuous in t, with the metric d_1.*
3. $T_\alpha\left((t+s)^{\frac{1}{\alpha}}\right)=T_\alpha\left((t)^{\frac{1}{\alpha}}\right)T_\alpha\left((s)^{\frac{1}{\alpha}}\right).$
4. *For all $\langle u,v\rangle$, $\langle u',v'\rangle\in\mathbb{F}_1$*

$$d_1\left(T_\alpha(t)\langle u,v\rangle,T_\alpha(t)\langle u',v'\rangle\right)\leq Me^{\omega t^\alpha}d_1\left(\langle u,v\rangle,\langle u',v'\rangle\right)\ \forall t\geq 0,$$

where $M>0$.

We call such a family $\{T_\alpha(t)\}$ simply intuitionistic fuzzy α-semigroup of type ω. The strict α-infinitesimal generator A_α of a intuitionistic fuzzy α-semigroup $\{T_\alpha(t)\}$ is defined by

$$A_\alpha x = \lim_{t \to 0} T_\alpha^{(\alpha)}(t)\langle u, v\rangle, \ \langle u, v\rangle \in I\!F_1.$$

The right side exists in $I\!F_1$.
We define the domain of A_α, by

$$D(A_\alpha) = \left\{ \langle u, v\rangle \in I\!F_1, \ \lim_{t \to 0} T_\alpha^{(\alpha)}(t)\langle u, v\rangle \textit{ exist} \right\}.$$

Lemma 5.3. *If the family $\{T_\alpha(t), \ t \geq 0\}$ is an intuitionistic fuzzy α-semigroup of type ω, then $jT_\alpha(t)j^{-1}$ is a nonlinear α-semigroup of type ω on $\mathcal{C}$.*

Proof. By [6], $jT_\alpha(t)j^{-1} : \mathcal{C} \longrightarrow \mathcal{C}$, since j is an isometric, which implies that $jT_\alpha(t)j^{-1}$ is a nonlinear α-semigroup of type ω on $\mathcal{C}$. □

Lemma 5.4. *If A_α is an intuitionistic fuzzy infinitesimal generator of an intuitionistic fuzzy α-semigroup of type ω $\{T_\alpha(t)\}_{t \geq 0}$. Then $jA_\alpha j^{-1}$ is the infinitesimal generator of $jT_\alpha(t)j^{-1}$.*

Proof. Let $x \in \mathcal{C}$ and put $R_\alpha(t) = jT_\alpha(t)j^{-1}$. We have $T(t) : \mathcal{C} \longrightarrow \mathcal{C}$, and $\langle u, v\rangle = j^{-1}x$

$$\begin{aligned}
&\lim_{t \to 0} \| \frac{R_\alpha(t + \epsilon t^{1-\alpha})x \ominus R_\alpha x}{\epsilon} - R_\alpha^{(\alpha)}(t)x \| = 0 \\
&\Longrightarrow \lim_{t \to 0} \| jT_\alpha(t)j^{-1}x \ominus jT_\alpha(t)j^{-1}x\epsilon - jT_\alpha^\alpha(t)j^{-1}x \| = 0 \\
&\Longrightarrow \lim_{t \to 0} d_1\Big(T_\alpha(t)j^{-1}x \ominus jT_\alpha(t)j^{-1}x\epsilon, jT_\alpha^\alpha(t)j^{-1}x\Big) = 0 \\
&\Longrightarrow \lim_{t \to 0} d_1\Big(T_\alpha(t)\langle u, v\rangle \ominus jT_\alpha(t)\langle u, v\rangle\epsilon, jT_\alpha^\alpha(t)\langle u, v\rangle\Big) = 0.
\end{aligned}$$

□

6 Intuitionistic fuzzy Caputo fractional evolution problem

In this section we consider the following problem

$$\begin{cases} {}_{gH}D^q x(t) = Ax(t) + f(t, x(t)), \ t \in [t_0, T] \\ x(0) =< u_0, v_0 >\in \mathbb{F}_1 \end{cases}, \tag{4}$$

where $0 < q < 1$ is a real number and the operator ${}_{gH}D^\alpha$ denotes the Caputo fractional generalized derivative of order α, and $f : [0, \infty) \times \mathbb{F}_1 \longrightarrow \mathbb{F}_1$, is a continuous fuzzy function. A is an operator of $\mathbb{F}_1$ from $\mathbb{F}_1$ generated an intuitionistic fuzzy α-semigroup T_α.

In this section, the existence and uniqueness of solutions of problem (1) with fuzzy initial conditions is proved under ${}^{cf}[gH]$-differentiability.

Lemma 6.1. *If $a \in I\!F_1$, then*

$$\int_{(0,t)} a ds = ta.$$

Proof. We set $[a]_\alpha = [a_-, a_+]$ and $[a]^\alpha = [a^-, a^+]$. We have

$$[\int_{(0,t)} ads]_\alpha = [ta_-, ta_+] = t[a_-, a_+] = t[a]_\alpha$$

and

$$[\int_{(0,t)} ads]^\alpha = [ta^-, ta^+] = t[a^-, a^+] = t[a]^\alpha.$$

□

Lemma 6.2. *Let $f : [a,b] \longrightarrow I\!F_1$ be a fuzzy-valued function such that $f'_{gH} \in \mathcal{C}^{I\!F_1}([a,b]) \cap L^{I\!F_1}([a,b])$, then*

$$I^\alpha \left({}_{gH}D^{\alpha f}\right)(t) = f(t) \ominus_{gH} f(a).$$

Proof. We set $f = \langle f_1, f_2 \rangle$, it become $I^\alpha \left({}_{gH}D^{\alpha f_1}\right)(t) = f_1(t) \ominus_{gH} f_1(a)$ and $I^\alpha \left({}_{gH}D^{\alpha f_2}\right)(t) = f_2(t) \ominus_{gH} f_2(a)$, but $0 \leq f_1 + f_2 \leq 1$, which implies that

$$I^\alpha \left({}_{gH}D^{\alpha f}\right)(t) = f(t) \ominus_{gH} f(a).$$

□

We denote

$$sgn(x) = \begin{cases} + & \text{if } x \text{ is } {}^{cf}\ [(i) - gH]\text{-differentiable} \\ \ominus(-1) & \text{if } x \text{ is } {}^{cf}\ [(ii) - gH]\text{-differentiable.} \end{cases} \tag{5}$$

