A New Approach to the Distances between Intuitionistic Fuzzy Sets

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1 Introduction and Preliminary Results

In a series of papers the author introduced a set of new negations and implications over Intuitionistic Fuzzy Sets (IFSs, for all notations about IFSs see [2]). Two of these negations were described in [6]. They generalize the classical negation over IFSs, but on the other hand, they have some non-classical properties. The set has the form

$$\mathcal{N} = \{\neg^{\varepsilon,\eta} \mid 0 \leq \varepsilon < 1 \ \& \ 0 \leq \eta < 1\},$$

where for each IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$:

$$\neg^{\varepsilon,\eta} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle | x \in E\}.$$

For numbers ε and η there are two cases.
• $\eta < \varepsilon$. As it is shown in [5], this case is impossible.
• $\eta \geq \varepsilon$. Let everywhere below $0 \leq \varepsilon \leq \eta < 1$ be fixed.
First, in [6] we showed that

$$\neg^{0,0} A = \{\langle x, \min(1, \nu_A(x)), \max(0, \mu_A(x))\rangle | x \in E\}$$

$$= \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} = \neg_1 A,$$

where $\neg_1$ is the classical intuitionistic fuzzy negation.
Second, we checked that set $\neg^{\varepsilon,\eta} A$ is an IFS, because

$$\min(1, b + \varepsilon) + \max(0, a - \eta) = \min(1, b + \varepsilon) \leq 1.$$

Third, we constructed a new implication, generated by the new negation:

$$A \rightarrow^{\varepsilon,\eta} B = \{\langle x, \max(c, \min(1, b + \varepsilon)), \min(d, \max(0, a - \eta))\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \max(c, b + \varepsilon)), \max(0, \min(d, a - \eta))\rangle | x \in E\}.$$

Fourth, in [2] the concepts of *Tautological Set (TS) and Intuitionistic Fuzzy Tautological Set* (IFTS) were introduced as follows: the IFS A is a TS iff for

E. Hüllermeier, R. Kruse, and F. Hoffmann (Eds.): IPMU 2010, Part I, CCIS 80, pp. 581–590, 2010.

every $x \in E : \mu_A(x) = 1, \nu_A(x) = 0$; the IFS A is an IFTS iff for every $x \in E$: $\mu_A(x) \geq \nu_A(x)$. Obviously, each TS is an IFTS.

In [6], we checked the axioms of intuitionistic logic (see, e.g., [18]) in the case, when A, B and C are IFSs.

Fifth, in [7], a series of new versions of operation "subtraction" was introduced. As a basis of the new versions of operation "subtraction" from [7], the well-known formula from set theory:

$$A - B = A \cap \neg B \tag{1}$$

was used, A and B being given sets.

On the other hand, as we discussed in [3], the Law for Excluded Middle is not always valid in IFS theory. By this reason, we can introduce a new series of "subtraction" operations, that will have the form:

$$A -'' B = \neg\neg A \cap \neg B. \tag{2}$$

In some papers, e.g., [7,19], the properties of some IF-subtractions were studied.

2 Main Results

2.1 In [8], using (1), we obtained the following two operations of subtraction:

$$A -'^{\varepsilon,\eta} B = A \cap \neg^{\varepsilon,\eta} B$$

$$= \{\langle x, \min(\mu_A(x), 1, \nu_B(x) + \varepsilon), \max(\nu_A(x), 0, \mu_B(x) - \eta)\rangle | x \in E\}$$
$$= \{\langle x, \min(\mu_A(x), \nu_B(x) + \varepsilon), \max(\nu_A(x), \mu_B(x) - \eta)\rangle | x \in E\}.$$

Also, using (2) and having in mind that

$$\neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta} A = \neg^{\varepsilon,\eta}\{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle x, | x \in E\}$$

$$= \{\langle x, \min(1, \max(0, \mu_A(x) - \eta) + \varepsilon), \max(0, \min(1, \nu_A(x) + \varepsilon) - \eta)\rangle x, | x \in E\}$$
$$= \{\langle x, \min(1, \max(\varepsilon, \mu_A(x) - \eta + \varepsilon)), \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta))\rangle x, | x \in E\}$$
$$= \{\langle x, \max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta))\rangle x, | x \in E\}$$

we obtain the following form of the operation $-''^{\varepsilon,\eta}$:

$$A -''^{\varepsilon,\eta} B = \neg^{\varepsilon,\eta}\neg^{\varepsilon,\eta} A \cap \neg^{\varepsilon,\eta} B$$

$$= \{\langle x, \max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta))\rangle x, | x \in E\}$$
$$\cap \{\langle x, \min(1, \nu_B(x) + \varepsilon), \max(0, \mu_B(x) - \eta)\rangle x, | x \in E\}$$
$$= \{\langle x, \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), 1, \nu_B(x) + \varepsilon),$$
$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)\rangle x, | x \in E\}$$
$$= \{\langle x, \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon),$$
$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)\rangle x, | x \in E\}.$$

