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Representation of complex intuitionistic fuzzy sets

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Abstract: In this paper, we propose the notion of complex intuitionistic fuzzy sets defined by complex-valued membership and non-membership functions in order to make extension the result presented in [6]. We first give a Cartesian representation, and then we discuss the polar representation.
Keywords: Complex intuitionistic fuzzy sets, Cartesian representation, Polar representation.
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1 Introduction

The concept of intuitionistic fuzzy sets is introduced by K. Atanassov (1984) [1,2]. This concept is a generalization of fuzzy theory introduced by L. Zadeh [3].

The concept of complex fuzzy sets as sets with complex membership functions was first introduced by Ramot et al., who in [8] demonstrated the increased expressive power gained by endowing a set S with a complex membership function $\mu_S(x) = r_S(x)e^{i\phi_S(x)}$, where $r_S(x)$ and $\phi_S(x)$ are real-valued functions with r_S solely responsible for the fuzzy information and ϕ_S functioning as a phase term containing additional crisp information.

In this work, we will be working on the same idea, but this time in the intuitionistic fuzzy theory, we can write the representation of complex membership function μ as $\mu_S(x) = r_S(x)e^{i\phi_S(x)}$

and the non-membership function ν as $\nu_S(x) = r'_S(x)e^{i\phi'_S(x)}$, where $(r_S(x), r'_S(x))$ and $(\phi_S(x), \phi'_S(x))$ are real-valued functions with (r_S, r'_S) solely responsible for the intuitionistic fuzzy information and (ϕ_S, ϕ'_S) functioning.

We attract the reader's attention to the difference between complex intuitionistic fuzzy sets and intuitionistic fuzzy complex numbers.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

Definition 1. *We denote*

$$I\!F = \left\{(u, v) : \mathbb{R} \to [0,1]^2 \;|\forall x \in \mathbb{R} \,/ 0 \leq u(x) + v(x) \leq 1\right\}$$

where

1. (u, v) *is normal i.e., there exists* $x_0, x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$*;*
2. u *is fuzzy convex and* v *is fuzzy concave;*
3. u *is upper semicontinuous and* v *is lower semicontinuous;*
4. $supp(u, v) = cl(\{x \in \mathbb{R} : v(x) < 1\})$ *is bounded.*

For $\alpha \in [0, 1]$ *and* $(u, v) \in I\!F$*, we define*

$$[(u, v)]^\alpha = \{x \in \mathbb{R} \mid v(x) \leq 1 - \alpha\}$$

and

$$[(u, v)]_\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$

Remark 1. *We can consider* $[(u, v)]_\alpha$ *as* $[u]^\alpha$ *and* $[(u, v)]^\alpha$ *as* $[1 - v]^\alpha$ *in the fuzzy case.*

Definition 2. *The intuitionistic fuzzy zero is the intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0), & x = 0 \\ (0, 1), & x \neq 0 \end{cases}$$

Definition 3. *Let* (u, v) *,*$(u', v') \in I\!F$ *and* $\lambda \in \mathbb{R}$*, we define the addition by:*

$$((u, v) \oplus (u', v'))(z) = \left(\sup_{z=x+y} \min(u(x), u'(y)); \inf_{z=x+y} \max(v(x), v'(y))\right)$$

$$\lambda(u, v) = \begin{cases} (\lambda u, \lambda v) & \text{if } \lambda \neq 0 \\ 0_{(0,1)} & \text{if } \lambda = 0 \end{cases}$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $\mathbb{F}$ as follows:

$$[(u,v)\oplus(z,w)]^\alpha = [(u,v)]^\alpha + [(z,w)]^\alpha, \tag{1}$$

$$[\lambda(u,v)]^\alpha = \lambda[(u,v)]^\alpha, \tag{2}$$

$$[(u,v)\oplus(z,w)]_\alpha = [(u,v)]_\alpha + [(z,w)]_\alpha, \tag{3}$$

$$[\lambda(u,v)]_\alpha = \lambda[(u,v)]_\alpha, \tag{4}$$

where $(u,v), (z,w) \in \mathbb{F}$ and $\lambda \in \mathbb{R}$.

