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On intuitionistic fuzzy ideals in Γ-near-rings

N. Palaniappan[1], P. S. Veerappan[2] and D. Ezhilmaran[3]

[1] Alagappa University, Karaikudi – 630 003, Tamilnadu, India,
e-mail: palaniappan.nallappan@yahoo.com

[2] Department of Mathematics, K. S. R. College of Technology
Tiruchengode-637215, Tamilnadu, India
e-mail: peeyesvee@yahoo.co.in

[3] Department of Mathematics, K. S. R. College of Technology
Tiruchengode-637215, Tamilnadu, India
e-mail: ezhil.devarasan@yahoo.com

Abstract: In this paper, we study some properties of intuitionistic fuzzy ideals of a Γ-near-ring and prove some results on these.

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1 Introduction

The notion of a fuzzy set was introduced by L. A. Zadeh [10], and since then this concept have been applied to various algebraic structures. The idea of intuitionistic fuzzy set was first published by K. T. Atanassov [1] as a generalization of the notion of fuzzy set. Γ-near-rings were defined by Bh. Satyanarayana [9] and G. L. Booth [2, 3] studied the ideal theory in Γ-near-rings. W. Liu [7] introduced fuzzy ideals and it has been studied by several authors. The notion of fuzzy ideals and its properties were applied to semi groups, BCK-algebras and semi rings. Y.B. Jun [5, 6] introduced the notion of fuzzy left (respectively, right) ideals.

In this paper, we introduce the notion of intuitionistic fuzzy ideals in Γ-near-rings and study some of its properties.

2 Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Definition 2.1. A non-empty set R with two binary operations " + " (addition) and "." (multiplication) is called a near-ring if it satisfies the following axioms:

(i) (R, +) is a group,

(ii) (R, **.**) is a semigroup,

(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$. It is a right near-ring because it satisfies the right distributive law.

Definition 2.2 A Γ-near-ring is a triple (M, +, Γ) where

(i) (M, +) is a group,

(ii) Γ is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma$, (M, +, α) is a near-ring,

(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all x, y, z $\in$ M and $\alpha, \beta \in \Gamma$.

Definition 2.3 A subset A of a Γ-near-ring M is called a left (respectively, right) ideal of M if

(i) (A, +) is a normal divisor of (M, +),

(ii) $u\alpha(x + v) - u\alpha v \in A$ (respectively, $x\alpha u \in A$) for all $x \in A$, $\alpha \in \Gamma$ and u, v $\in$ M.

Definition 2.4 A fuzzy set μ in a Γ-near-ring M is called a fuzzy left (respectively, right) ideal of M if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(y + x - y) \geq \mu(x)$, for all x, y $\in$ M.

(iii) $\mu(u\alpha(x + v) - u\alpha v) \geq \mu(x)$ (respectively, $\mu(x\alpha u) \geq \mu(x)$) for all x, u, v $\in$ M and $\alpha \in \Gamma$.

Definition 2.5 [1] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A: X \to [0, 1]$ denote the degree of membership and degree of non membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Notation. For the sake of simplicity, we shall use the symbol $A = \langle\mu_A, \nu_A\rangle$ for the IFS

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}.$$

Definition 2.6 [1]. Let X be a non-empty set and let $A = < \mu_A, \nu_A >$ and $B = < \mu_B, \nu_B >$ be IFSs in X. Then,

1. $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
2. $A = B$ iff $A \subset B$ and $B \subset A$.
3. $A^c = \langle\nu_A, \mu_A\rangle$.
4. $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
5. $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
6. $\Box A = (\mu_A, 1-\mu_A)$ $\Diamond A = (1-\nu_A, \nu_A)$.

Definition 2.7. Let μ and ν be two fuzzy sets in a Γ-near-ring. For s, t $\in$ [0, 1] the set $U(\mu,s) = \{x \in \mu(x) \geq s\}$ is called upper level of μ. The set $L(\nu, t) = \{x \in \nu(x) \leq t\}$ is called lower level of ν.

Definition 2.8. Let A be an IFS in a Γ-near-ring M. For each pair $\langle t, s\rangle \in [0, 1]$ with $t + s \leq 1$, the set $A_{\langle t, s\rangle} = \{x \in X \mid \mu_A(x) \geq t \text{ and } \nu_A(x) \leq s\}$ is called a $\langle t, s\rangle$–level subset of A.

Definition 2.9. Let $A = \langle\mu_A, \nu_A\rangle$ be an intuitionistic fuzzy set in M and let $t \in [0,1]$. Then, the sets $U(\mu_A ; t) = \{x \in M \mid \mu_A(x) \geq t\}$ and $L(\nu_A ; t) = \{x \in M \mid \nu_A(x) \leq t\}$ are called upper level set and lower level set of A, respectively.

3 Intuitionistic fuzzy ideals

In what follows, let M denote a Γ-near-ring, unless otherwise specified.