Here, using the subtraction of sets A and $\neg^{\varepsilon,\eta} A$, we will introduce the following two (ε,η)-norms for element $x \in E$:

$$||x||'_{\varepsilon,\eta} = \langle \min(\mu_A(x), \nu_A(x) + \varepsilon), \max(\nu_A(x), \mu_A(x) - \eta)\rangle, \tag{3}$$

$$\begin{aligned} ||x||''_{\varepsilon,\eta} = \langle &\min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon), \\ &\max(0, \min(1-\eta, \nu_A(x) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle. \end{aligned} \tag{4}$$

Theorem 1. The two (ε,η)-norms are intuitionistic fuzzy pairs.

Proof. Let

$$X \equiv \min(\mu_A(x), \nu_A(x) + \varepsilon) + \max(\nu_A(x), \mu_A(x) - \eta).$$

If $\nu_A(x) \geq \mu_A(x) - \eta$, then

$$X = \min(\mu_A(x), \nu_A(x) + \varepsilon) + \nu_A(x) \leq \mu_A(x) + \nu_A(x) \leq 1.$$

If $\nu_A(x) > \mu_A(x) - \eta$, then

$$X = \min(\mu_A(x), \nu_A(x) + \varepsilon) + \mu_A(x) - \eta$$

$$\leq \nu_A(x) + \varepsilon + \mu_A(x) - \eta \leq 1 + \varepsilon - \eta < 1.$$

Therefore, the first norm is an intuitionistic fuzzy pair.

Let

$$Y \equiv \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon)$$
$$+ \max(0, \min(1-\eta, \nu_A(x) + \varepsilon - \eta), \mu_A(x) - \eta).$$

If $\max(0, \min(1-\eta, \nu_A(x) + \varepsilon - \eta), \mu_A(x) - \eta) = 0$, then

$$Y = \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon) + 0 \leq \max(\varepsilon, \mu_A(x) - \eta + \varepsilon) \leq 1.$$

If $\max(0, \min(1-\eta, \nu_A(x) + \varepsilon - \eta), \mu_A(x) - \eta) = \min(1-\eta, \nu_A(x) + \varepsilon - \eta)$, then

$$\min(1-\eta, \nu_A(x) + \varepsilon - \eta) \geq \mu_A(x) - \eta,$$

i.e., $\nu_A(x) + \varepsilon \geq \mu_A(x)$. Hence,

$$\mu_A(x) - \eta) \leq \nu_A(x) + \varepsilon - \eta) \leq \nu_A(x)$$

and therefore

$$Y = \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon) + \min(1-\eta, \nu_A(x) + \varepsilon - \eta)$$

$$= \min(\varepsilon + \max(0, \mu_A(x) - \eta), \nu_A(x) + \varepsilon) + \min(1-\eta, \nu_A(x) + \varepsilon - \eta)$$

$$= \varepsilon + \min(\max(0, \mu_A(x) - \eta), \nu_A(x)) + \min(1-\eta, \nu_A(x) + \varepsilon - \eta)$$

$$= \varepsilon + \max(0, \mu_A(x) - \eta) + \min(1, \nu_A(x) + \varepsilon) - \eta$$

$$\leq \max(0, \mu_A(x) - \eta) + \min(1, \nu_A(x) + \varepsilon).$$

If $\mu_A(x) \geq \eta$, then

$$Y \leq \mu_A(x) - \eta + \min(1, \nu_A(x) + \varepsilon) \leq \mu_A(x) - \eta + \nu_A(x) + \varepsilon \leq 1 - \eta + \varepsilon \leq 1;$$

if $\mu_A(x) < \eta$, then $Y \leq \min(1, \nu_A(x) + \varepsilon) \leq 1$.