We denote

$$\begin{aligned}
[(u,v)]_l^+(\alpha) &= \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\},\\
[(u,v)]_r^+(\alpha) &= \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\},\\
[(u,v)]_l^-(\alpha) &= \inf\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\},\\
[(u,v)]_r^-(\alpha) &= \sup\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}.
\end{aligned}$$

Remark 2.

$$\begin{aligned}
\big[(u,v)\big]_\alpha &= \big[[(u,v)]_l^+(\alpha), [(u,v)]_r^+(\alpha)\big],\\
\big[(u,v)\big]^\alpha &= \big[[(u,v)]_l^-(\alpha), [(u,v)]_r^-(\alpha)\big].
\end{aligned}$$

Theorem 1. *Let $\mathcal{M} = \{M_\alpha,\ M^\alpha\ :\ \alpha \in [0,1]\}$ be a family of subsets in $\mathbb{R}$ satisfying Conditions $(i)-(iv)$:*

i) $\alpha \leq \beta \Rightarrow M_\beta \subset M_\alpha$ *and* $M^\beta \subset M^\alpha$

ii) M_α *and* M^α *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha \in [0,1]$.

iii) *for any nondecreasing sequence* $\alpha_i \to \alpha$ *on* $[0,1]$*, we have* $M_\alpha = \bigcap_i M_{\alpha_i}$ *and* $M^\alpha = \bigcap_i M^{\alpha_i}$.

iv) *For each* $\alpha \in [0,1]$*,* $M_\alpha \subset M^\alpha$ *let* u *and* v *be defined by*

$$u(x) = \begin{cases} 0 & if\ x \notin M_0 \\ \sup\{\alpha \in [0,1]\ :\ x \in M_\alpha\} & if x \in M_0 \end{cases}$$

$$v(x) = \begin{cases} 1 & if\ x \notin M^0 \\ 1-\sup\{\alpha \in [0,1]\ :\ x \in M^\alpha\} & if x \in M^0 \end{cases}$$

Then $(u,v) \in \mathbb{F}$.

Proof. See [7]. □

On $\mathbb{F}$, we define the metric d_∞ by:

$$d_\infty\left((u,v),(z,w)\right) = \frac{1}{4}\sup_{0<\alpha\le 1}\left|[(u,v)]_r^+(\alpha) - [(z,w)]_r^+(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|[(u,v)]_l^+(\alpha) - [(z,w)]_l^+(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|[(u,v)]_r^-(\alpha) - [(z,w)]_r^-(\alpha)\right| + \frac{1}{4}\sup_{0<\alpha\le 1}\left|[(u,v)]_l^-(\alpha) - [(z,w)]_l^-(\alpha)\right|$$

where $|.|$ denotes the usual Euclidean norm in $\mathbb{R}$.

Theorem 2. *$(\mathbb{F}, d_\infty)$ is a complete metric space.*

Proof. See [7] □

$(\mathbb{F}, d_\infty)$ is a complete metric space which can be embedded isomorphically as a cone in a Banach space (see theorem 3.1 in [5]). We recall the definition of a complex fuzzy set:

Definition 4. *A complex fuzzy set A, defined on a universe of discourse X, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in A. By definition $\mu_A(x)$ is a value in the unit circle in the complex plane in the polar case, and a value in the unit square in $\mathbb{C}$ in the Cartesian case.*

3 Main results

3.1 Complex intuitionistic fuzzy set

Similarly to the definition of complex fuzzy set, we give here a definition of complex intuitionistic fuzzy set:

Definition 5. *A complex intuitionistic fuzzy set A, defined on a universe of discourse X, is characterized by a membership function $\mu_A(x)$ and non-membership function $\nu_A(x)$ that assign any element $x \in X$ a complex-valued grade of membership and non-membership in A. By definition, the values that $\mu_A(x)$, $\nu_A(x)$ and $\mu_A(x) + \nu_A(x)$ may obtain, are all lying within the unit circle in the complex plane in the polar case, and μ_A, ν_A and $\mu_A(x) + \nu_A(x)$ obtain a value in the unit square in $\mathbb{C}$ in the Cartesian case.*

