Notes on Intuitionistic Fuzzy Sets
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Note on one inequality and its application in intuitionistic fuzzy sets theory. Part 1

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Abstract: The inequality $\mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} \leq \frac{1}{2}$ is introduced and proved, where μ and ν are real numbers, for which $\mu, \nu \in [0, 1]$ and $\mu + \nu \leq 1$. The same inequality is valid for $\mu = \mu_A(x)$, $\nu = \nu_A(x)$, where μ_A and ν_A are the membership and the non-membership functions of an arbitrary intuitionistic fuzzy set A over a fixed universe E and $x \in E$. Also, a generalization of the above inequality for arbitrary $n \geq 2$ is proposed and proved.
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1 Introduction

The Intuitionistic Fuzzy Sets (IFSs) are introduced by K. Atanassov in [1,2] as follows. Let E be a universal set, $\mu_A, \nu_A : E \to I := [0, 1]$ be mappings and for each $x \in E$:

$$\mu_A(x) + \nu_A(x) \leq 1. \tag{1}$$

Then the set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called an IFS.

Mappings μ_A and ν_A are called membership and non-membership functions for the element $x \in E$ to the set $A \subseteq E$.

When for each $x \in E$:

$$\mu_A(x) + \nu_A(x) = 1, \tag{2}$$

the set A is transformed to the ordinary fuzzy (Zadeh's) set [4].

2 Main results

The main result of the paper is the following Theorem 1.

Theorem 1. *Let* $\mu, \nu \in I$ *be real numbers satisfying inequality*

$$\mu + \nu \leq 1. \tag{3}$$

Then the inequality

$$\mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} \leq \frac{1}{2} \tag{4}$$

holds, where the equality is possible if and only if $\mu = \nu = \frac{1}{2}$.

Before giving the proof of Theorem 1, we need the following lemma.

Lemma. *Let the function* f *be given by*

$$f(x) = (1-x)^{\frac{1}{x}} := e^{\frac{\ln(1-x)}{x}}. \tag{5}$$

Then function f *is strictly concave on interval* $(0,1)$ *and also strictly decreasing on the same interval. Also,* $f(1) = 0$ *and if we define* $f(0) := \lim_{x \to o^+} f(x)$*, then* $f(0) = \frac{1}{e}$.

Proof. Using (5) we obtain:

$$\left(\frac{d}{dx}\right) f(x) = f(x) . \frac{d}{dx}\left(\frac{\ln(1-x)}{x}\right) \tag{6}$$

and

$$\left(\frac{d}{dx}\right)^2 f(x) = f(x) . \left(\frac{d}{dx}\left(\frac{\ln(1-x)}{x}\right)\right)^2 + \left(\frac{d}{dx}\right)^2 \left(\frac{\ln(1-x)}{x}\right). \tag{7}$$

For $x \in (0,1)$ we have

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Hence,

$$\frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1}. \tag{8}$$

The series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ converges uniformly on each compact set $[\delta_1, \delta_2]$, where $0 < \delta_1 < \delta_2 < 1$. This follows from Weierstrass criterion for uniform convergence of series (see [3]), since

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \leq \sum_{n=0}^{\infty} \frac{\delta_2^n}{n+1},$$

and the series $\sum_{n=0}^{\infty} \frac{\delta_2^n}{n+1}$ converges (from D'Alembert criterion for convergence of series (see [3]). Therefore, (8) yields

$$\frac{d}{dx}\left(\frac{\ln(1-x)}{x}\right) = -\sum_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n. \tag{9}$$

Since, $f(x) > 0$, formulas (6) and (9) imply that f is strictly decreasing on $(0,1)$.

The series (9) converges uniformly on each compact set $[\delta_1, \delta_2]$, where $0 < \delta_1 < \delta_2 < 1$. Therefore,

$$\left(\frac{d}{dx}\right)^2 \left(\frac{\ln(1-x)}{x}\right) = -\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3} x^n \tag{10}$$

and the series (10) converges (from D'Alembert criterion). Then, to prove that f is strictly concave, we need to prove that

$$\left(\frac{d}{dx}\right)^2 f(x) < 0 \tag{11}$$

for $x \in (0,1)$. Because of $(7), (9)$ and (10), inequality (11) is equivalent to the inequality

$$\left(\sum_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n\right)^2 < \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3} x^n. \tag{12}$$