If $\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(x) - \eta) = \mu_A(x) - \eta$, then

$$Y = \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon) + \mu_A(x) - \eta$$

$$\leq \nu_A(x) + \varepsilon + \mu_A(x) - \eta \leq 1 + \varepsilon - \eta \leq 1.$$

Therefore, the second norm is also an intuitionistic fuzzy pair.

All norms and distances, defined over IFSs up to now (see, e.g. [10, 11, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]), have been real numbers, that may in some cases be normalized to the [0, 1] interval. As we see, the two norms (3) and (4) have the form of intuitionistic fuzzy pairs and they are the first of this form.

Let $e^*, o^*, u^* \in E$ so that $\mu_A(e^*) = 1,\ \nu_A(e^*) = 0,\ \mu_A(o^*) = 0,\ \nu_A(o^*) = 1,\ \mu_A(u^*) = 0,\ \nu_A(u^*) = 0$. Then the norms of these three elements are:

x	$\|\|x\|\|'_{\varepsilon,\eta}$	$\|\|x\|\|''_{\varepsilon,\eta}$
e^*	$\langle \varepsilon, 1-\eta \rangle$	$\langle \varepsilon, 1-\eta)\rangle$
o^*	$\langle 0, 1 \rangle$	$\langle \varepsilon, 1-\eta \rangle$
u^*	$\langle 0, 0 \rangle$	$\langle \varepsilon, 0 \rangle$

2.2 There are two ways for introducing distances between elements of a fixed universe E, as it is shown on Fig. 1.

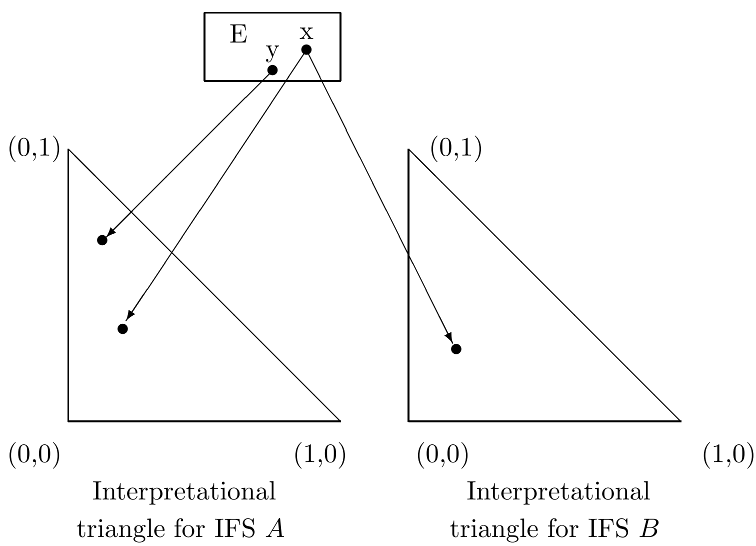

Fig. 1.

Up to now, all distances, similarly to all norms, have been real numbers. Here, and below, we will introduce distances, having forms of intuitionistic fuzzy pairs.

First, we will introduce the following five (ε, η)-distances between the values of one element $x \in E$ about two IFSs A and B, by analogy with the first (ε, η)-norm:

$$\begin{aligned}
d'_{\varepsilon,\eta;str_opt}(A, B)(x) = \langle &\min(\mu_A(x), \nu_B(x) + \varepsilon) + \min(\mu_B(x), \nu_A(x) + \varepsilon) \\
&- \min(\mu_A(x), \nu_B(x) + \varepsilon). \min(\mu_B(x), \nu_A(x) + \varepsilon), \\
&\max(\nu_A(x), \mu_B(x) - \eta). \max(\nu_B(x), \mu_A(x) - \eta)\rangle, \\
d'_{\varepsilon,\eta;opt}(A, B)(x) = \langle &\max(\min(\mu_A(x), \nu_B(x) + \varepsilon), \min(\mu_B(x), \nu_A(x) + \varepsilon)), \\
&\min(\max(\nu_A(x), \mu_B(x) - \eta), \max(\nu_B(x), \mu_A(x) - \eta))\rangle, \\
d'_{\varepsilon,\eta;aver}(A, B)(x) = \langle &\frac{\min(\mu_A(x), \nu_B(x) + \varepsilon) + \min(\mu_B(x), \nu_A(x) + \varepsilon)}{2}, \\
&\frac{\max(\nu_A(x), \mu_B(x) - \eta) + \max(\nu_B(x), \mu_A(x) - \eta)}{2}\rangle, \\
d'_{\varepsilon,\eta;pes}(A, B)(x) = \langle &\min(\min(\mu_A(x), \nu_B(x) + \varepsilon), \min(\mu_B(x), \nu_A(x) + \varepsilon)), \\
&\max(\max(\nu_A(x), \mu_B(x) - \eta), \max(\nu_B(x), \mu_A(x) - \eta))\rangle, \\
d'_{\varepsilon,\eta;str_pes}(A, B)(x) = \langle &\min(\mu_A(x), \nu_B(x) + \varepsilon). \min(\mu_B(x), \nu_A(x) + \varepsilon), \\
&\max(\nu_A(x), \mu_B(x) - \eta) + \max(\nu_B(x), \mu_A(x) - \eta) \\
&- \max(\nu_A(x), \mu_B(x) - \eta). \max(\nu_B(x), \mu_A(x) - \eta)\rangle.
\end{aligned}$$