Definition 3.1. An IFS $A = \langle\mu_A, \nu_A\rangle$ in M is called an intuitionistic fuzzy left (respectively, right) ideal of a Γ-near-ring M if

(i) $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$,

(ii) $\mu_A(y + x - y) \geq \mu_A(x)$

(iii) $\mu_A(u\alpha(x + v) - u\alpha v) \geq \mu_A(x)$ (respectively, $\mu_A(x\alpha u) \geq \mu_A(x)$),

(iv) $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$,
(v) $\nu_A(y + x - y) \leq \nu_A(x)$,
(vi) $\nu_A(u\alpha(x + v) - u\alpha v) \leq \nu_A(x)$ (respectively, $\nu_A(x\alpha u) \leq \nu_A(x)$),
for all x, y, u, v $\in$ M and $\alpha \in \Gamma$.

Example 3.2. Let R be the set of all integers then R is a ring. Take M = Γ = R. Let a, b $\in$ M, $\alpha \in \Gamma$, suppose $a\alpha b$ is the product of a, α, b $\in$ R. Then, M is a Γ-near-ring.

Define an IFS A = $\langle \mu_A, \nu_A \rangle$ in R as follows.
$\mu_A(0) = 1$ and $\mu_A(\pm 1) = \mu_A(\pm 2) = \mu_A(\pm 3) = ... = t$ and
$\nu_A(0) = 0$ and $\nu_A(\pm 1) = \nu_A(\pm 2) = \nu_A(\pm 3) = ... = s$, where $t \in [0, 1]$, $s \in [0, 1]$ and $t + s \leq 1$.
By routine calculations, clearly A is an intuitionistic fuzzy ideal of a Γ-near-ring R.

Theorem 3.3. If A is an ideal of a Γ-near-ring M, then the IFS $\hat{A} = \langle \chi_A, \bar{\chi}_A \rangle$ is an intuitionistic fuzzy ideal of M.
Proof. Let x, y $\in$ M.
If x, y, u, v $\in$ A and $\alpha \in \Gamma$, then $x - y \in A$, $(y + x - y) \in A$ and $(u\alpha(x + v) - u\alpha v) \in A$, since A is an ideal of M.
Hence, $\chi_A(x - y) = 1 \geq \{\chi_A(x) \wedge \chi_A(y)\}$,
$\chi_A(y + x - y) = 1 \geq \chi_A(x)$ and
$\chi_A(u\alpha(x + v) - u\alpha v) = 1 \geq \chi_A(x)$ (respectively, $\chi_A(x\alpha u) \geq \chi_A(x)$) .
Also, we have

$0 = 1 - \chi_A(x - y) = \bar{\chi}_A(x - y) \leq \{\chi_A(x) \vee \bar{\chi}_A(y)\}$,
$0 = 1 - \chi_A(y + x - y) = \bar{\chi}_A(y + x - y) \leq \chi_A(x)$, and
$0 = 1 - \chi_A(u\alpha(x + v) - u\alpha v) = \bar{\chi}_A(u\alpha(x + v) - u\alpha v) \leq \bar{\chi}_A(x)$ (respectively, $\chi_A(x\alpha u) \leq \bar{\chi}_A(x)$).
If $x \notin A$ or $y \notin A$, then $\chi_A(x) = 0$ or $\chi_A(y) = 0$. Thus, we have
$\chi_A(x - y) \geq \{\chi_A(x) \wedge \chi_A(y)\}$,
$\chi_A(y + x - y) \geq \chi_A(x)$ and
$\chi_A(u\alpha(x + v) - u\alpha v) \geq \chi_A(x)$ (respectively, $\chi_A(x\alpha u) \geq \chi_A(x)$) for all $\alpha \in \Gamma$.
Also
$\bar{\chi}_A(x - y) \leq \{\chi_A(x) \vee \bar{\chi}_A(y)\}$,
$= \{(1 - \chi_A(x)) \vee (1 - \chi_A(y))\} = 1$
$\bar{\chi}_A(y + x - y) \leq \bar{\chi}_A(x)$
$= (1 - \chi_A(x)) = 1$
and
$\bar{\chi}_A(u\alpha(x + v) - u\alpha v) = 1 - \chi_A(u\alpha(x + v) - u\alpha v) \leq 1 - \chi_A(x) = \bar{\chi}_A(x)$.
This completes the proof.

Definition 3.4[3]. An intuitionistic fuzzy left (respectively, right) ideal A = $\langle \mu_A, \nu_A \rangle$ of a Γ-near-ring M is said to be normal if $\mu_A(0) = 1$ and $\nu_A(0) = 0$.

Theorem 3.5. Let A = $\langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy left (respectively, right) ideal of a Γ-near-ring M and let (x) = $\mu_A(x) + 1 - \mu_A(0)$, (x) = $\nu_A(x) - \nu_A(0)$. If (x) + (x) $\leq$ 1 for all x$\in$M, then $A^+ = \langle\ ,\ \rangle$ is a normal intuitionistic fuzzy left (respectively, right) ideal of M.