Theorem 6.3. *The initial value problem (1) is equivalent to one of the following integral equations*

$$x(t) = T_\alpha(t)x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)), \tag{6}$$

if $x(t)$ be ${}^{cf}[(i) - gH]$-differentiable

$$x(t) = T_\alpha(t)x_0 \ominus \frac{-1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1,} f(s, x(s)) \tag{7}$$

if $x(t)$ be ${}^{cf}[(ii) - gH]$-differentiable

$$x(t) = \begin{cases} T_\alpha(t)x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)),\ if\ t \in [a,c] \\ T_\alpha(t)x_0 \ominus \frac{-1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)),\ if\ t \in [c,b] \end{cases}. \tag{8}$$

Proof. We have ${}_{gH}D^q x(t) = Ax(t) + f(t, x(t))$, wich implies $[{}_{gH}D^q x(t)]^\alpha = [Ax(t) + f(t, x(t))]^\alpha$ and $[{}_{gH}D^q x(t)]_\alpha = [Ax(t) + f(t, x(t))]_\alpha$, by [4] we get

$$[x(t)]^\alpha = [T_\alpha(t)x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s))]^\alpha$$

and

$$[x(t)]_\alpha = [T_\alpha(t)x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s))]_\alpha.$$

Now using Theorems 2.3, 6.2 and [2] for completing the proof. □

Theorem 6.4. *[7] Consider $U : T \longrightarrow X$ to be a set of continuous function. Then U is a relative compact set if and only if U is equicontinuous and for any $t \in T$, $U(t)$ is a relative compact set in X.*

Theorem 6.5. *[7] Let U be a closed convex subset of a Banach space X. If $A : U \longrightarrow U$ is continuous and $U = A(U)$ is compact, then A has a fixed point in U.*

Theorem 6.6. *Let f continue on $R_0 = \{(t_0, x), t \in [t_0, t_0 + h^*],\ ||x, x_0|| \leq \eta\}$ such that $\sup_{t\in[t_0,T]} d_1 (f(t, x), 0) = M$, where $\eta > d_1(T_\alpha(t)x_0, x_0)$ and f is $\frac{\Gamma(q+1)}{(t-t_0)^q}$-Lipschitz. Then the problem (1) has a unique solution.*

Proof. We set

$$(Ax)(t) = T_\alpha(t)x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s))ds.$$

The proof is presented in several steps.
Step 1: $(Ax)(t) \in B(x_0, \eta)$ is continuous

$$\begin{aligned} d_1 ((Ax)(t), x_0) &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} d_1 \left(f(s, x(s), \widetilde{0}) \right) ds + d_1 (T_\alpha(t)x_0, x_0) \\ &\leq \eta. \end{aligned}$$

So if $x \in B$ then $Ax \in B$.
Step 2: $t \longrightarrow (Ax)(t)$ is contraction. For $t_0 \leq t_1 \leq t_2 \leq t_0 + h$,

$$\begin{aligned} d_1\Big((Ax)(t_1), (Ax)(t_2)\Big) &\leq \frac{1}{\Gamma(q)} d_1 \left(\int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s)), \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) \right) \\ &\leq \frac{1}{\Gamma(q)} \Bigg\{ \int_{t_0}^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| d_1 \left(f(s, x(s)), \widetilde{0} \right) ds \\ &\qquad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} d_1(f(s, x(s)), \widetilde{0}) ds \Bigg\}. \end{aligned}$$

Since $q < 1$ then $q - 1 < 0$, so we have

$$\int_{t_0}^{t_1} \Big|(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\Big| ds \leq \frac{1}{q}(t_2 - t_1)^q,$$

which implies that

$$d_1\Big((Ax)(t_1), (Ax)(t_2)\Big) \leq \frac{2\,M}{\Gamma(q + 1)}.$$

By Lemmas 5.3 and 5.4, A has a fixed point, which is the solution of the problem.
Step 3: By the same proof, we prove that $T(t)x_0 \ominus \frac{-1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s))ds$ has a fixed point.
Step 4: Uniqueness. Suppose that x and y two solution of (4), we have

$$
\begin{aligned}
d_1\Big(x(t), y(t)\Big) &= d_1\Big(Ax(t), Ay(t)\Big) \\
&\le \frac{1}{\Gamma(q)} \int_{t_0}^{t} |(t-s)^{q-1}| d_1\Big(f(s, x(s), f(s, y(s)))\Big) ds \\
&< q \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^{-q} d_1\Big(x(s), y(s)\Big) ds.
\end{aligned}
$$

By setting $\psi(t) = (t-t_0)^{-q} d_1\left(x(t), y(t)\right)$ and $m = \sup_{[t_0,T]} \psi(t)$, we get

$$
m < mq(t_1 - t_0)^{-q} \int_{t_0}^{t_1} (t_1 - s)^{q-1} ds = m,
$$

which completes the proof. □

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