Second, we will introduce the following five (ε, η)-distances between the values of two elements $x, y \in E$ about the IFS A, by analogy with the first (ε, η)-norm:

$$\begin{aligned}
d'_{\varepsilon,\eta;str_opt}(A)(x, y) = \langle &\min(\mu_A(x), \nu_A(y) + \varepsilon) + \min(\mu_A(y), \nu_A(x) + \varepsilon) \\
&- \min(\mu_A(x), \nu_A(y) + \varepsilon). \min(\mu_A(y), \nu_A(x) + \varepsilon), \\
&\max(\nu_A(x), \mu_A(y) - \eta). \max(\nu_A(y), \mu_A(x) - \eta)\rangle, \\
d'_{\varepsilon,\eta;opt}(A)(x, y) = \langle &\max(\min(\mu_A(x), \nu_A(y) + \varepsilon), \min(\mu_A(y), \nu_A(x) + \varepsilon)), \\
&\min(\max(\nu_A(x), \mu_A(y) - \eta), \max(\nu_A(y), \mu_A(x) - \eta))\rangle, \\
d'_{\varepsilon,\eta;aver}(A)(x, y) = \langle &\frac{\min(\mu_A(x), \nu_A(y) + \varepsilon) + \min(\mu_A(y), \nu_A(x) + \varepsilon)}{2}, \\
&\frac{\max(\nu_A(x), \mu_A(y) - \eta) + \max(\nu_A(y), \mu_A(x) - \eta)}{2}\rangle, \\
d'_{\varepsilon,\eta;pes}(A)(x, y) = \langle &\min(\min(\mu_A(x), \nu_A(y) + \varepsilon), \min(\mu_A(y), \nu_A(x) + \varepsilon)), \\
&\max(\max(\nu_A(x), \mu_A(y) - \eta), \max(\nu_A(y), \mu_A(x) - \eta))\rangle, \\
d'_{\varepsilon,\eta;str_pes}(A)(x, y) = \langle &\min(\mu_A(x), \nu_A(y) + \varepsilon). \min(\mu_A(y), \nu_A(x) + \varepsilon), \\
&\max(\nu_A(x), \mu_A(y) - \eta) + \max(\nu_A(y), \mu_A(x) - \eta) \\
&- \max(\nu_A(x), \mu_A(y) - \eta). \max(\nu_A(y), \mu_A(x) - \eta)\rangle.
\end{aligned}$$

Theorem 2. The ten (ε, η)-distances are intuitionistic fuzzy pairs.

The proof is similar to the above one.

As illustration, we will calculate the distances between the pairs (e^*, o^*), (e^*, u^*) and (u^*, o^*).

	(e^*, o^*)	(e^*, u^*)	(o^*, u^*)
$d'_{\varepsilon,\eta;str\_opt}(A)$	$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0, 0\rangle$
$d'_{\varepsilon,\eta;aver}(A)$	$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0, 0\rangle$
$d'_{\varepsilon,\eta;opt}(A)$	$\langle \frac{1}{2}, \frac{1}{2}\rangle$	$\langle 0, 1\rangle$	$\langle 0, \frac{1}{2}\rangle$
$d'_{\varepsilon,\eta;pes}(A)$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
$d'_{\varepsilon,\eta;str\_pes}(A)$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$