3.2 Cartesian representation

The complex membership function in [9], μ, is defined as

$$\mu(V, z) = \mu_R(V) + i\mu_I(z),$$

likewise, we can define the complex non-membership function as

$$\nu(V, z) = \nu_R(V) + i\nu_I(z),$$

where V is to be interpreted as a set in an intuitionistic fuzzy set of sets and z as an element of V . This definition can be easily extended to $\mathbb{R}$, for $x \in \mathbb{R}$, let

$$f_1(x) = u(x) + iv(x) \qquad \text{and} \qquad f_2(x) = u'(x) + iv'(x),$$

where $f = (u, u') : \mathbb{R} \longrightarrow [0, 1]^2$ and $g = (v, v') : \mathbb{R} \longrightarrow [0, 1]^2$. For ease of notation, denote $\mathcal{F}$ by (f, g). Thus, f_1, f_2 assigns to each $x \in \mathbb{R}$ a value in the unit square in $\mathbb{C}$, representing a complex grade of membership and non-membership. Note that u, v, u' and v' considered individually define non-complex fuzzy sets in $\mathbb{R}$.

Now, for $f = (u, u')$, $g = (v, v') : \mathbb{R} \longrightarrow [0, 1]^2$, α-level sets are classically defined as follows:

$$[f]^\alpha = [(u, u')]^\alpha = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha\}\,; \qquad [f]_\alpha = [(u, u')]_\alpha = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha\}$$

and

$$[f]^0 = [(u, u')]^0 = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1\}}; \qquad [f]_0 = [(u, u')]_0 = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0\}}.$$

We use the above to define (α, β)-level sets for $\mathcal{F} = (f, g)$, $0 < \alpha,\ \beta \leq 1$:

$$[\mathcal{F}]^{(\alpha,\beta)} = [(f, g)]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta, \tag{5}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = [(f, g)]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta, \tag{6}$$

Consider the following set of conditions as an alternative definition of $[\mathcal{F}]^{(\alpha,\beta)}$ and $[\mathcal{F}]_{(\alpha,\beta)}$:

$$[\mathcal{F}]^{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) \leq 1 - \beta\}\,, \tag{7}$$

$$[\mathcal{F}]^{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) < 1\}}, \tag{8}$$

$$[\mathcal{F}]^{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) \leq 1 - \beta\}}, \tag{9}$$

$$[\mathcal{F}]^{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) < 1\}}, \tag{10}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) \geq \beta\}\,, \tag{11}$$

$$[\mathcal{F}]_{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) > 0\}}, \tag{12}$$

$$[\mathcal{F}]_{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\ v(x) \geq \beta\}}, \tag{13}$$

$$[\mathcal{F}]_{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\ v(x) > 0\}}. \tag{14}$$

Note that (7) and (10) are equivalent to definition (5), likewise (11) and (14) are equivalent to definition (6) for the corresponding α, β, but (8), (9) and (12), (13) are not: (5) and (6) may not yield closed sets in the case when exactly one of α, β is equal to 0, but (8), (9) and (12), (13) would yield the respective closures of those sets.

For $f, g \in IF$, we have $[f]^\alpha \cap [g]^\beta$, $[f]_\alpha \cap [g]_\beta$ are always compact and $[f]^1 \cap [g]^1 \subset [f]^\alpha \cap [g]^\beta \subset [f]^0 \cap [g]^0$ and $[f]_1 \cap [g]_1 \subset [f]_\alpha \cap [g]_\beta \subset [f]_0 \cap [g]_0$ are nonempty as in order to ensure this, it is sufficient that $[f]^1 \cap [g]^1$ and $[f]_1 \cap [g]_1$ be nonempty, meaning that there should exist some $x_0,\, x_1 \in \mathbb{R}$ such that $x_0 \in [f]^1$, i.e.,$u'(x_0) = 0$, $x_0 \in [g]^1$,i.e. $v'(x_0) = 0$ and $x_1 \in [f]_1$, i.e.,$u(x_1) = 1$, $x_1 \in [g]_1$,i.e. $v(x_1) = 1$. With that in mind, we define the following set:

$$\hat{\mathbb{F}}^2 = \Big\{\Big((u,u'),\,(v,v')\Big) \in \mathbb{F}^2 \,|\, \exists x_0,\, x_1 \in \mathbb{R},\, /\, u(x_1) = v(x_1) = 1,\, u'(x_0) = v'(x_0) = 0\Big\}. \quad (15)$$