The left-hand side of inequality (12) must be understood as a Cauchy–Mertens multiplication of the series $\sum\limits_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n$ by itself, i.e.,

$$\begin{aligned}\left(\sum_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n\right)^2 &= \left(\sum_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(n+1)}{n+2} x^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{k+1}{k+2} . \frac{n-k+1}{n-k+2}\right) x^n. \end{aligned} \tag{13}$$

From (13), inequality (12) is equivalent to

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{k+1}{k+2} . \frac{n-k+1}{n-k+2}\right) x^n < \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3} x^n. \tag{14}$$

To prove (14), it is enough to prove that the inequality

$$\sum_{k=0}^{n} \frac{k+1}{k+2} . \frac{n-k+1}{n-k+2} < \frac{(n+1)(n+2)}{n+3} \tag{15}$$

holds. For this aim, we use the equality

$$\frac{1}{n+4} . \left(\frac{1}{k+2} + \frac{1}{n-k+2}\right) = \frac{1}{(k+2)(n-k+2)}. \tag{16}$$

Because of (16), we may rewrite (15) in the form

$$\sum_{k=0}^{n} \frac{1}{n+4} . \left(\frac{1}{k+2} + \frac{1}{n-k+2}\right) .(k+1)(n-k+1) < \frac{(n+1)(n+2)}{n+3}. \tag{17}$$

But (17) is equivalent to

$$\sum_{k=0}^{n} \frac{1}{n+4} . \frac{k+1}{k+2} .(n-k+1) + \sum_{k=0}^{n} \frac{1}{n+4} .(k+1). \frac{n-k+1}{n-k+2} < \frac{(n+1)(n+2)}{n+3}. \tag{18}$$

Since $\frac{k+1}{k+2} < 1$ and $\frac{n-k+1}{n-k+2} < 1$, (18) will be proved if we know that the inequality

$$\frac{1}{n+4}.\left(\sum_{k=0}^{n}(n-k+1)+\sum_{k=0}^{n}(k+1)\right) < \frac{(n+1)(n+2)}{n+3}. \tag{19}$$

holds. But after a simple calculation, the left-hand side of (19) equals to $\frac{(n+1)(n+2)}{n+4}$.

This proves (19) since

$$\frac{(n+1)(n+2)}{n+4} < \frac{(n+1)(n+2)}{n+3}.$$

Then the above mentioned function f is strictly concave on interval $(0,1)$. Also,

$$\lim_{x\to o^+} f(x) = \lim_{x\to o^+}(1-x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^y = \frac{1}{e}$$

and the Lemma is proved. □

Corollary 1. *If* $x \in (0,1)$ *and* $x \neq \frac{1}{2}$*, then*

$$f(x)+f(1-x) < 2f\left(\frac{1}{2}\right) = 2.\frac{1}{4} = \frac{1}{2}. \tag{20}$$

When $x = \frac{1}{2}$, (20) *is an equality.*
Proof. Since f is strictly concave on $(0,1)$, then

$$\frac{f(x)+f(1-x)}{2} < f\left(\frac{x+(1-x)}{2}\right) = f\left(\frac{1}{2}\right)$$

and (20) holds. □

Corollary 2. *If* $\mu,\nu \in (0,1)$ *and* $\mu+\nu = 1$*, then* (4) *holds.*
Proof. If $\mu = \nu = \frac{1}{2}$, then, obviously, (4) is an equality.
Let $\mu \neq \nu$. We put $\mu = 1-x,\ \nu = x$. Hence, $x \in (0,1)$. Since

$$f(x)+f(1-x) = (1-x)^{\frac{1}{x}} + x^{\frac{1}{1-x}},$$

then (20) yields exactly (4). □

We must note that Corollary 2 means that for fuzzy sets (4) is always true for $\mu,\nu \in I$, since

$$f(0) = \frac{1}{e} < \frac{1}{2}$$

and $f(1) = 0$.