Third, we will introduce the following five (ε, η)-distances between the values of two elements $x, y \in E$ about the IFS A, by analogy with the second (ε, η)-norm:

$$d''_{\varepsilon,\eta;str\_opt}(A,B)(x) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon)$$

$$+ \min(\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon) - \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon)$$

$$. \min(\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon), \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)$$

$$. \max(0, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle,$$

$$d''_{\varepsilon,\eta;opt}(A,B)(x) = \langle \max(\min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon),$$

$$\min(\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon)),$$

$$\min(\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta),$$

$$\max(0, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta))\rangle,$$

$$d''_{\varepsilon,\eta;aver}(A,B)(x) = \langle \frac{1}{2}.(\min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon)$$

$$+ \min(\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon)),$$

$$\frac{1}{2}.(\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)$$

$$+ \max(0, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta))\rangle,$$

$$d''_{\varepsilon,\eta;pes}(A,B)(x) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon, \nu_B(x) + \varepsilon),$$

$$\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon),$$

$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle,$$

$$d''_{\varepsilon,\eta;str\_pes}(A,B)(x) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_B(x) + \varepsilon).$$

$$\min(\max(\varepsilon, \mu_B(x) - \eta + \varepsilon), \nu_A(x) + \varepsilon),$$

$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)$$

$$+ \max(0, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta)$$

$$- \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_B(x) - \eta)$$

$$. \max(0, \min(1 - \eta, \nu_B(x) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle.$$

Fourth, we will introduce the following five (ε, η)-distances between the values of two elements $x, y \in E$ about the IFS A, by analogy with the second (ε, η)-norm:

$$d''_{\varepsilon,\eta;str\_opt}(A)(x,y) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(y) + \varepsilon)$$

$$+ \min(\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon)$$

$$- \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(y) + \varepsilon)$$

$$. \min(\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon),$$

$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta)$$

$$. \max(0, \min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle,$$

$$d''_{\varepsilon,\eta;opt}(A)(x,y) = \langle \max(\min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(y) + \varepsilon),$$

$$\min(\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon)),$$

$$\min(\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta),$$

$$\max(0, \min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta))\rangle,$$

$$d''_{\varepsilon,\eta;aver}(A)(x,y) = \langle \frac{1}{2}.(\min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(y) + \varepsilon)$$

$$+ \min(\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon)),$$

$$\frac{1}{2}.(\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta)$$

$$+ \max(0, \min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta))\rangle,$$

$$d''_{\varepsilon,\eta;pes}(A)(x,y) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon, \nu_A(y) + \varepsilon),$$

$$\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon),$$

$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta,$$

$$\min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle,$$

$$d''_{\varepsilon,\eta;str\_pes}(A)(x,y) = \langle \min(\max(\varepsilon, \mu_A(x) - \eta + \varepsilon), \nu_A(y) + \varepsilon)$$

$$. \min(\max(\varepsilon, \mu_A(y) - \eta + \varepsilon), \nu_A(x) + \varepsilon),$$

$$\max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta)$$

$$+ \max(0, \min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta)$$

$$- \max(0, \min(1 - \eta, \nu_A(x) + \varepsilon - \eta), \mu_A(y) - \eta)$$

$$. \max(0, \min(1 - \eta, \nu_A(y) + \varepsilon - \eta), \mu_A(x) - \eta)\rangle.$$

Theorem 3. The last ten (ε, η)-distances are intuitionistic fuzzy pairs.

The proof is similar to the above one.

2.3 Now, we will discuss new distances between two given IFSs A and B. Up to now, they have also been real numbers. Here, for the first time, we will introduce a whole IFS, representing the distances between the origins of each element $x \in E$ with respect to the two sets. This set-form of distances can have different forms, but we will describe the following ten of them:

$$D'(A,B)_{\varepsilon,\eta;type} = \{\langle x, \mu_{d'(A,B)_{\varepsilon,\eta;type}(x)}, \nu_{d'(A,B)_{\varepsilon,\eta;type}(x)}\rangle | x \in E\},$$

$$D''(A,B)_{\varepsilon,\eta;type} = \{\langle x, \mu_{d''(A,B)_{\varepsilon,\eta;type}(x)}, \nu_{d''(A,B)_{\varepsilon,\eta;type}(x)}\rangle | x \in E\},$$

where

$$\langle \mu_{d'(A,B)_{\varepsilon,\eta;type}(x)}, \nu_{d'(A,B)_{\varepsilon,\eta;type}(x)}\rangle = d'_{\varepsilon,\eta;type}(A,B)(x),$$

$$\langle \mu_{d''(A,B)_{\varepsilon,\eta;type}(x)}, \nu_{d''(A,B)_{\varepsilon,\eta;type}(x)}\rangle = d''_{\varepsilon,\eta;type}(A,B)(x)$$

for "*type*" $\in$ {"*str_opt*", "*opt*", "*aver*", "*pes*", "*str_pes*"}.