Proof. We first observe that $\mu_A^+(0) = 1$, $\nu_A^+(0) = 0$ and $\mu_A^+(x)$, $\nu_A^+(x) \in [0,1]$ for every x $\in$ M. Hence,
$A^+ = \langle \mu_A^+, \nu_A^+ \rangle$ is a normal intuitionistic fuzzy set. To prove that it is an intuitionistic fuzzy left (respectively, right) ideal, let x, y $\in$ M and $\alpha \in \Gamma$. Then,
$\mu_A^+(x - y) = \mu_A(x - y) + 1 - \mu_A(0)$

$\geq \{\mu_A(x) \wedge \mu_A(y)\} + 1 - \mu_A(0)$
$= \{\mu_A(x) + 1 - \mu_A(0)\} \wedge \{\mu_A(y) + 1 - \mu_A(0)\}$
$= \mu_A^+(x) \wedge \mu_A^+(y)$
$\nu_A^+(x - y) = \nu_A(x - y) - \nu_A(0)$
$\leq \{\nu_A(x) \vee \nu_A(y)\} - \nu_A(0)$
$= \{\nu_A(x) - \nu_A(0)\} \vee \{\nu_A(y) - \nu_A(0)\}$
$= \nu_A^+(x) \vee \nu_A^+(y)$,
$\mu_A^+(y + x - y) = \mu_A(y + x - y) + 1 - \mu_A(0)$
$\geq \{\mu_A(x) + 1 - \mu_A(0)\}$
$= \mu_A^+(x)$
$\nu_A^+(y + x - y) = \nu_A(y + x - y) - \nu_A(0)$
$\leq \{\nu_A(x) - \nu_A(0)\}$
$= \nu_A^+(x)$
and
$\mu_A^+(u\alpha(x + v) - u\alpha v) = \mu_A(u\alpha(x + v) - u\alpha v) + 1 - \mu_A(0)$
$\geq \mu_A(x) + 1 - \mu_A(0) = \mu_A^+(x)$
$\nu_A^+(u\alpha(x + v) - u\alpha v) = \nu_A (u\alpha(x + v) - u\alpha v) - \nu_A(0)$
$\leq \nu_A(x) - \nu_A(0) = \nu_A^+(x)$.
This shows that A^+ is an intuitionistic fuzzy left (respectively, right) ideal of M. So, A^+ is a normal intuitionistic fuzzy left (respectively, right) ideal of M.

Definition 3.6. Let I be an ideal of a Γ-near-ring M. If for each a+I, b+I in the factor group M/I and each $\alpha \in \Gamma$, we define $(a+I)\alpha(b+I) = a\alpha b+I$, then M/I is a Γ-near-ring which we shall call the Γ-residue class ring of M with respect to I.

Theorem 3.7. Let I be an ideal of a Γ-near-ring M. If A is an intuitionistic fuzzy left (respectively, right) ideal of M, then the IFS Ã of M/I defined by $\mu_{\tilde{A}}(a+I) = \vee_{x\in I} \mu_A(a+x)$ and $\nu_{\tilde{A}}(a+I) = \wedge_{x\in I} \nu_A(a+x)$ is an intuitionistic fuzzy left (respectively, right) ideal of the Γ-residue class ring M/I of M with respect to I.

Proof. Let a, b ∈ M be such that a + I = b + I.
Then b = a + y for some y ∈ I and so
$\mu_{\tilde{A}}(b+I) = \vee_{x\in I} \mu_A(b+x) = \vee_{x\in I} \mu_A(a+y+x) = \bigvee_{x+y=z\in I} \mu_A(a + z) = \mu_{\tilde{A}}(a + I)$,
$\nu_{\tilde{A}}(b+I) = \wedge_{x\in I} \nu_A(b+x) = \wedge_{x\in I} \nu_A(a+y+x) = \bigwedge_{x+y=z\in I} \nu_A(a + z) = \nu_{\tilde{A}}(a + I)$.

Hence, Ã is well defined.
For any x + I, y + I ∈ M/I and α ∈ Γ, we have
$\mu_{\tilde{A}}((x + I) - (y + I)) = \mu_{\tilde{A}}((x-y) + I)$
$= \bigvee_{z\in I} \mu_A((x-y) + z)$
$= \bigvee_{z=u-v\in I} \mu_A((x-y) + (u-v))$
$= \bigvee_{u,v\in I} \mu_A((x+u) - (y+v))$
$\geq \bigvee_{u,v\in I} (\mu_A(x+u) \wedge \mu_A(y+v))$
$= (\bigvee_{u\in I} \mu_A(x+u)) \wedge (\bigvee_{v\in I} \mu_A(y+v))$
$= \mu_{\tilde{A}}(x + I) \wedge \mu_{\tilde{A}}(y + I)$.