Then for $(f,g) \in \hat{\mathbb{F}}^2$, $[\mathcal{F}]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta$, $[\mathcal{F}]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta \in \mathcal{P}_k(\mathbb{R})$ for all $\alpha,\, \beta \in [0,1]$. And the compactness of the $[\mathcal{F}]^{(\alpha,\beta)}$ sets guarantees the complete equivalence of definition (5) and the set of definitions (7)-(10), and the complete equivalence of definition (6) and the set of definitions (11)-(14).

We recall that $\mathbb{F}$ is closed under addition and scalar multiplication, to establish a similar result for $\hat{\mathbb{F}}^2$. For functions $f = (u,u')$, $g = (v,v') \in \mathbb{F}$, addition and scalar multiplication can be defined via level sets as (1)-(4).

For $\mathcal{F} = (f,g) = \big((u,u'),(v,v')\big)$, $\mathcal{G} = (f',g') = \big((x,x'),(y,y')\big) \in \hat{\mathbb{F}}^2$ and a scalar λ, let

$$\mathcal{F} + \mathcal{G} = (f,g) + (f',g') = (f+f', g+g'), \quad (16)$$

$$\lambda\mathcal{F} = \lambda(f,g) = (\lambda f, \lambda g). \quad (17)$$

It is clear that for $\mathcal{F},\, \mathcal{G} \in \mathbb{F}$, $\lambda\mathcal{F},\, \mathcal{F}+\mathcal{G} \in \mathbb{F}\times\mathbb{F}$. We need to show that there exist $c_0,\, c_1 \in \mathbb{R}$ such that

$$(u+x)(c_0) = (v+y)(c_0) = 1, \text{ and } (u'+x')(c_1) = (v'+y')(c_1) = 0.$$

That is to say, $[\mathcal{F}+\mathcal{G}]^{(1,1)}$ and $[\mathcal{F}+\mathcal{G}]_{(1,1)}$ are nonempty, and also that $[\lambda\mathcal{F}]^{(1,1)}$ and $[\lambda\mathcal{F}]_{(1,1)}$ are likewise nonempty.

We know that there exist $a_0, a_1, b_0, b_1 \in \mathbb{R}$ such that $u(a_1) = v(a_1) = 1$, $u'(a_0) = v'(a_0) = 0$ and $x(b_1) = y(b_1) = 1$, $x'(b_0) = y'(b_0) = 0$, then

$$\begin{aligned}
[\lambda\mathcal{F}]^{(1,1)} &= [\lambda f]^1 \cap [\lambda g]^1 \\
&= \big\{\lambda z \,|\, z \in [f]^1\big\} \cap \big\{\lambda z \,|\, z \in [g]^1\big\} \\
&= \{\lambda z \,|\, u'(z) = 0\} \cap \{\lambda z \,|\, v'(z) = 0\}
\end{aligned}$$

Clearly $\lambda a_0 \in [\lambda\mathcal{F}]^{(1,1)}$. And

$$\begin{aligned}
[\lambda\mathcal{F}]_{(1,1)} &= [\lambda f]_1 \cap [\lambda g]_1 \\
&= \{\lambda z \,|\, z \in [f]_1\} \cap \{\lambda z \,|\, z \in [g]_1\} \\
&= \{\lambda z \,|\, u(z) = 1\} \cap \{\lambda z \,|\, v(z) = 1\}
\end{aligned}$$

$\lambda a_1 \in [\lambda\mathcal{F}]_{(1,1)}$. Also,

$$\begin{aligned}
[\mathcal{F}+\mathcal{G}]^{(1,1)} &= [(f+f', g+g')]^{(1,1)} \\
&= [f+f']^1 \cap [g+g']^1 \\
&= \big([f]^1 + [f']^1\big) \cap \big([g]^1 + [g']^1\big) \\
&= \big\{z_0 + z_1 \,|\, z_0 \in [f]^1,\, z_1 \in [f']^1\big\} \cap \big\{z_0 + z_1 \,|\, z_0 \in [g]^1,\, z_1 \in [g']^1\big\} \\
&= \{z_0 + z_1 \,|\, u'(z_0) = x'(z_1) = 0\} \cap \{z_0 + z_1 \,|\, v'(z_0) = y'(z_1) = 0\}
\end{aligned}$$