The so constructed sets are IFSs and the proof of this fact is similar to the above proof. Now, we can introduce the numerical form of the distances between IFSs A and B by analogy with ordinary intuitionistic fuzzy distances (see, e.g., [2]).

3 Conclusion

The so constructed norms and distances generalize the norms and distances from [2]. The latter are obtained from the above ones in the case when $\varepsilon = \eta = 0$. On the other hand, the new objects generate different problems, e.g., for construction of their integral representations. The idea for set-forms of the distances can be transformed for all other forms of the intuitionistic fuzzy norms and distances, discussed in the literature.

On the other hand, the above idea was generated in respect of author's research on Conway's Game of Life (see, e.g., [12]). It is a popular zero-player game, devised by John Horton Conway in 1970, and it is the best-known example of a cellular automaton. Its "universe" is an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, live or dead. Every cell interacts with its eight neighbours, which are the cells that are directly horizontally, vertically, or diagonally adjacent. In a stepwise manner, the state of each cell in the grid preserves or alternates with respect to a given list of rules.

In future we shall propose an intuitionistic fuzzy estimation of the cells' state in a modified Game of Life. For each cell we can define its IF estimation as a pair consisting of the degrees l_p and l_a, namely degrees of presence and absence of life, where $l_p + l_a \leq 1$. In the classical Conway's Game of Life, the live and dead states correspond to the elementary IF estimations $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. In a future research, using the above formulas for norms and distances, we can calculate the IF state of liveliness of each cell, as functions of the current states of the cell's neighbours. Criteria of liveliness will also be determined in terms of IFS.

Acknowledgements

The author is very thankful to Prof. Janusz Kacprzyk for his advices for improving of present paper.

The author is grateful for the support provided by the projects DID-02-29 “Modelling processes with fixed development rules” and BIn-2/09 “Design and development of intuitionistic fuzzy logic tools in information technologies” funded by the National Science Fund, Bulgarian Ministry of Education, Youth, and Science.

References

1. Atanassov, K.: Two variants of intuitionistic fuzzy propositional calculus, Preprint IM-MFAIS-5-88, Sofia (1988)
2. Atanassov, K.: Intuitionistic Fuzzy Sets. Springer, Berlin (1999)
3. Atanassov, K.: On intuitionistic fuzzy negations and De Morgan Laws. In: Proc. of Eleventh International Conf., IPMU 2006, Paris, July 2-7, pp. 2399–2404 (2006)
4. Atanassov, K.: On some intuitionistic fuzzy implication. Comptes Rendus de l'Academie bulgare des Sciences 59(1), 19–24 (2006)
5. Atanassov, K.: Intuitionistic fuzzy implication $\rightarrow^{\varepsilon,\eta}$ and intuitionistic fuzzy negation $\neg^{\varepsilon,\eta}$. Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics 1, 1–10 (2008)
6. Atanassov, K.: 25 years of intuitionistic fuzzy sets or the most interesting results and the most important mistakes of mine. In: Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. I: Foundations, pp. 1–35. Academic Publishing House EXIT, Warszawa (2008)
7. Atanassov, K.: Remark on operations “subtraction” over intuitionistic fuzzy sets. Notes on Intuitionistic Fuzzy Sets 15(3), 24–29 (2009), `http://ifigenia.org/wiki/issue:nifs/15/3/24-29`
8. Atanassov, K.: Intuitionistic fuzzy subtractions $-'^{\varepsilon,\eta}$ and $-''^{\varepsilon,\eta}$. In: Eight Polish Conf. on Intuitionistic Fuzzy Sets and Generalized Nets, Warsaw, October 16 (in press, 2009)
9. Atanassov, K., Dimitrov, D.: On the negations over intuitionistic fuzzy sets. Part 1. Annual of “Informatics” Section Union of Scientists in Bulgaria 1, 49–58 (2008), `http://ifigenia.org/wiki/issue:usb-2008-1-49-58`
10. Ban, A.: Intuitionistic Fuzzy Measures. Theory and Applications. Nova Sci. Publishers, New York (2006)
11. Chen, T.-Y.: A note on distances between intuitionistic fuzzy sets and/or interval valued fuzzy sets based on the Hausdorff metric. Fuzzy Sets and Systems 158(22), 2523–2525 (2007)
12. Gardner, M.: Mathematical Games: The fantastic combinations of John Conway's new solitaire game “Life”. Scientific American 223, 120–123 (1970)
13. Grzegorzewski, P.: Distances and orderings in a family of intuitionistic fuzzy numbers. In: Proc. of the Third Conf. of the European Society for Fuzzy Logic and Technology, EUSFLAT 2003, Zittau, September 10-12, pp. 223–227 (2003)
14. Grzegorzewski, P.: The Hamming distance between intuitionistic fuzzy sets. In: De Baets, B., Kaynak, O., Bilgiç, T. (eds.) IFSA 2003. LNCS (LNAI), vol. 2715, pp. 35–38. Springer, Heidelberg (2003)