Proof of Theorem 1. From Corollary 1, we have that Theorem 1 is valid for $\mu+\nu = 1$.
Let $\mu,\nu \in (0,1)$ and $\mu+\nu < 1$. Also, let $\alpha = 1-\mu$. Therefore, $\alpha \in (0,1)$. Then we have

$$\mu^{\frac{1}{\alpha}} + \alpha^{\frac{1}{\mu}} \leq \frac{1}{2}. \tag{21}$$

But, also, we have

$$0 < \nu < \alpha < 1 \tag{22}$$

and

$$1 < \frac{1}{\alpha} < \frac{1}{\nu}. \tag{23}$$

From $0 < \mu < 1$ and (23) we obtain

$$\mu^{\frac{1}{\nu}} < \mu^{\frac{1}{\alpha}}. \tag{24}$$

From (22) we obtain

$$\nu^{\frac{1}{\mu}} < \alpha^{\frac{1}{\mu}}. \tag{25}$$

Now, (24) and (25) yield

$$\mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} < \mu^{\frac{1}{\alpha}} + \alpha^{\frac{1}{\mu}}. \tag{26}$$

From (21) and (26) inequality (4) holds immediately. Thus, Theorem 1 holds for the case $\mu, \nu \in (0, 1)$.

It remains only to consider the following cases:

1. $\mu = 0$,
2. $\mu = 1$.

If Case 1 holds, we consider the subcases:

1.1. If $\nu = 0$, then $\mu^{\frac{1}{\nu}}$ and $\nu^{\frac{1}{\mu}}$ takes the form $0^{\frac{1}{0}}$, which we may consider (after putting $\mu = x$) as

$$\lim_{x\to 0^+} x^{\frac{1}{x}} = \lim_{y\to\infty} \left(\frac{1}{y}\right)^y = 0.$$

Therefore, (4) is obviously true.

1.2. If $\nu \neq 0$ is true, then $\mu^{\frac{1}{\nu}} = 0^{\frac{1}{\nu}} = 0$.

Also, when $0 < \nu < 1$, $\nu^{\frac{1}{\mu}} = \nu^{+\infty} = 0$.

If $\nu = 1$, then $\nu^{\frac{1}{\mu}}$ takes the form $1^{+\infty}$, which we understand (after putting $\mu = x$) as

$$\lim_{x\to 0^+} (1-x)^{\frac{1}{x}} = \lim_{y\to\infty} \left(1 - \frac{1}{y}\right)^y = \frac{1}{e}.$$

Therefore, (4) is true, since $\frac{1}{e} < \frac{1}{2}$.

Let Case 2 hold. Then $\nu = 0$, because of the conditions $0 \leq \nu$ and $\mu + \nu = 1$. Therefore,

$$\nu^{\frac{1}{\mu}} = 0^1 = 0.$$

Also, we have that $\mu^{\frac{1}{\nu}}$ takes the form $1^{+\infty}$, which (after putting $\nu = x$) we consider as

$$\lim_{x\to 0^+} (1-x)^{\frac{1}{x}} = \lim_{y\to\infty} \left(1 - \frac{1}{y}\right)^y = \frac{1}{e}.$$

Therefore, (4) is again true and Theorem 1 is proved. □

Finally, we will give an unexpected form of Theorem 1 for the case of fuzzy sets.

Theorem 2. *Let $\mu, \nu \in (0,1)$ and $\mu + \nu = 1$. Then the inequality*

$$\mu^{1+\mu+\mu^2+\cdots} + \nu^{1+\nu+\nu^2+\cdots} \leq \frac{1}{2} \tag{27}$$

holds and the equality is possible if and only if $\mu = \nu = \frac{1}{2}$.

Proof. We use (4) that is proved in Theorem 1. But since $\mu + \nu = 1$, we may rewrite (4) in the form

$$\mu^{\frac{1}{1-\mu}} + \nu^{\frac{1}{1-\nu}} \leq \frac{1}{2}. \tag{28}$$

Since $\mu, \nu \in (0,1)$, we have

$$\frac{1}{1-\mu} = 1 + \mu + \mu^2 + \ldots, \quad \frac{1}{1-\nu} = 1 + \nu + \nu^2 + \ldots. \tag{29}$$

Then, from (28) and (29) inequality (27) holds. □

Let us now look at one generalization of (4) for arbitrary $n \geq 2$, proving the following theorem.