$\nu_{\tilde{A}}((x + I) - (y + I)) = \nu_{\tilde{A}}((x-y) + I)$

$= \bigwedge_{z\in I} \nu_A((x-y) + z)$

$= \bigwedge_{z=u-v\in I} \nu_A((x-y) + (u-v))$

$= \bigwedge_{u,v\in I} \nu_A((x+u) - (y+v))$

$\leq \bigwedge_{u,v\in I} (\nu_A(x+u) \vee \nu_A(y+v))$

$= (\bigwedge_{u\in I} \nu_A(x+u)) \vee (\bigwedge_{v\in I} \nu_A(y+v))$

$= \nu_{\tilde{A}}(x + I) \vee \nu_{\tilde{A}}(y + I),$

$\mu_{\tilde{A}}((y + I) + ((x + I) - (y + I)) = \mu_{\tilde{A}}((y + x - y) + I)$

$= \bigvee_{z\in I} \mu_A((y + x - y) + z)$

$= \bigvee_{z=v+(u-v)\in I} \mu_A((y + x - y) + (v + (u - v)))$

$= \bigvee_{u,v\in I} \mu_A((y + v) + ((x + u) - (y + v)))$

$\geq \bigvee_{u\in I} (\mu_A(x+u))$

$= \mu_{\tilde{A}}(x + I) .$

$\nu_{\tilde{A}}((y + I) + ((x + I) - (y + I)) = \nu_{\tilde{A}}((y + x - y) + I)$

$= \bigwedge_{z\in I} \nu_A((y + x - y) + z)$

$= \bigwedge_{z=u-v\in I} \nu_A((y + x - y) + (v + (u - v)))$

$= \bigwedge_{u,v\in I} \nu_A((y + v) + ((x + u) - (y + v)))$

$\leq (\bigwedge_{u\in I} \nu_A(x+u))$

$= \nu_{\tilde{A}}(x + I),$

$\mu_{\tilde{A}}((a + I)\alpha((x + I) + (b + I)) - ((a + I)\alpha(b + I))) = \mu_{\tilde{A}}((a\alpha(x + b) -a\alpha b) + I)$

$= \bigvee_{z\in I} \mu_A((a\alpha(x + b) -a\alpha b) + z)$

$\geq \bigvee_{z\in I} \mu_A(a\alpha x + a\alpha z)$ because $a\alpha z \in I$

$= \bigvee_{z\in I} \mu_A(a\alpha(x + z)) \geq \bigvee_{z\in I} \mu_A(x + z) = \mu_{\tilde{A}}(x + I),$

$\nu_{\tilde{A}}((a + I)\alpha((x + I) + (b + I)) - ((a + I)\alpha(b + I))) = \nu_{\tilde{A}}((a\alpha(x + b) -a\alpha b) + I)$

$= \bigwedge_{z\in I} \nu_A((a\alpha(x + b) -a\alpha b) + z)$

$\leq \bigwedge_{z\in I} \nu_A(a\alpha x + a\alpha z)$ because $a\alpha z \in I$

$= \bigwedge_{z\in I} \nu_A(a\alpha(x+z)) \le \bigwedge_{z\in I} \nu_A(x+z) = \nu_{\tilde{A}}(x+I)$.

Similarly,

$$\mu_{\tilde{A}}((x+I)\alpha(a+I)) \ge \mu_{\tilde{A}}(x+I) \text{ and } \nu_{\tilde{A}}((x+I)\alpha(a+I)) \le \nu_{\tilde{A}}(x+I).$$

Hence, $\tilde{A}$ is an intuitionistic fuzzy left (respectively, right) ideal of M/I.

Theorem 3.8. If the IFS $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy left (respectively, right) ideal of M, then the set $M_A = \{x \in M \mid \mu_A(x) = \mu_A(0) \text{ and } \nu_A(x) = \nu_A(0)\}$ is an ideal of M.

Proof. Let x, y $\in M_A$. Then $\mu_A(x) = \mu_A(y) = \mu_A(0)$ and $\nu_A(x) = \nu_A(y) = \nu_A(0)$. Since A is an intuitionistic fuzzy ideal of M, it follows that

$$\mu_A(x-y) \ge \{\mu_A(x) \wedge \mu_A(y)\} = \{\mu_A(0) \wedge \mu_A(0) = \mu_A(0),$$
$$\nu_A(x-y) \le \{\nu_A(x) \vee \nu_A(y)\} = \{\nu_A(0) \vee \nu_A(0)\} = \nu_A(0).$$

Hence, $\mu_A(x-y) = \mu_A(0)$ and $\nu_A(x-y) = \nu_A(0)$. So, $x - y \in M_A$.

$$\mu_A(y+x-y) \gtrsim \mu_A(x) = \mu_A(0),$$
$$\nu_A(y+x-y) \le \nu_A(x) = \nu_A(0).$$

Hence, $\mu_A(y+x-y) = \mu_A(0)$ and $\nu_A(y+x-y) = \nu_A(0)$. So $y + x - y \in M_A$.

Let $x \in M, \alpha \in \Gamma$ and $y \in M_A$. Therefore, $\mu_A(x\alpha(y+z) - x\alpha z) \ge \mu_A(x) = \mu_A(0)$ (respectively, $\mu_A(y\alpha x) \ge \mu_A(x) = \mu_A(0)$) and $\nu_A(x\alpha(y+z) - x\alpha z) \le \nu_A(y) = \nu_A(0)$ (respectively, $\nu_A(y\alpha x) \le \nu_A(x) = \nu_A(0)$). Hence, $\mu_A(x\alpha(y+z) - x\alpha z) = \mu_A(0)$ and $\nu_A(x\alpha(y+z) - x\alpha z) = \nu_A(0)$.