15. Grzegorzewski, P.: On measuring distances between intuitionistic fuzzy sets. In: Atanassov, K., Kacprzyk, J., Krawczak, M. (eds.) Issues in Intuitionistic Fuzzy Sets and Generalized Nets, vol. 2, pp. 107–115. Wydawnictwo WSISiZ, Warszawa (2004)
16. Guo, X.J.: Li: Entropy, distance measure and similarity measure of intuitionistic fuzzy sets and their relations. Mathematics in Practice and Theory 37(4), 109–113 (2007)
17. Klir, G.: Bo Yuan: Fuzzy Sets and Fuzzy Logic. Prentice Hall, New Jersey (1995)
18. Rasiova, H., Sikorski, R.: The Mathematics of Metamathematics. Warszawa, Pol. Acad. of Sci. (1963)
19. Riečan, B., Boyadzhieva, D., Atanassov, K.: On intuitionistic fuzzy subtraction related to intuitionistic fuzzy negation $\neg_{11}$. Notes on Intuitionistic Fuzzy Sets 15(4), 1–6 (2009), http://ifigenia.org/wiki/issue:nifs/15/4/1-6
20. Song, Y., Zhang, Q., Zhou, X.: Supplier selection model based on distance measure between intuitionistic fuzzy sets. In: Proc. of IEEE International Conference on Systems, Man and Cybernetics, vol. 5, pp. 3795–3799
21. Szmidt, E., Kacprzyk, J.: Distances between intuitionistic fuzzy sets. Fuzzy Sets and Systems 114(4), 505–518 (2000)
22. Szmidt, E., Kacprzyk, J.: Distance from consensus under intuitionistic fuzzy preferences. In: Proceedings of EUROFUSE Workshop on Preference Modelling and Applications, EUROFUSE-PM 2001, Granada, April 25-27 (2001)
23. Szmidt, E., Kacprzyk, J.: Analysis of agreements in a group of experts via distances between intuitionistic fuzzy preferences. In: Proc. of the Ninth International Conf., IPMU 2002, Annecy, France, July 1-5, vol. III, pp. 1859–1865 (2002)
24. Szmidt, E., Kacprzyk, J.: Group agreement analysis via distances and entropy of intuitionistic fuzzy sets. In: Proceedings of Int. Conf. on Fuzzy Information Processing: Theories and Applications, Beijing, China, March 1-4, vol. I, pp. 631–635 (2003)
25. Szmidt, E., Kacprzyk, J.: Distances between intuitionistic fuzzy sets and their applications in reasoning. In: Halgamude, S., Wang, L. (eds.) Computational Intelligence for Modelling and Prediction, pp. 101–116. Springer, Berlin (2005)
26. Szmidt, E., Kacprzyk, J.: Distances between intuitionistic fuzzy sets: straightforward approaches may not work. In: Proc. of 3rd Int. IEEE Conf. "Intelligent Systems" IS 2006, London, September 4-6, pp. 716–721 (2006)
27. Tan, C., Zhang, Q.: Aggregation of opinion in group decision making based on intuitionistic fuzzy distances. Mathematics in Practice and Theory 36(2), 119–124 (2006) (in Chinese)
28. Wagn, W., Xin, X.: Distance measure between intuitionistic fuzzy sets. Pattern Recognition Letters 26(13), 2063–2069 (2005)
29. Yang, Y., Chiclana, F.: Intuitionistic fuzzy sets: Spherical representation and distances. Int. J. of Intelligent Systems 24, 399–420 (2009)