$a_0 + b_0 \in [\mathcal{F} + \mathcal{G}]^{(1,1)}$. And

$$\begin{aligned}
[\mathcal{F} + \mathcal{G}]_{(1,1)} &= [(f + f', g + g')]_{(1,1)} \\
&= [f + f']_1 \cap [g + g']_1 \\
&= ([f]_1 + [f]_1) \cap ([g]_1 + [g']_1) \\
&= \{z_0 + z_1 \,|\, z_0 \in [f]_1,\ z_1 \in [f']_1\} \cap \{z_0 + z_1 \,|\, z_0 \in [g]_1,\ z_1 \in [g']_1\} \\
&= \{z_0 + z_1 \,|\, u(z_0) = x(z_1) = 1\} \cap \{z_0 + z_1 \,|\, v(z_0) = y(z_1) = 1\}
\end{aligned}$$

Clearly $a_1 + b_1 \in [\mathcal{F} + \mathcal{G}]_{(1,1)}$.

We have thus shown that $\hat{\mathbb{F}}^2$ is closed under addition and scalar multiplication.

Consider the product metric on $\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, $\hat{d}_\infty : \mathbb{F}^2 \times \mathbb{F}^2 \longrightarrow \mathbb{R}^+$ by:

$$\hat{d}_\infty(\mathcal{F}, \mathcal{G}) = \max\{d_\infty(f, f'),\, d_\infty(g, g')\}, \quad \mathcal{F} = (f, g),\, \mathcal{G} = (f', g') \in \hat{\mathbb{F}}^2. \tag{18}$$

Since $\hat{\mathbb{F}}^2 \subset \mathbb{F}^2$, $\hat{d}_\infty$ is also a metric for $\hat{\mathbb{F}}^2$. Hence, $(\hat{\mathbb{F}}^2, \hat{d}_\infty)$ is a complete metric space.

It will also prove useful to define a zero element in $\hat{\mathbb{F}}^2$. Recall that on $\mathbb{F}$ we define zero element $0_{(1,0)} \in \mathbb{F}$ by

$$0_{(1,0)}(x) = \begin{cases} (1,0) & , x = 0 \\ (0,1) & , x \neq 0 \end{cases}$$

The zero element on $\hat{\mathbb{F}}^2$ then reads

$$\hat{0} = \big(0_{(1,0)}, 0_{(1,0)}\big) \in \mathbb{F}^2.$$

We have $\hat{0}(0) = \big((1,0), (1,0)\big)$, verifying that $\hat{0} \in \hat{\mathbb{F}}^2$.

From [5], there exists a Banach space $\mathcal{B}$ and an embedding $j : \mathbb{F} \longrightarrow \mathcal{B}$.

Thus, $\hat{\mathbb{F}}^2 \subset \mathbb{F} \times \mathbb{F}$ is embedded into a Banach space.

Remark 3. *In the same manner can be defined $\hat{\mathbb{F}}^n$, $n \geq 3$ and it is shown that is embedding into a Banach space.*

3.3 Polar representation of complex grades of membership and non-membership

The polar representation of the membership function as presented in [9], μ, is defined as

$$\mu(V, z) = r(V)e^{i\sigma\phi(z)}.$$

Likewise, we can define the polar representation of complex non-membership function as

$$\nu(V, z) = r'(V)e^{i\sigma\phi'(z)},$$

where σ is a scaling factor, which does not translate directly to and from the respective Cartesian representation. Therefore, the two representations of the corresponding extension to $\mathbb{R}$ are not equivalent as defined, which will be seen below. Thus, depending on the application, one may be more appropriate to use than the other.