So, $(x\alpha(y+z) - x\alpha z) \in M_A$. Hence, M_A is an intuitionistic fuzzy ideal of M.

Theorem 3.9. Let A be an intuitionistic fuzzy left (respectively, right) ideal of a Γ-near-ring M. For each pair $\langle t, s\rangle \in [0, 1]$, the level set $A_{\langle t, s\rangle}$ is an ideal of M.

Proof. Let x, y $\in A_{\langle t, s\rangle}$. Then $\mu_A(x) \ge t$, $\mu_A(y) \ge t$ and $\nu_A(x) \lesssim s$, $\nu_A(y) \le s$. Since A is an intuitionistic fuzzy left (respectively, right) ideal, we have

$\mu_A(x-y) \ge \{\mu_A(x) \wedge \mu_A(y)\} \ge t$ and $\nu_A(x-y) \le \{\nu_A(x) \vee \nu_A(y)\} \lesssim s$.

So $x - y \in A_{\langle t, s\rangle}$.

$\mu_A(y+x-y) \ge \mu_A(x) \ge t$ and $\nu_A(y+x-y) \le \nu_A(x) \lesssim s$.

So $y + x - y \in A_{\langle t, s\rangle}$.

Let $x \in M$, $y \in A_{\langle t, s\rangle}$ and $\alpha \in \Gamma$.

Then $\mu_A(x\alpha(y+z) - x\alpha z) \ge \mu_A(y) \ge t$ and $\nu_A(x\alpha(y+z) - x\alpha z) \le \nu_A(y) \lesssim s$.

So $(x\alpha(y+z) - x\alpha z) \in A_{\langle t, s\rangle}$.

Hence, $A_{\langle t, s\rangle}$ is an ideal of M.

Definition 3.10. Let A and B be two intuitionistic fuzzy subsets of a Γ-near-ring M and $\alpha \in \Gamma$. The product AΓB is defined by

$$\mu_{A\Gamma B}(x) = \begin{cases} \bigvee_{x=(u\gamma(v+w)-u\gamma w)} \left(\mu_A(u) \wedge \mu_B(v)\right) & \textit{for } u, v \in M, \quad \gamma \in \Gamma \\ 0 & \textit{otherwise}, \end{cases}$$

$$\nu_{A\Gamma B}(x) = \begin{cases} \bigwedge_{x=(u\gamma(v+w)-u\gamma w)} \left(\nu_A(u) \vee \nu_B(v)\right) & \textit{for } u, v \in M, \quad \gamma \in \Gamma \\ 1 & \textit{otherwise}. \end{cases}$$

Definition 3.11 Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be two IFSs in a Γ-near-ring M. Then, the composition of A and B is defined to be the intuitionistic fuzzy set $A_oB = \langle \mu_{A_oB}, \nu_{A_oB} \rangle$ in M given by

$$\mu_{AoB}(x)=\bigvee\begin{cases}\bigwedge_{1\le i\le k}\big(\mu_A(u_i)\wedge\mu_B(v_i)\big): x=\sum_1^k(u_i\gamma_i(v_i+w_i)-u_i\gamma_i w_i), u_i, v_i\in M, \gamma_i\in\Gamma, k\in N\\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \textit{Otherwise}\end{cases}$$

$$\nu_{AoB}(x)=\bigwedge\begin{cases}\bigvee_{1\le i\le k}\big(\nu_A(u_i)\vee\nu_B(v_i)\big): x=\sum_1^k(u_i\gamma_i(v_i+w_i)-u_i\gamma_i w_i), u_i, v_i\in M, \gamma_i\in\Gamma, k\in N\\ 1 \qquad\qquad\qquad\qquad\qquad\qquad \textit{Otherwise}\end{cases}$$

Theorem 3.12. If $A=\langle\mu_A,\nu_A\rangle$ and $B=\langle\mu_B,\nu_B\rangle$ are intuitionistic fuzzy ideals in a Γ-near-ring M, then AoB is an intuitionistic fuzzy ideal in M.