Theorem 3. *Let $n \geq 2$ be an arbitrary integer and $x_i \in (0,1), i = 1, 2, \ldots, n$ be real numbers, such that*

$$\sum_{i=1}^{n} x_i = 1. \tag{30}$$

Then the inequality

$$\frac{(1-x_1)^{\frac{1}{x_1}} + (1-x_2)^{\frac{1}{x_2}} + \cdots + (1-x_n)^{\frac{1}{x_n}}}{n} \leq \left(1 - \frac{1}{n}\right)^n \tag{31}$$

holds and the equality is possible if and only if $x_1 = x_2 = \cdots = \frac{1}{n}$.

Proof. Since $f(x) = (1-x)^{\frac{1}{x}}$ is concave on $(0,1)$, then for arbitrary $x_i \in (0,1), i = 1, 2, \ldots, n$ and from (30), we have

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = f\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n$$

and the equality holds if and only if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

If we rewrite (31) in the form

$$(1-x_1)^{\frac{1}{x_1}} + (1-x_2)^{\frac{1}{x_2}} + \cdots + (1-x_n)^{\frac{1}{x_n}} \leq n\left(1 - \frac{1}{n}\right)^n, \tag{32}$$

then, for $n = 2$, putting $x_1 = \nu, x_2 = \mu$, we obtain exactly (4). So, (32) is a generalization of (4) for arbitrary $n \geq 2$.

Since the sequence $\left\{\left(1 - \frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is strictly increasing and $\lim\limits_{n\to+\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$, then as a corollary of Theorem 3, we obtain the inequality

$$\frac{(1-x_1)^{\frac{1}{x_1}} + (1-x_2)^{\frac{1}{x_2}} + \cdots + (1-x_n)^{\frac{1}{x_n}}}{n} < \frac{1}{e}. \tag{33}$$

But now, we observe that if $x_i \in (0,1)$, the (33) holds, i.e., we do not need the condition $x_1 + x_2 + \cdots + x_n = 1$.

Indeed, since $f(x) = (1-x)^{\frac{1}{x}}$ is strictly decreasing on $(0,1)$ (see the Lemma), then

$$(1-x_i)^{\frac{1}{x_i}} < \lim_{x \to 0^+} f(x) = \frac{1}{e}.$$

Therefore, we have

$$\frac{(1-x_1)^{\frac{1}{x_1}} + (1-x_2)^{\frac{1}{x_2}} + \cdots + (1-x_n)^{\frac{1}{x_n}}}{n} < \frac{\frac{1}{e} + \frac{1}{e} + \cdots + \frac{1}{e}}{n} = \frac{1}{e}$$

and (33) is proved. □

Finally, we must mention that if A is a fixed IFS over a universe E, then we can construct the following two new sets

$$B = \{\langle x, \mu_A(x)^{\frac{1}{\nu_A(x)}}, \nu_A(x)^{\frac{1}{\mu_A(x)}} \rangle | x \in E\}$$

and

$$C = \{\langle x, \mu_A(x)^{\frac{1}{\nu_A(x)}} + \nu_A(x)^{\frac{1}{\mu_A(x)}}, \pi_A(x)^{\frac{1}{\pi_A(x)}} \rangle | x \in E\}.$$

These sets are IFSs, because for each $x \in E$: $\mu_A(x)^{\frac{1}{\nu_A(x)}}, \nu_A(x)^{\frac{1}{\mu_A(x)}} \in [0,1]$ and

$$\mu_A(x)^{\frac{1}{\nu_A(x)}} + \nu_A(x)^{\frac{1}{\mu_A(x)}} \leq \mu_A(x) + \nu_A(x) \leq 1;$$

and $\mu_A(x)^{\frac{1}{\nu_A(x)}} + \nu_A(x)^{\frac{1}{\mu_A(x)}}, \pi_A(x)^{\frac{1}{\pi_A(x)}} \in [0,1]$ and

$$\mu_A(x)^{\frac{1}{\nu_A(x)}} + \nu_A(x)^{\frac{1}{\mu_A(x)}} + \pi_A(x)^{\frac{1}{\pi_A(x)}} \leq \mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

3 Conclusion

In the second part of the present paper, we will represent a new inequality which one may deduce with the help of (4) and the well-known Young's inequality for product. This new inequality also allows IFS interpretation.

References

[1] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer, Heidelberg.

[2] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[3] Fikhtengolts, G. (1965). *The Fundamentals of Mathematical Analysis*. Vol. 2, Elsevier.

[4] Zadeh, L. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353.