For $x \in \mathbb{R}$, the polar form of f_1 and f_2 is defined as follows:

$$f_1(x) = r(x)e^{2\pi\phi(x)i}, \qquad f_2(x) = r'(x)e^{2\pi\phi'(x)i},$$

where $f = (r, r'), g = (\phi, \phi') : \mathbb{R} \longrightarrow [0, 1]^2$.

We denote $f_1 = (r, \phi)$ and $f_2 = (r', \phi')$. The scaling factor is taken to be 2π , allowing the range of f_1 and f_2 to be the entire unit circle. Because $e^{2\pi i\phi}$ is periodic, we take the value of ϕ giving the maximum distance from e^0, $\phi = 0.5$, to be the "maximum" membership or non-membership value.

The level sets for $f = (r, r')$, $[f]^\alpha$ and $[f]_\alpha$ are defined just as

$$[f]^\alpha = [(r, r')]^\alpha = \{x \in \mathbb{R} \,|\, r'(x) \leq 1 - \alpha\},$$

and

$$[f]_\alpha = [(r, r')]_\alpha = \{x \in \mathbb{R} \,|\, r(x) \geq \alpha\}.$$

And we define the level sets for $g = (\phi, \phi')$, denoted $[g]^{\langle\alpha\rangle}$ and $[g]_{\langle\alpha\rangle}$, must be defined differently to account for the periodicity:

$$[g]^{\langle\alpha\rangle} = \{x \in \mathbb{R} \,|\, \phi'(x) \in [\alpha, 1 - \alpha],\ \alpha \in (0, 0.5]\}, \tag{19}$$

$$[g]_{\langle\alpha\rangle} = \{x \in \mathbb{R} \,|\, \phi(x) \in [\alpha, 1 - \alpha],\ \alpha \in (0, 0.5]\}, \tag{20}$$

$$[g]^{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi'(x) < 1\}}, \tag{21}$$

$$[g]_{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi(x) < 1\}}, \tag{22}$$

$$[g]^{\langle\alpha\rangle} = [(\phi, \phi')]^{\langle 1-\alpha\rangle}, \quad [g]_{\langle\alpha\rangle} = [(\phi, \phi')]_{\langle 1-\alpha\rangle}, \text{ for all } \alpha \in [0, 1]. \tag{23}$$

For $\mathcal{F} = (f, g)$, we can then define the level sets $[\mathcal{F}]^{\langle\alpha,\beta\rangle}$ and $[\mathcal{F}]_{\langle\alpha,\beta\rangle}$ as

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [(f, g)]^{\langle\alpha,\beta\rangle} = [f]^{\langle\alpha\rangle} \cap [g]^{\langle\beta\rangle}, \quad \text{and} \quad [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [(f, g)]_{\langle\alpha,\beta\rangle} = [f]_{\langle\alpha\rangle} \cap [g]_{\langle\beta\rangle}, \tag{24}$$

or by the relations:

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r'(x) \leq 1 - \alpha,\ \phi'(x) \in [\beta, 1 - \beta]\}, \tag{25}$$

$$[\mathcal{F}]_{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\ \phi(x) \in [\beta, 1 - \beta]\}, \tag{26}$$

$$[\mathcal{F}]^{\langle\alpha,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) \leq 1 - \alpha,\ 0 < \phi'(x) < 1\}}, \tag{27}$$

$$[\mathcal{F}]_{\langle\alpha,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\ 0 < \phi(x) < 1\}}, \tag{28}$$

$$[\mathcal{F}]^{\langle 0,\beta\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\ \phi'(x) \in [\beta, 1 - \beta]\}}, \tag{29}$$

$$[\mathcal{F}]_{\langle 0,\beta\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\ \phi(x) \in [\beta, 1 - \beta]\}}, \tag{30}$$

$$[\mathcal{F}]^{\langle 0,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\ 0 < \phi'(x) < 1\}}, \tag{31}$$

$$[\mathcal{F}]_{\langle 0,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\ 0 < \phi(x) < 1\}}, \tag{32}$$

together with

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [\mathcal{F}]^{\langle\alpha,1-\beta\rangle}, \quad \text{and} \quad [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [\mathcal{F}]_{\langle\alpha,1-\beta\rangle}, \quad \text{for all}\, \alpha,\beta \in [0,1]. \tag{33}$$