Proof. For any $x, y\in M$, we have

$$\mu_{AoB}(x-y)=\vee\{\bigwedge_{1\le i\le k}\mu_A(u_i)\wedge\mu_B(v_i) : x-y=\sum_1^k(u_i\alpha(v_i+u')-u_i\alpha u'), u_i, v_i, u'\in M, \alpha\in\Gamma \text{ and } k\in N\}$$

$$\ge\vee\{(\bigwedge_{1\le i\le m}\mu_A(a_i)\wedge\mu_B(b_i))\wedge(\bigwedge_{1\le i\le n}\mu_A(-c_i)\wedge\mu_B(d_i)) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'),$$

$$-y=\sum_1^n-(c_i\alpha(d_i+c')-c_i\alpha c'), a_i, b_i, c_i, d_i, a', c'\in M, \alpha\in\Gamma \text{ and } m, n\in N\}$$

$$=\vee\{(\bigwedge_{1\le i\le m}\mu_A(a_i)\wedge\mu_B(b_i))\wedge(\bigwedge_{1\le i\le n}\mu_A(c_i)\wedge\mu_B(d_i)) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'), y=\sum_1^n(c_i\alpha(d_i+c')-c_i\alpha c'),$$
$$a_i, b_i, c_i, d_i, a', c'\in M, \alpha\in\Gamma \text{ and } m, n\in N\}$$

$$=\vee\{\bigwedge_{1\le i\le m}\mu_A(a_i)\wedge\mu_B(b_i) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'), a_i, b_i, a'\in M, \alpha\in\Gamma \text{ and } m\in N\}\wedge$$

$$\vee\{\bigwedge_{1\le i\le m}\mu_A(c_i)\wedge\mu_B(d_i) : y=\sum_1^n(c_i\alpha(d_i+c')-c_i\alpha c'), c_i, d_i, c'\in M, \alpha\in\Gamma \text{ and } n\in N\}$$

$$=\mu_{AoB}(x)\wedge\mu_{AoB}(y)$$

$$\nu_{AoB}(x-y)=\wedge\{\bigvee_{1\le i\le k}\nu_A(u_i)\vee\nu_B(v_i) : x-y=\sum_1^k(u_i\alpha(v_i+u')-u_i\alpha u'), u_i, v_i, u'\in M, \alpha\in\Gamma \text{ and } k\in N\}$$

$$\le\wedge\{(\bigvee_{1\le i\le k}\nu_A(a_i)\vee\nu_B(b_i))\vee(\bigvee_{1\le i\le n}\nu_A(-c_i)\vee\nu_B(d_i)) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'),$$

$$-y=\sum_1^n-(c_i\alpha(d_i+c')-c_i\alpha c'), a_i, b_i, c_i, d_i, a', c'\in M, \alpha\in\Gamma \text{ and } m, n\in N\}$$

$$=\wedge\{(\bigvee_{1\le i\le k}\nu_A(a_i)\vee\nu_B(b_i))\vee(\bigvee_{1\le i\le n}\nu_A(c_i)\vee\nu_B(d_i)) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'),$$

$$y=\sum_1^n(c_i\alpha(d_i+c')-c_i\alpha c'), a_i, b_i, c_i, d_i, a', c'\in M, \alpha\in\Gamma \text{ and } m, n\in N\}$$

$$=\wedge\{\bigvee_{1\le i\le k}\nu_A(a_i)\vee\nu_B(b_i) : x=\sum_1^m(a_i\alpha(b_i+a')-a_i\alpha a'), a_i, b_i, a'\in M, \alpha\in\Gamma \text{ and } m\in N\}\vee$$

$$\wedge\{\bigvee_{1\le i\le m}\nu_A(c_i)\vee\nu_B(d_i) : y=\sum_1^n(c_i\alpha(d_i+c')-c_i\alpha c'), c_i, d_i, c'\in M, \alpha\in\Gamma \text{ and } n\in N\}$$

$$=\nu_{AoB}(x)\vee\nu_{AoB}(y).$$

$$\mu_{AoB}(y+x-y)\ge\vee\{\bigwedge_{1\le i\le k}\mu_A(u_i) : x=\sum_1^k(u_i\alpha(v_i+u')-u_i\alpha u'), u_i, v_i, u'\in M, \alpha\in\Gamma \text{ and } k\in N\}$$

$= \vee\{(\bigwedge_{1\le i\le m} \mu_A(a_i) \wedge \mu_B(b_i)) : x = \sum_1^m (a_i\alpha(b_i+a') - a_i\alpha a'), a_i, b_i, a'\in M, \alpha\in\Gamma \text{ and } m \in N \}$
$= \mu_{A\circ B}(x)$

$\nu_{A\circ B}(y + x - y) \le \wedge\{ \bigvee_{1\le i\le k} \nu_A(u_i) : x = \sum_1^k (u_i\alpha(v_i + u') - u_i\alpha u'), u_i, v_i, u' \in M, \alpha\in\Gamma \text{ and } k \in N \}$
$= \wedge\{(\bigvee_{1\le i\le k} \nu_A(a_i) \vee \nu_B(b_i)): x = \sum_1^m (a_i\alpha(b_i+a') - a_i\alpha a'), a_i, b_i, a'\in M, \alpha\in\Gamma \text{ and } m \in N \}$
$= \nu_{A\circ B}(x)$.

Also
$\mu_{A\circ B}(x) = \vee\{ \bigwedge_{1\le i\le m} \mu_A(a_i) \wedge \mu_B(b_i) : x = \sum_1^m (a_i\alpha(b_i+a') - a_i\alpha a') , a_i, b_i, a' \in M, \alpha\in\Gamma \text{ and } m \in N \}$
$\le \vee\{ \bigwedge_{1\le i\le m} \mu_A(a_i) \wedge \mu_B(b_i\alpha y) : (x\alpha(y+z) - x\alpha z) = \sum_1^m ((a_i\alpha(b_i+a') - a_i\alpha a')\alpha y), a_i, b_i\alpha y \in M, \alpha\in \Gamma \text{ and } m \in N\}$
$= \vee\{ \bigwedge_{1\le i\le m} \mu_A(u_i) \wedge \mu_B(v_i) : (x\alpha(y + z) - x\alpha z) = \sum_1^m (u_i\alpha(v_i + u') - u_i\alpha u'), u_i, v_i, u' \in M, \alpha \in \Gamma \text{ and } m \in N\}$
$= \mu_{A\circ B}((x\alpha(y + z) - x\alpha z))$

$\nu_{A\circ B}(x) = \wedge\{ \bigvee_{1\le i\le m} \nu_A(a_i) \vee \nu_B(b_i) : x = \sum_1^m (a_i\alpha(b_i + a') - a_i\alpha a') , a_i, b_i \in M, \alpha\in\Gamma \text{ and } m \in N \}$
$\ge \wedge\{ \bigvee_{1\le i\le m} \nu_A(a_i) \vee \nu_B(b_i\alpha y) : (x\alpha(y + z) - x\alpha z) = \sum_1^m ((a_i\alpha(b_i+a') - a_i\alpha a')\alpha y), a_i, b_i\alpha y\in M, \alpha\in\Gamma \text{ and } m\in N\}$
$= \wedge\{ \bigvee_{1\le i\le m} \nu_A(u_i) \vee \nu_B(v_i) : (x\alpha(y + z) - x\alpha z) = \sum_1^m (u_i\alpha(v_i + u') - u_i\alpha u'), u_i, v_i, u' \in M, \alpha\in\Gamma \text{ and } m \in N\}$
$= \nu_{A\circ B}(x\alpha(y + z) - x\alpha z)$.

That is, $\mu_{A\circ B}(x\alpha(y + z) - x\alpha z) \ge \mu_{A\circ B}(x)$ and $\nu_{A\circ B}(x\alpha(y + z) - x\alpha z) \le \nu_{A\circ B}(x)$.
Similarly, we get $\mu_{A\circ B}(y\alpha x) \ge \mu_{A\circ B}(x)$ and $\nu_{A\circ B}(y\alpha x) \le \nu_{A\circ B}(x)$.
Hence, $A\circ B$ is an intuitionistic fuzzy ideal of M.

Definition 3.13. A function $f : M \to N$, where M and N are Γ-near-ring, is said to be a Γ-homomorphism if $f(a +b) = f(a) + f(b)$, $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 3.14. A function $f : M \to N$, where f is a Γ-homomorphism and M and N are Γ-near-ring, is said to be a Γ-endomorphism if $N \subseteq M$.

Definition 3.15. Let $f : X \to Y$ be a mapping of Γ-near-rings and A be an intuitionistic fuzzy set of Y. Then, the map $f^{-1}(A)$ is the pre-image of A under f, if $\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$ and $\nu_{f^{-1}(A)}(x) = \nu_A(f(x))$, for all $x \in X$.

Theorem 3.16. Let f be a Γ-homomorphism of M. If the IFS $A = \langle\mu_A, \nu_A\rangle$ is an intuitionistic fuzzy left (respectively, right) ideal of M, then $B = \langle\mu_{f^{-1}(A)}, \nu_{f^{-1}(A)}\rangle$ is an intuitionistic fuzzy left (respectively, right) ideal of M.

Proof. For any $x, y \in M$, $\alpha \in \Gamma$, we have
$\mu_{f^{-1}(A)}(x - y) = \mu_A(f(x - y)) = \mu_A(f(x) - f(y))$
$\ge \{\mu_A(f(x)) \wedge \mu_A(f(y))\}$
$= \{\mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(A)}(y)\}$,
$\mu_{f^{-1}(A)}(y + x - y) = \mu_A(f(y + x - y)) = \mu_A(f(y) + f(x) - f(y))$

$. \geq \mu_A(f(x))$
$= \mu_{f^{-1}(A)}(x)$

and

$\mu_{f^{-1}(A)}(x\alpha(y + z) - x\alpha z) = \mu_A(f(x\alpha y))$
$= \mu_A(f(x)\alpha f(y))$
$. \geq \mu_{f^{-1}(A)}(x).$

Similarly,

$\nu_{f^{-1}(A)}(x - y) = \nu_A(f(x - y)) = \nu_A(f(x) - f(y))$
$\leq \{\nu_A(f(x)) \vee \nu_A(f(y))\}$
$= \{\nu_{f^{-1}(A)}(x) \vee \nu_{f^{-1}(A)}(y)\}$ and
$\nu_{f^{-1}(A)}(y + x - y) = \nu_A(f(y + x - y)) = \nu_A(f(y) + f(x) - f(y))$
$\leq \nu_A(f(x))$
$= \nu_{f^{-1}(A)}(x)$
$\nu_{f^{-1}(A)}((x\alpha(y + z) - x\alpha z)) = \nu_A(f(x\alpha y))$
$= \nu_A(f(x)\alpha f(y))$
$\leq \nu_{f^{-1}(A)}(y).$

Hence, B is an intuitionistic fuzzy left (respectively, right) ideal of M.