It is clear that, for $g = (\phi,\phi') \in IF$, $[g]^{\langle\alpha\rangle} \subset [g]^{\alpha}$ and $[g]_{\langle\alpha\rangle} \subset [g]_{\alpha}$ for all $\alpha \in [0, 0.5]$. However, $[g]^{\langle\alpha\rangle}$, $[g]_{\langle\alpha\rangle}$ need not be compact or convex. In order to address this issue, we define

$$\hat{\mathcal{G}} = \left\{(u,v) : \mathbb{R} \longrightarrow [0,1]^2 \,\text{satisfying all of the following conditions}\right\},$$

1. There exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = v(x_1) = 0.5$.
2. u and v are monotone.
3. u is upper semi-continuous on K_1 and lower semi-continuous on K_2, with
$$K_1 = \{x \in \mathbb{R} \,|\, 0 < u(x) \le 0.5\}, \quad \text{and} \quad K_2 = \{x \in \mathbb{R} \,|\, 0.5 \le u(x) < 1\}.$$
4. v is lower semi-continuous on K_1' and upper semi-continuous on K_2', with
$$K_1' = \{x \in \mathbb{R} \,|\, 0 < v(x) \le 0.5\}, \quad \text{and} \quad K_2' = \{x \in \mathbb{R} \,|\, 0.5 \le v(x) < 1\}.$$
5. $\overline{K_1 \cup K_2}$ and $\overline{K_1' \cup K_2'}$ are compact.

Now, we define

$$\hat{\mathbb{F}}^2_* = \left\{\Big((r,r'),(\phi,\phi')\Big) \in \mathbb{F} \times \hat{\mathcal{G}} \,|\, \exists\, x_0, x_1 \in \mathbb{R}\,\text{s.t}\; r(x_0) = 1,\; r'(x_1) = 0,\; \phi(x_0) = \phi'(x_1) = 0.5\right\}.$$

Note that, for $\mathcal{F} \in \hat{\mathbb{F}}^2_*$, definition (24) is equivalent to the set of definitions (25)-(33).

For $f = (u,v) \in \hat{\mathcal{G}}$, we may write

$$f(x) = (u,v)(x) = \begin{cases} (u_1(x), v_1(x)), & x \in K_1 \cap K_1', \\ (u_2(x), v_2(x)), & x \in K_1 \cap K_2', \\ (u_3(x), v_3(x)), & x \in K_2 \cap K_1', \\ (u_4(x), v_4(x)), & x \in K_2 \cap K_2', \end{cases}$$

where for some (z_1, z_2), $(z_3, z_4) \in \mathbb{F}$,

$$u_1 = \frac{1}{2}z_1, \quad v_1 = \frac{1}{2}z_2 \quad u_2 = \frac{1}{2}z_1 \quad v_2 = \frac{1}{2}(2 - z_2),$$
$$u_3 = \frac{1}{2}(2 - z_3), \quad v_3 = \frac{1}{2}z_4 \quad u_4 = \frac{1}{2}(2 - z_3) \quad \text{and}\, v_2 = \frac{1}{2}(2 - z_4).$$

Thus, there exists an embedding l such that $l : \hat{\mathcal{G}} \longrightarrow \mathbb{F} \times IF$ by $f = (u,v) \longrightarrow ((z_1,z_2), (z_3,z_4))$, which implies there exists an embedding $k \equiv (id, l) : \mathbb{F} \times \hat{\mathcal{G}} \longrightarrow \mathbb{F} \times \mathbb{F} \times \mathbb{F}$, where id is the canonical identity map.

Now, if $u(x_0) = v(x_1) = 0.5$, we can choose (z_1, z_2), (z_3, z_4) so that

$$z_1(x_0) = z_2(x_1) = z_3(x_0) = z_4(x_1) = 1$$

hence, $\hat{\mathbb{F}}^2_*$ is embedded into $\hat{\mathbb{F}}^3$.

Since, as shown in the remark (3), $\hat{IF}^3$ is embedded into a Banach space, then so is $\hat{\mathbb{F}}^2_*$.

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