Theorem 3.17. If $A = \langle\mu_A, \nu_A\rangle$ is an intuitionistic fuzzy set in M such that the non-empty sets $U(\mu_A; t)$ and $L(\nu_A; t)$ are ideals of M for all $t \in [0, 1]$, then A is an intuitionistic fuzzy left (respectively, right) ideal of M.

Proof. Suppose that there exists $x_0, y_0 \in M$ such that $\mu_A(x_0 - y_0) < \{\mu_A(x_0) \wedge \mu_A(y_0)\}$. Let $t_0 = ½ \{\mu_A(x_0 - y_0) +(\mu_A(x_0) \wedge \mu_A(y_0))\}$. Then, $(\mu_A(x_0) \wedge \mu_A(y_0)) \geq t_0 > \mu_A(x_0 - y_0)$. It follows that $x_0, y_0 \in U(\mu_A; t_0)$ and $x_0 - y_0 \notin U(\mu_A; t_0)$. This is a contradiction.
Hence, $\mu_A(x - y) \geq \{\mu_A(x) \wedge \mu_A(y)\}$, for all $x , y \in M$.
Suppose that there exists $x_0, y_0 \in M$ such that $\mu_A(y_0 + x_0 - y_0) < \mu_A(x_0)$.
Let $t_0 = ½ \{\mu_A(y_0 + x_0 - y_0) + \mu_A(x_0)\}$. Then $\mu_A(x_0) \geq t_0 > \mu_A(y_0 + x_0 - y_0)$. It follows that $x_0, y_0 \in U(\mu_A ; t_0)$ and $y_0 + x_0 - y_0 \notin U(\mu_A ; t_0)$. This is a contradiction.
Hence, $\mu_A(y + x - y) \geq \mu_A(x)$, for all $x , y \in M$.
Now let $x_0, y_0 \in M$ and $\alpha \in \Gamma$ such that $\mu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) < \mu_A(x_0)$.
Let $t_0 = ½ \{\mu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) + \mu_A(x_0)\}$.
Then we get $\mu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) \leq t_0 < \mu_A(x_0)$. It follows that $y_0 \in U(\mu_A ; t_0)$ and $x_0\alpha((y_0 + z_0) - x_0\alpha z_0) \notin U(\mu_A ; t_0)$. This is a contradiction.
Thus, $\mu_A(x\alpha(y + z) - x\alpha z) \geq \mu_A(x)$ (respectively, $\mu_A(y\alpha x) \geq \mu_A(x)$).
Similarly, suppose that there exists $x_0, y_0 \in M$ such that $\nu_A(x_0 - y_0) > \{\nu_A(x_0) \vee \nu_A(y_0)\}$.
Let $t_0 = ½ \{\nu_A(x_0 - y_0) + (\nu_A(x_0) \vee \nu_A(y_0))\}$. Then $(\nu_A(x_0) \vee \nu_A(y_0)) \leq t_0 < \nu_A(x_0 - y_0)$.
It follows that $x_0, y_0 \in L(\mu_A ; t_0)$ and $x_0 - y_0 \notin L(\mu_A ; t_0)$. This is a contradiction.
Hence, $\nu_A(x - y) \leq \{\nu_A(x) \vee \nu_A(y)\}$, for all $x, y \in M$.
Suppose that there exists $x_0, y_0 \in M$ such that $\nu_A(y_0 + x_0 - y_0) < \nu_A(x_0)$.
Let $t_0 = ½ \{\nu_A(y_0 + x_0 - y_0) + \nu_A(x_0)\}$. Then, $\nu_A(x_0) \geq t_0 > \nu_A(y_0 + x_0 - y_0)$.
It follows that $x_0, y_0 \in U(\mu_A ; t_0)$ and $y_0 + x_0 - y_0 \notin U(\mu_A ; t_0)$. This is a contradiction.
Hence, $\nu_A(y + x - y) \geq \nu_A(x)$, for all $x, y \in M$.
Now let $x_0, y_0 \in M$ and $\alpha \in \Gamma$ such that $\nu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) > \nu_A(x_0)$.
Let $t_0 = ½ \{\nu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) + \nu_A(x_0)\}$.
Then we get $\nu_A(x_0\alpha((y_0 + z_0) - x_0\alpha z_0)) > t_0 > \nu_A(x_0)$. It follows that $y_0 \in L(\mu_A ; t_0)$ and $x_0\alpha y_0 \notin L(\nu_A ; t_0)$. This is a contradiction. Thus, $\nu_A(x\alpha(y + z) - x\alpha z) \leq \nu_A(x_0)$ (respectively, $\nu_A(y\alpha x) \leq \nu_A(x_0)$). Hence, A is an intuitionistic fuzzy left (respectively, right) ideal of M.

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